Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 10 – French, Chapter 4

Spring 2013 Semester

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Forced Oscillators and Resonance

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

- Natural oscillation frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \omega_{free} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

- Amplitude of steady-state oscillations:

$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

- Phase difference:

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2}\right)$$

Resonance Phenomena

- Change of variables:

$$\gamma = \frac{b}{m} \qquad Q = \frac{\omega_0}{\gamma}$$

Strong damping force ➔ large γ
Strong damping force ➔ small Q

- Natural oscillation frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \omega_{free} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2}} = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

- Amplitude of steady-state oscillations:

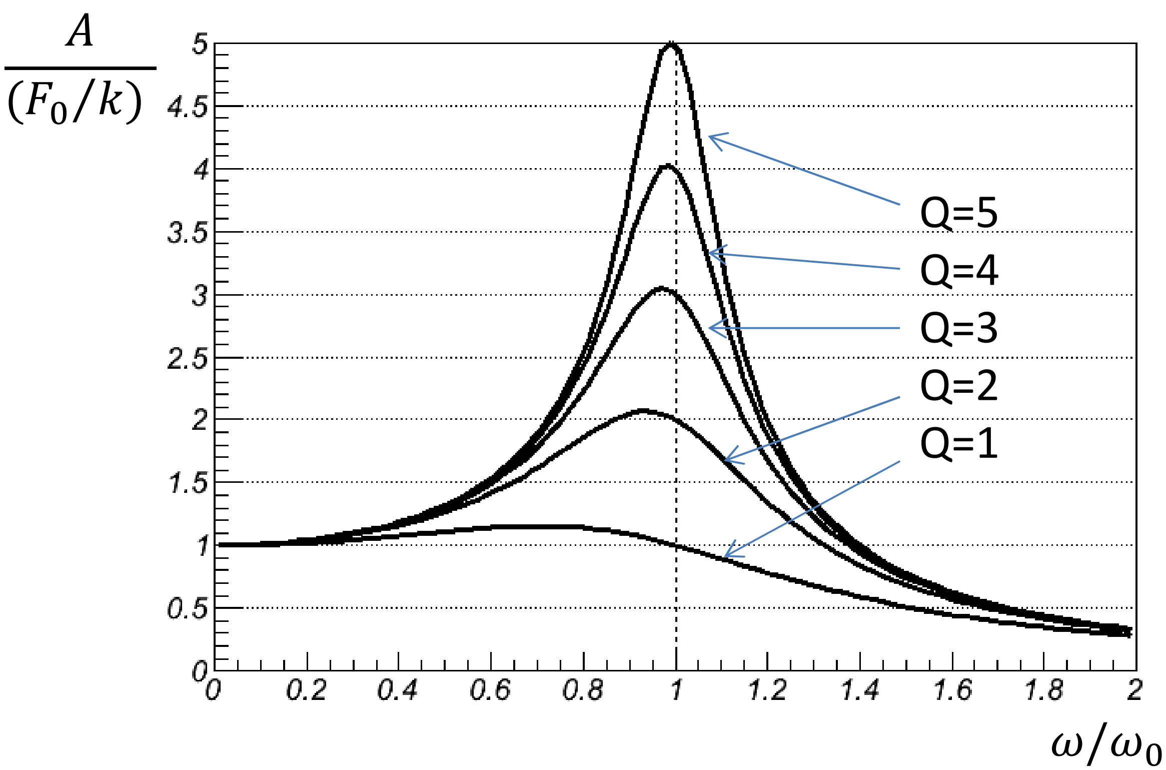
$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Phase difference:

$$\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$

Resonance Phenomena

- Steady state amplitude:

