Coupled Oscillators

- Simple pendulum:

\[ \ddot{\theta} + \omega^2 \sin \theta = 0 \]
\[ \ddot{\theta} + \omega^2 \theta \approx 0 \]
\[ \omega = \sqrt{\frac{l}{g}} \]

\[ x \approx l\theta \]
\[ \ddot{x} + \omega^2 x \approx 0 \]
\[ x(t) = A \cos(\omega t + \alpha) \]
Two Independent Oscillators

• Two simple pendula:

\[
\ddot{x}_A + \omega^2 x_A \approx 0 \\
\ddot{x}_B + \omega^2 x_B \approx 0
\]

\[
x_A(t) = A \cos(\omega t + \alpha) \\
x_B(t) = B \cos(\omega t + \beta)
\]
Two Coupled Oscillators

- Two simple pendula connected to a spring:

- There are many types of motion possible now.
- The solutions are not independent.
- We can consider two “modes” of oscillation.
Two Coupled Oscillators

- The spring is at its relaxed length and exerts no force on A or B.
- Each pendulum oscillates at its natural frequency

\[ \omega_0 = \sqrt{\frac{g}{\ell}} \]

\[ x_A(t) = x_B(t) = A \cos(\omega t + \alpha) \]

One differential equation describes both pendula.
Two Coupled Oscillators

- In this case, $x_A = -x_B$
- The spring is stretched or compressed and produces $F_A = -k(x_A - x_B) = -2kx_A$
- Differential equation for A:
  $$\ddot{x}_A + \left[ (\omega_0)^2 + \frac{2k}{m} \right] x_A = 0$$
- Differential equation for B:
  $$\ddot{x}_B + \left[ (\omega_0)^2 + \frac{2k}{m} \right] x_B = 0$$
Two Coupled Oscillators

\[ \ddot{x}_A + [(\omega_0)^2 + 2(\omega_c)^2]x_A = 0 \]

- This is just the differential equation for simple harmonic motion:
  \[ \ddot{x}_A + \omega'^2 x_A = 0 \]
- Oscillation frequency is
  \[ \omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2} \]
- The spring increases the restoring force and increases the frequency.
Two Coupled Oscillators

- We have identified two modes of the system:
  - One oscillates with frequency $\omega_0 = \sqrt{g/\ell}$
  - The other with frequency $\omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$
- These are the only two normal modes of the system.
- But we can superimpose the solutions to describe arbitrary motion.
Two Coupled Oscillators

- The spring is stretched by the amount $x_A - x_B$
- Restoring force on pendulum A:
  $$F_A = -k(x_A - x_B)$$
- Restoring force on pendulum B:
  $$F_B = k(x_A - x_B)$$

\[
\begin{align*}
    m\ddot{x}_A + \frac{mg}{\ell} x_A + k(x_A - x_B) &= 0 \\
    m\ddot{x}_B + \frac{mg}{\ell} x_B - k(x_A - x_B) &= 0
\end{align*}
\]
Two Coupled Oscillators

\[ \ddot{x}_A + (\omega_0)^2 x_A + \frac{k}{m} (x_A - x_B) = 0 \]
\[ \ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2] x_A - (\omega_c)^2 x_B = 0 \]
\[ \ddot{x}_B + (\omega_0)^2 x_B - k(x_A - x_B) = 0 \]
\[ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2] x_B - (\omega_c)^2 x_A = 0 \]

- Each equation contains a term in the other coordinate
- The motion of A affects B and the motion of B affects A
- They must be solved simultaneously
Two Coupled Oscillators

\[ \ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0 \]
\[ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0 \]

- Add equations for A and B together:
  \[ \frac{d^2}{dt^2} (x_A + x_B) + (\omega_0)^2 (x_A + x_B) = 0 \]

- Subtract equations A and B:
  \[ \frac{d^2}{dt^2} (x_A - x_B) + [(\omega_0)^2 + 2(\omega_c)^2](x_A - x_B) = 0 \]
Two Coupled Oscillators

- We have successfully “decoupled” the differential equations:

\[
\frac{d^2}{dt^2} (x_A + x_B) + (\omega_0)^2 (x_A + x_B) = 0
\]

\[
\frac{d^2}{dt^2} (x_A - x_B) + (\omega')^2 (x_A - x_B) = 0
\]

where \( \omega_0 = \sqrt{g/\ell} \) and \( \omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2} \)

- We just need to re-label the coordinates:

\[
q_1 = x_A + x_B
\]

\[
q_2 = x_A - x_B
\]
Two Coupled Oscillators

• Decoupled equations:
  \[ \ddot{q}_1 + (\omega_0)^2 q_1 = 0 \]
  \[ \ddot{q}_2 + (\omega')^2 q_2 = 0 \]

• Solutions are
  \[ q_1(t) = A \cos(\omega_0 t + \alpha) \]
  \[ q_2(t) = B \cos(\omega' t + \beta) \]

• The variables \( q_1 \) and \( q_2 \) are called “normal coordinates”.

...
Initial Conditions

• Suppose we had the initial conditions:
  \[x_A = A_0 \quad x_A' = 0\]
  \[x_B = 0 \quad x_B' = 0\]

• Try to satisfy these when \(\alpha = \beta = 0\):
  \[x_A(t) = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}A \cos \omega_0 t + \frac{1}{2}B \cos \omega' t\]
  \[x_B(t) = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}A \cos \omega_0 t - \frac{1}{2}B \cos \omega' t\]

• At time \(t = 0\),
  \[\frac{1}{2}(A + B) = A_0 \quad \frac{1}{2}(A - B) = 0\]

• Now we know that \(A = B = A_0\).
Initial Conditions

• Velocity:

\[
\dot{x}_A(t) = -\frac{1}{2}A_0 \omega_0 \sin \omega_0 t - \frac{1}{2}A_0 \omega' \sin \omega' t
\]

\[
\dot{x}_B(t) = -\frac{1}{2}A_0 \omega_0 \sin \omega_0 t + \frac{1}{2}A_0 \omega' \sin \omega' t
\]

• Initial conditions are satisfied at \( t = 0 \).
Initial Conditions

• Complete solution:

\[ x_A(t) = \frac{1}{2} A_0 (\cos \omega_0 t + \cos \omega' t) \]

\[ x_B(t) = \frac{1}{2} A_0 (\cos \omega_0 t - \cos \omega' t) \]