1. The boundary condition $\Psi(R, t) = 0$ requires that $r = R$ is a node in the function $J_0(kr)$ which we approximate by

$$J_0(z) \sim \frac{1}{\sqrt{\pi}} \cos \left( z - \frac{\pi}{4} \right)$$

where $z = kr$.

If $r = R$ is a node, then $kR - \pi/4 = \pi/2, 3\pi/2, ...$

Thus,

$$k = \frac{\frac{\pi}{2} + \frac{\pi}{4}}{R}, \frac{3\pi}{2} + \frac{\pi}{4}, \ldots$$

which we can write as

$$k_n = \frac{(2n-1)\pi/2 + \pi/4}{R}$$

$$= \frac{n\pi - \pi}{R - 4R}$$

The frequencies would be

$$\omega_n = \nu k_n = \frac{\pi\nu}{R} \left( n - \frac{1}{4} \right)$$

Although the question did not ask for a comparison, we can compare these approximate roots with the true roots of $J_0(z)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Rk_n$ (approx)</th>
<th>$z_n$ (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3\pi/4 = 2.3562$</td>
<td>$2.4048$</td>
</tr>
<tr>
<td>2</td>
<td>$7\pi/4 = 5.4978$</td>
<td>$5.5201$</td>
</tr>
<tr>
<td>3</td>
<td>$11\pi/4 = 8.6399$</td>
<td>$8.6537$</td>
</tr>
<tr>
<td>4</td>
<td>$15\pi/4 = 11.7810$</td>
<td>$11.7915$</td>
</tr>
<tr>
<td>5</td>
<td>$19\pi/4 = 14.9226$</td>
<td>$14.9309$</td>
</tr>
</tbody>
</table>
2. Pressure waves inside the spherical balloon will satisfy the wave equation in spherical coordinates:

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = -\frac{\omega^2 \psi}{v^2}$$

provided \( \frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial \phi} = 0 \), as we assume.

We can write this as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \psi \right) = -\frac{\omega^2 \psi}{v^2}$$

It is useful to check that this is the case:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \psi \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( \psi + r \frac{\partial \psi}{\partial r} \right)$$

$$= \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

$$= \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

but also,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r^2} \left( 2r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial r^2} \right)$$

$$= \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

So now we can comfortably write

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \psi \right) = -\frac{\omega^2 \psi}{v^2}$$

If we let \( \psi(r, t) = \frac{1}{r} f(r, t) \) then this is

$$\frac{1}{r} \frac{\partial \psi}{\partial r} = -\frac{\omega^2 \psi}{v^2} \Rightarrow \frac{\partial f}{\partial r} = -\frac{\omega^2 f}{v^2}$$
Solutions are $f(r,t) = A \sin(kr) \cos(\omega t)$ but not $\cos(kr) \cos(\omega t)$ because the function $\Psi(r,t) = \frac{f(r,t)}{r}$ must remain finite as $r \to 0$. Since $r = R$ is a node, we must have $kR = n\pi$.

The frequencies of oscillations are then

$$\omega_n = \nu k_n = \frac{n\pi \nu}{R}$$

and when we write $\nu = \sqrt{\frac{\gamma p}{\rho}}$ we have

$$\omega_n = \frac{n\pi}{R} \sqrt{\frac{\gamma p}{\rho}}$$
3. (a) The power carried by a pulse in one transmission line is

\[ P = Z I^2 \]

If there are equal currents \( I' \) propagating in the two transmission lines on the right, then the total power is

\[ P = 2Z(I')^2 = Z'(2I')^2 \]

So the effective impedance is

\[ Z' = \frac{2}{Z} \]

(b) The reflection coefficient is

\[ \rho = \frac{Z' - Z}{Z' + Z} \]

\[ = \frac{\frac{Z}{2} - \frac{Z}{3}}{\frac{Z}{2} + \frac{Z}{3}} = \frac{\frac{1}{2}}{-\frac{1}{3}} = -\frac{1}{3} \Rightarrow V_r = -\frac{V_i}{3} \]

The reflected pulse is inverted.

(c) The transmitted pulse has an amplitude

\[ V_t = \tau V_i \] \[ \text{where} \quad \tau = \frac{Z'}{Z + Z'} = \frac{\frac{Z}{2}}{\frac{Z}{2} + \frac{Z}{3}} = \frac{2}{3} \]

The pulses on both transmission lines have this amplitude: \( V_t = \frac{2V_i}{3} \)

\[ V'_t = \frac{2V_i}{3} \]
Here is another way to reason this out:

The current and voltage in the transmission line are related by

$$V = IT.$$  

The reflected power is then

$$P_r = \frac{V_r^2}{Z} = \left(\frac{-\frac{1}{3}V_i}{Z}\right)^2$$

The transmitted power is then

$$P_t = P_i - P_r = \frac{V_i^2}{Z} - \left(\frac{\frac{1}{3}V_i}{Z}\right)^2$$

$$= \frac{V_i^2}{Z} \left(1 - \frac{1}{9}\right)$$

$$= \frac{8V_i^2}{9Z}.$$  

Since this power will be split equally on each transmission line, the power on one line is

$$P_t' = \frac{4V_i^2}{9Z} = \left(\frac{\frac{2}{3}V_i}{Z}\right)^2$$

Hence, the voltage on each line is

$$V_t = \frac{2}{3}V_i.$$