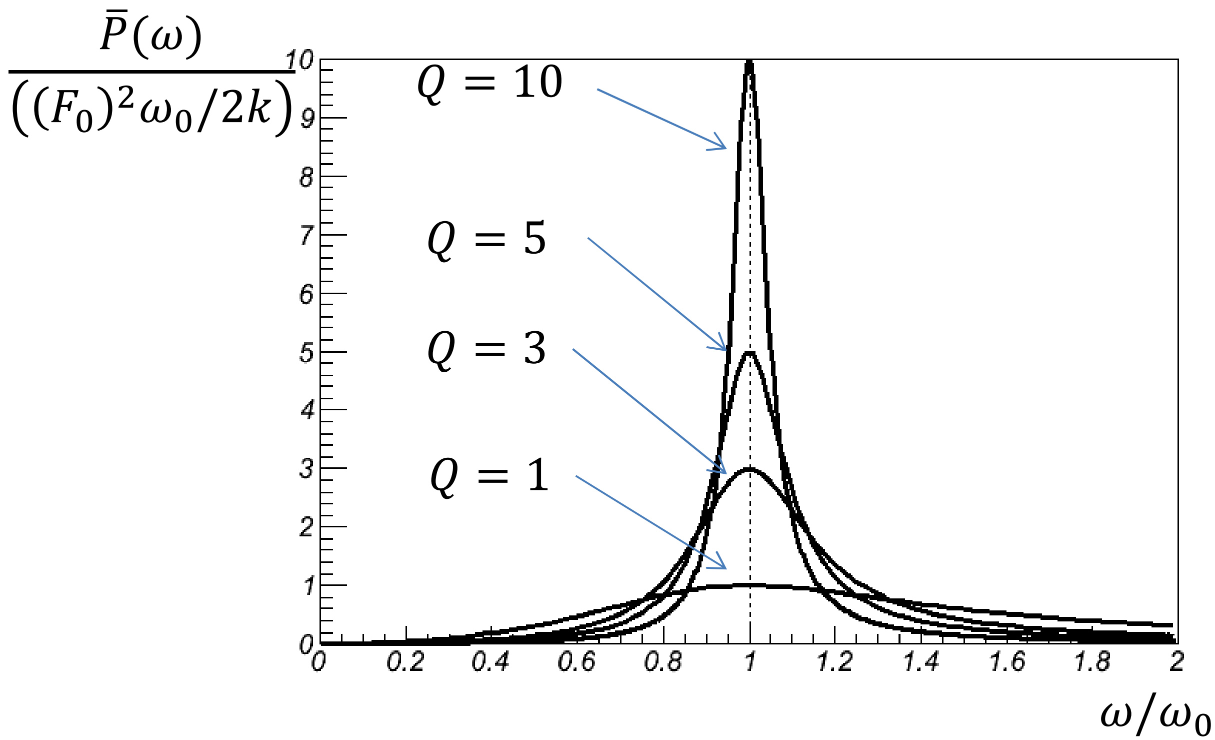


Peak is near $\omega/\omega_0 \approx 1$. The peak occurs at *exactly* $\omega/\omega_{free} = 1$.

Average Power

- The rate at which the oscillator absorbs energy is:

$$\bar{P}(\omega) = \frac{(F_0)^2\omega_0}{2kQ}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

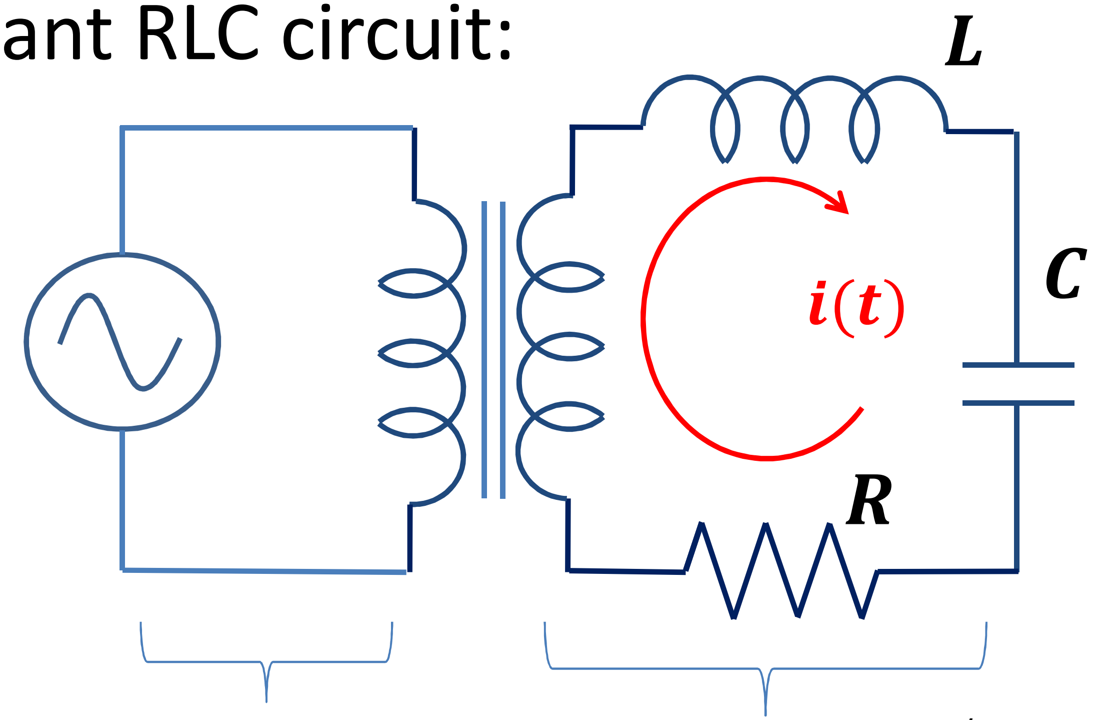


Full-Width-at-Half-Max:

$$FWHM = \frac{\omega_0}{Q} = \gamma$$

Examples

- Resonant RLC circuit:



$$\mathcal{E}_0 \cos \omega t$$

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\int_0^t i(t)dt$$

The transformer does not play a role in the analysis of the circuit. It is just a convenient way to isolate the driving voltage source from the part of the circuit that oscillates.

Resonant Circuit

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\int_0^t i(t)dt = \mathcal{E}_0 \cos \omega t$$

- Differentiate once:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = -\mathcal{E}_0 \omega \sin \omega t$$

- Redefine what we man by "$t = 0$":

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = \mathcal{E}_0 \omega \cos \omega t$$

- Change of variables:

$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + (\omega_0)^2 i(t) = \frac{\mathcal{E}_0 \omega}{L}\cos \omega t$$

Resonant Circuit

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + (\omega_0)^2 i(t) = \frac{\mathcal{E}_0 \omega}{L} \cos \omega t$$

- Amplitude of steady state current oscillations:

$$A(\omega) = \mathcal{E}_0 \omega C \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Voltage across the capacitor:

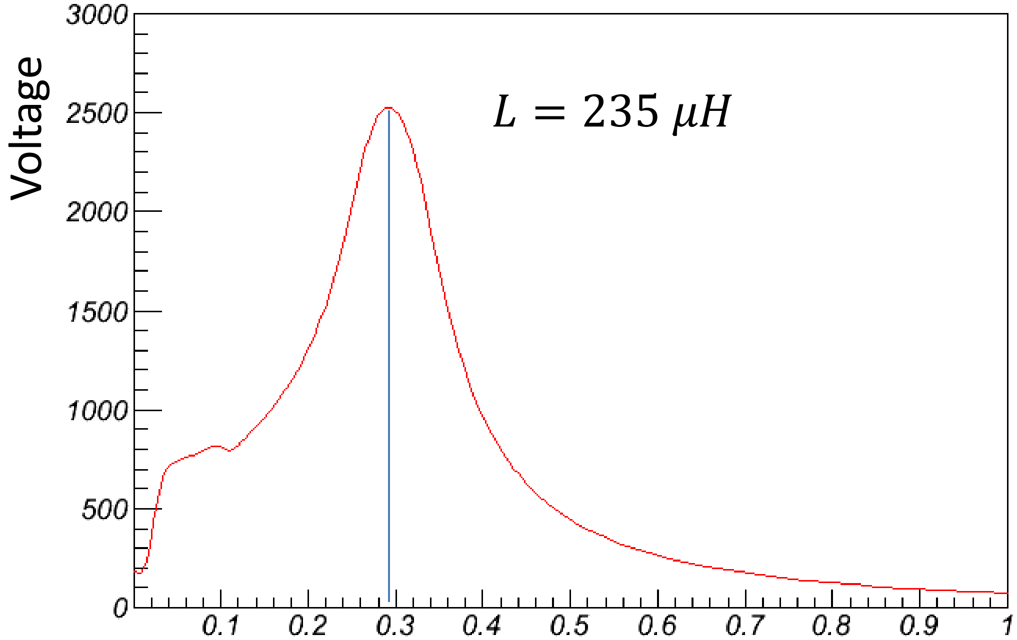
$$V(t) = i(t) X_C = \frac{i(t)}{\omega C}$$

- Amplitude of voltage oscillations measured across C:

$$V(\omega) = \mathcal{E}_0 \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Actual Data

- If we know that $L = 235\ \mu H$, can we estimate R and C?



$f = \omega/2\pi$
[Mhz]

Peak voltage is at

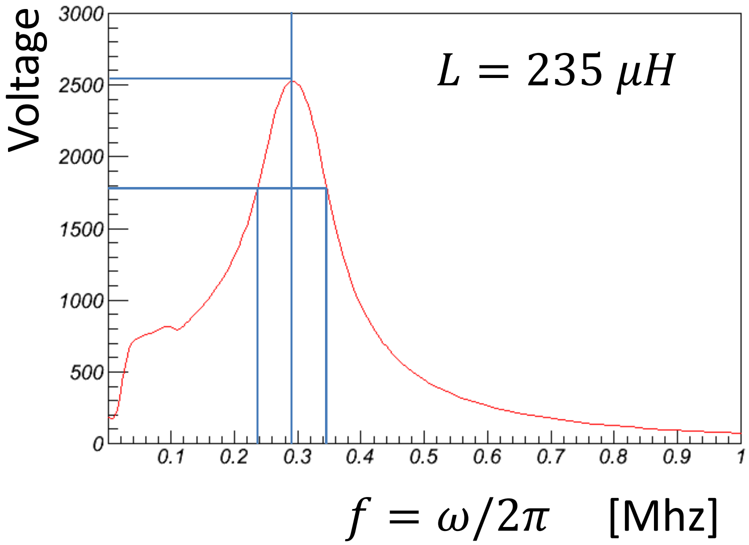
$$f_{peak} = 290\ kHz$$

$$\omega_{peak} = 2\pi(f_{peak})$$

Energy

- Energy stored on a capacitor is $E = \frac{1}{2}CV^2$
- The graph of the stored energy is proportional to the square of the voltage graph.
- If we defined $2\Delta\omega$ as the FWHM on the graph of power *vs* frequency, then it will correspond to $1/\sqrt{2}$ of the peak voltage.

Resonant Circuit



$$2\Delta f = 110\ kHz$$

$$\gamma = 2\pi(\Delta f) = 6.91 \times 10^5\ s^{-1}$$

But $\gamma = R/L$ so we can find R:

$$R = \gamma L = (6.91 \times 10^5\ s^{-1})(235\ \mu H)$$
$$= 162\ \Omega$$

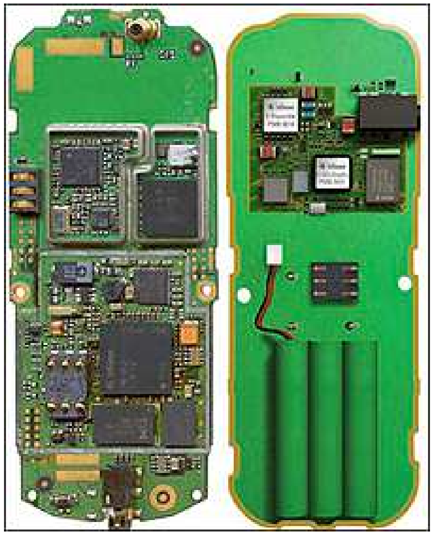
Peak position is $\omega_{peak} = \sqrt{\frac{1}{LC} - \frac{\gamma^2}{2}}$ ➔ $C = \frac{1}{L\left(\left(\omega_{peak}\right)^2 + \frac{\gamma^2}{2}\right)}$

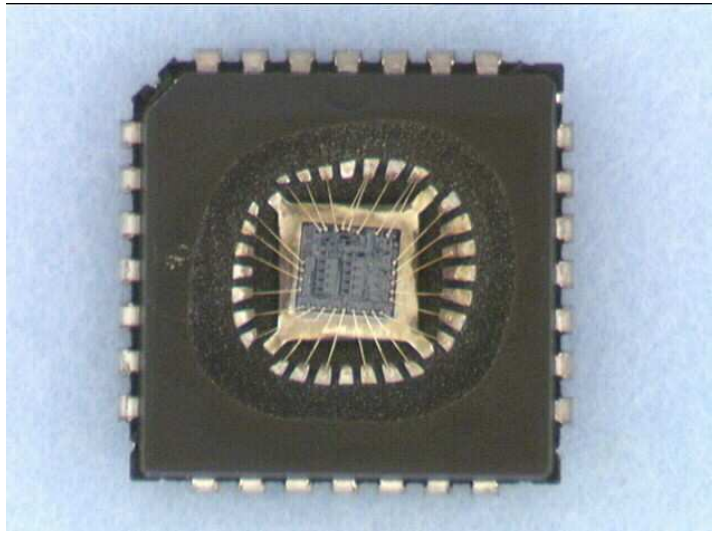
$$C = 1.20\ \text{nF}$$

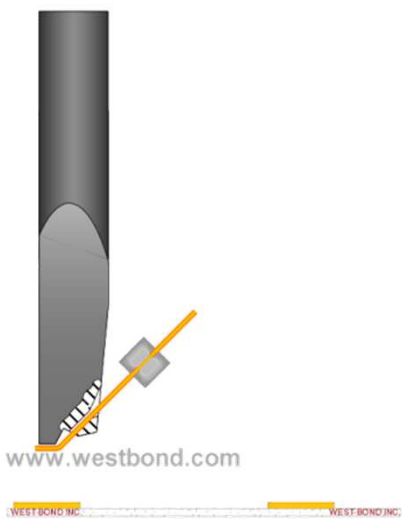
Lifetime of Oscillations

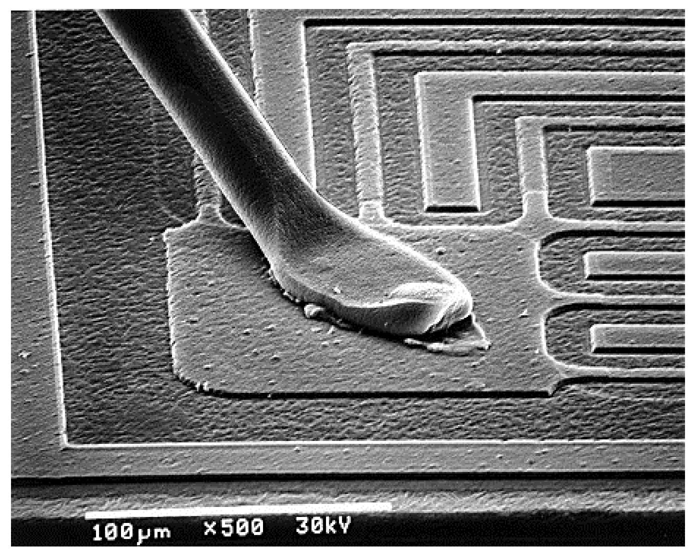
- Amplitude of a damped harmonic oscillator:
$$x(t) = A\, e^{-\gamma t/2} \cos \omega t$$
- Maximum potential energy:
$$U = \frac{1}{2} k\, x^2 \propto e^{-\gamma t}$$
- After time $t = 1/\gamma$, the energy is reduced by the factor $1/e$.
- We call $\tau = 1/\gamma$ the “lifetime” of the oscillator.

Other Resonant Systems

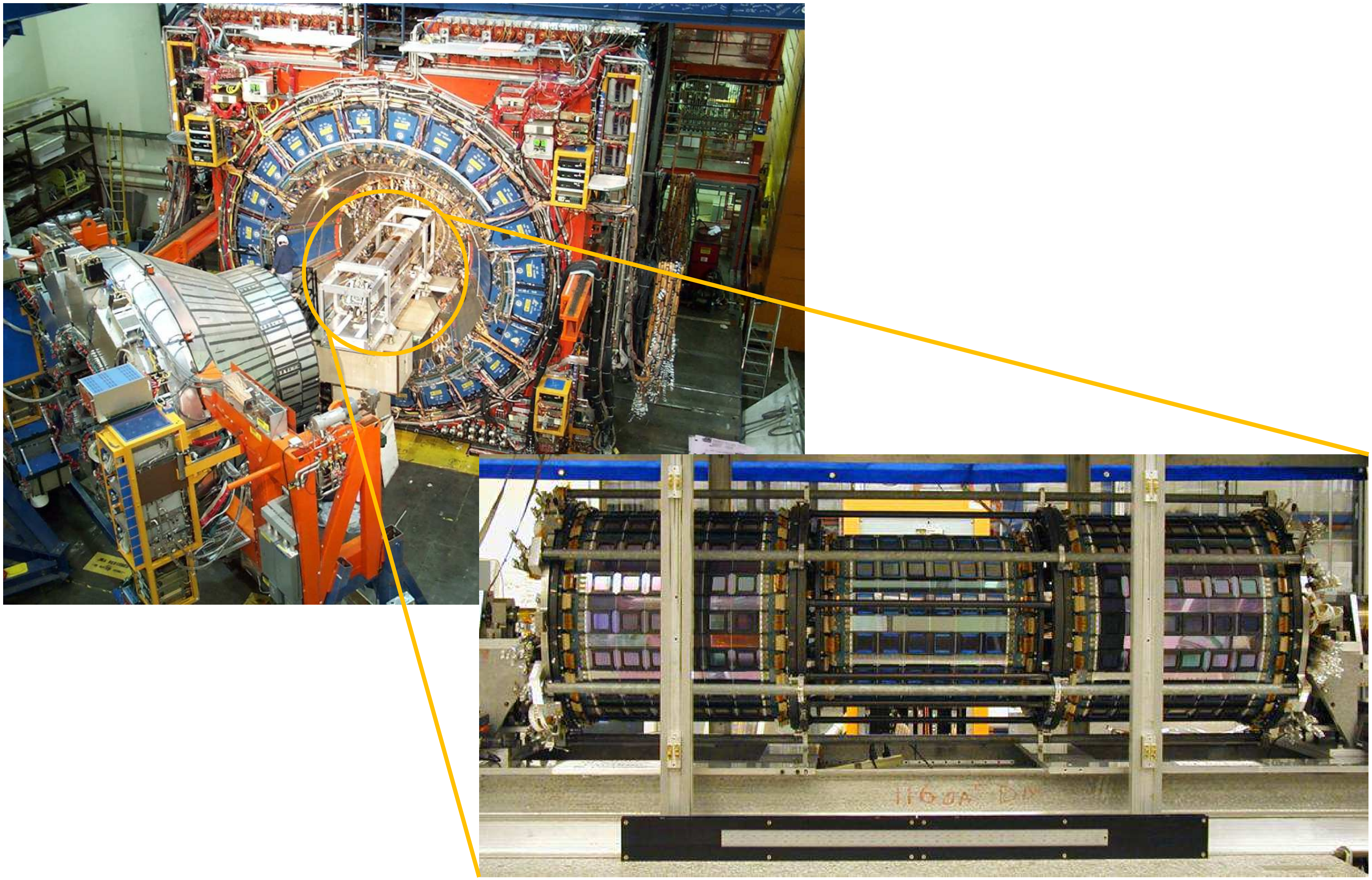




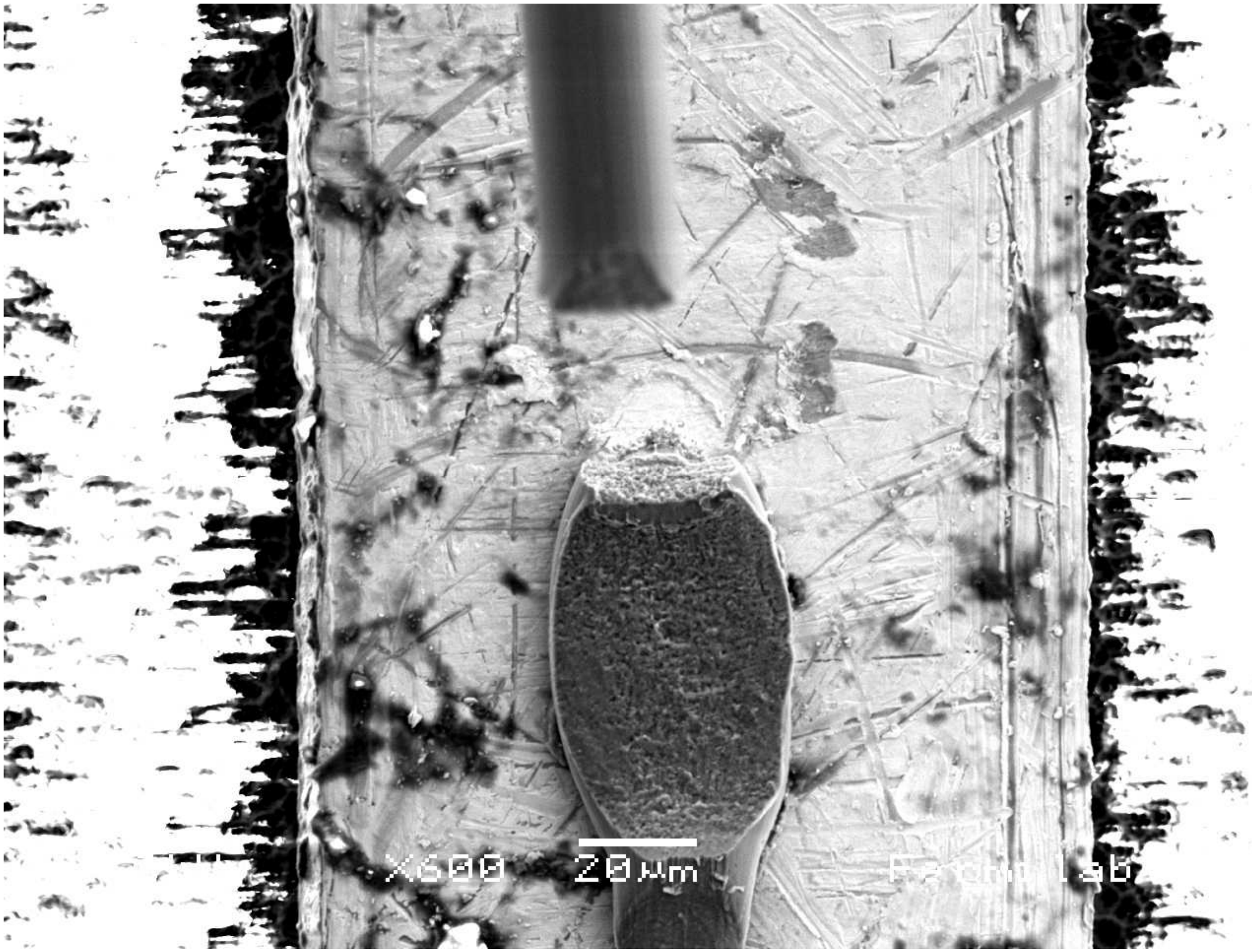




Other Resonant Systems

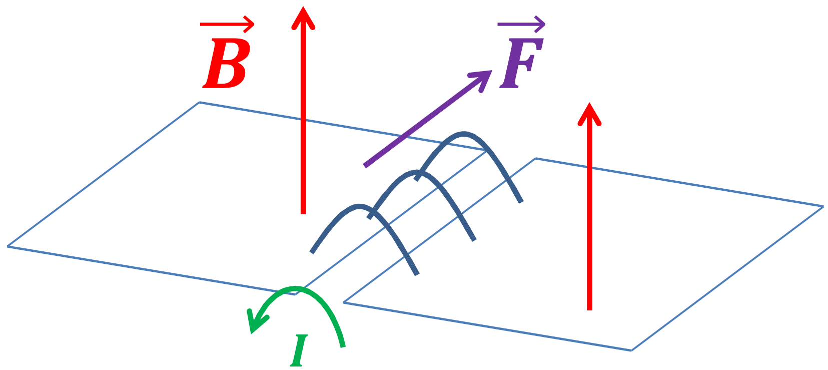


Other Resonant Systems



Wire Bond Resonance

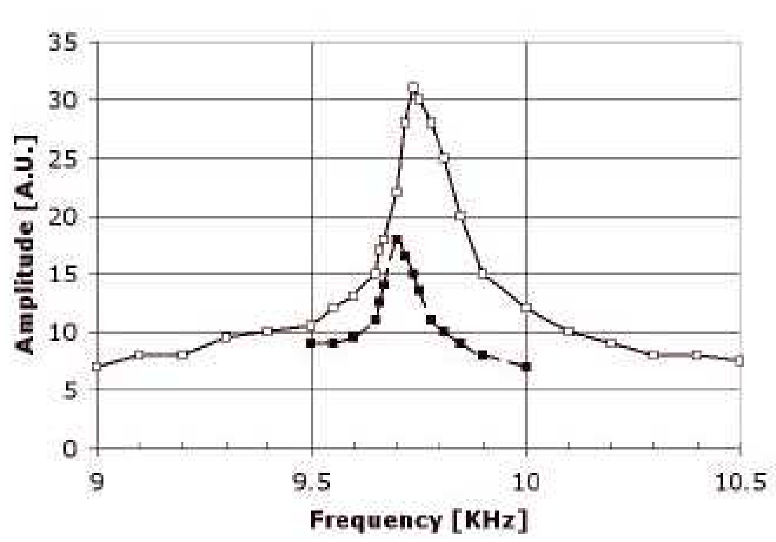
- Wire bonds in a magnetic field:



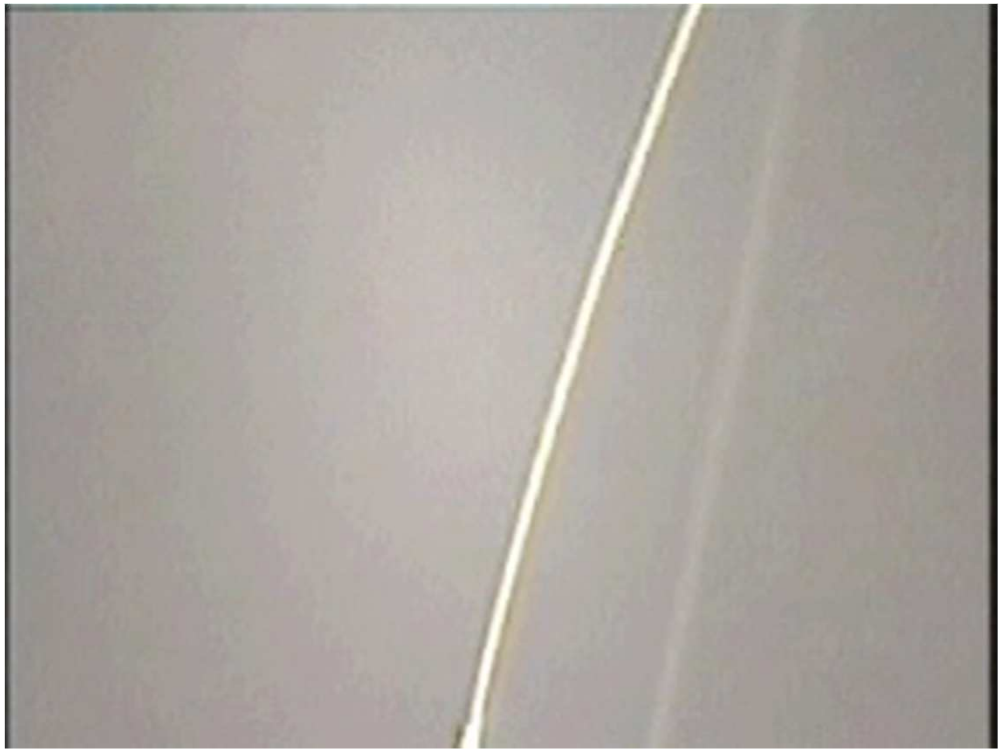
Lorentz force is $\vec{F} = I \int d\vec{\ell} \times \vec{B}$

The tiny wire is like a spring.

A periodic current produces the driving force.

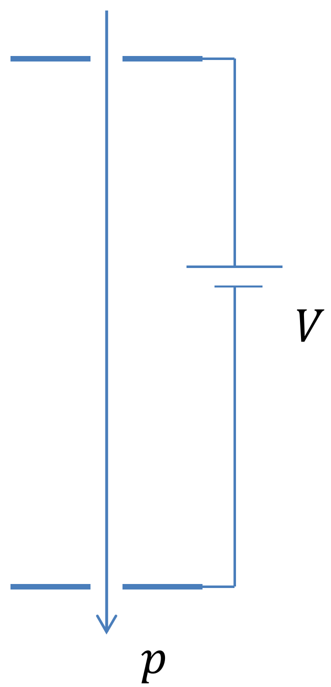


Wire Bond Resonance



Resonance in Nuclear Physics

- A proton accelerated through a potential difference V gains kinetic energy $T = eV$:



Phys. Rev. 75, 246 (1949).

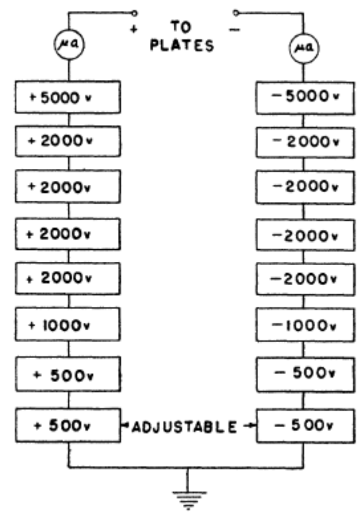


FIG. 1. Block diagram of battery stacks. Adjustable 500-volt boxes were set to any voltage below 500 volts by means of a potentiometer. The polarity of any of the boxes except the adjustable 500-volt box could be selected at will for comparison purposes.

* 5000 volt boxes used Eveready No. 493, 300 volt batteries. All other batteries were of the Burgess XX45, 67½ volt type except for several heavier duty batteries under continuous drain to provide continuous range of adjustments.

** The actual voltage is 504.08 Int. volts and is determined by the resistor divider ratio and the 1.50000 volt setting on the potentiometer.

Resonance in Nuclear Physics

- In quantum mechanics, energy and frequency are proportional:

$$E = \hbar\omega$$

- A given energy corresponds to a driving force with frequency ω.
- When a nucleus resonates at this frequency, the proton energy is easily absorbed.

Nuclear Resonance

ABSOLUTE VOLTAGE DETERMINATION 255

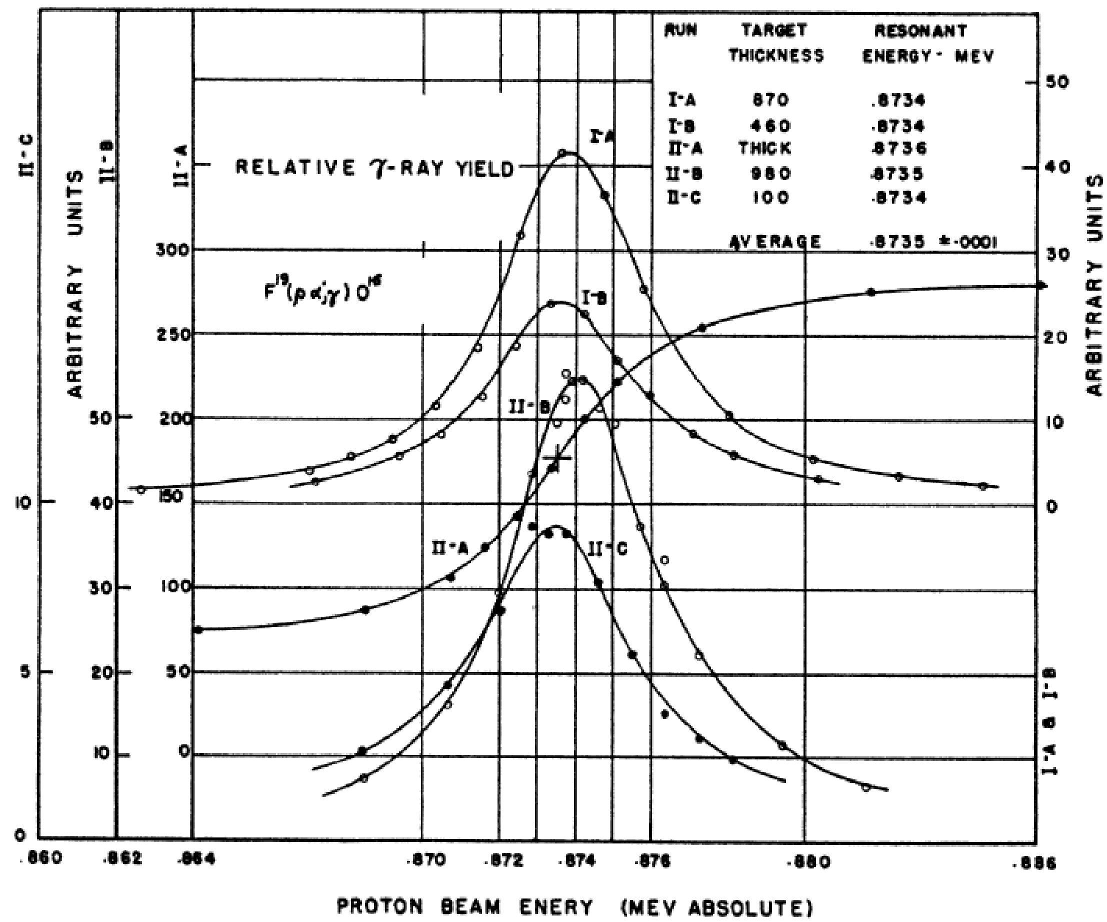
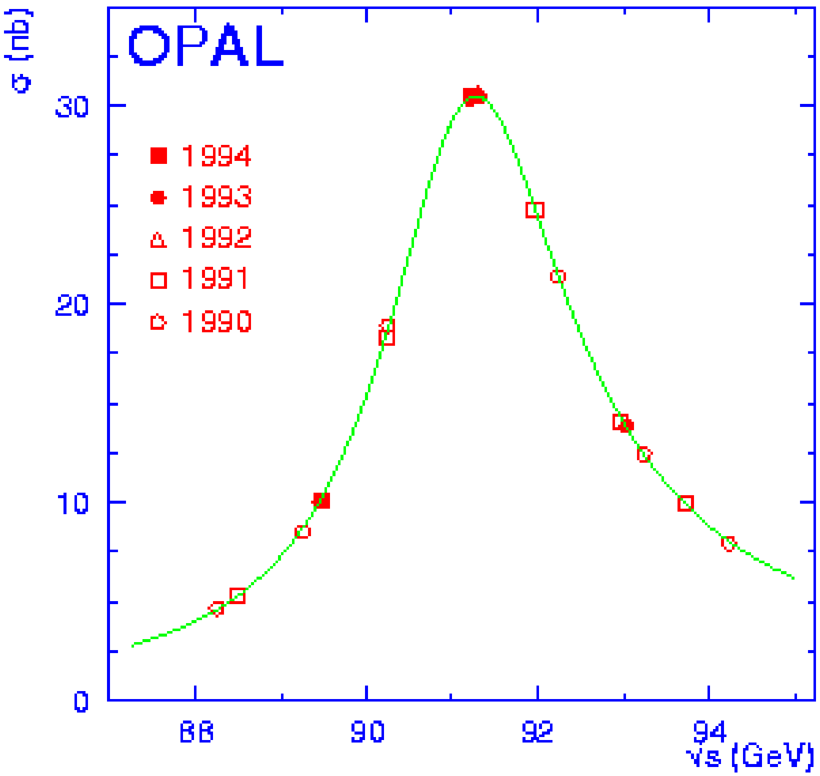
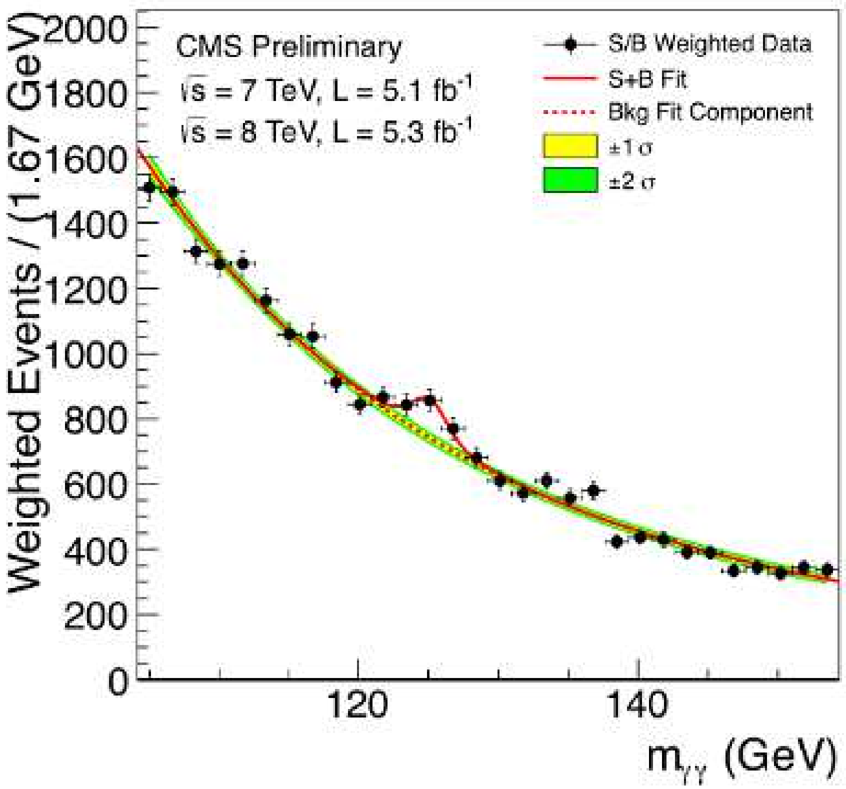


FIG. 7. γ-ray yield curves for both series of measurements of F(pγ) resonance reaction. Yield values are all on same relative scale.

“Lifetime” is defined in terms of the width of the resonance.

Resonance





- Resonances are the main way we observe fundamental particles.