Purdue University Department of Physics

Physics 24100

Electricity & Optics

Lecture 3 – Chapter 22 sec. 1-2

Fall 2012 Semester

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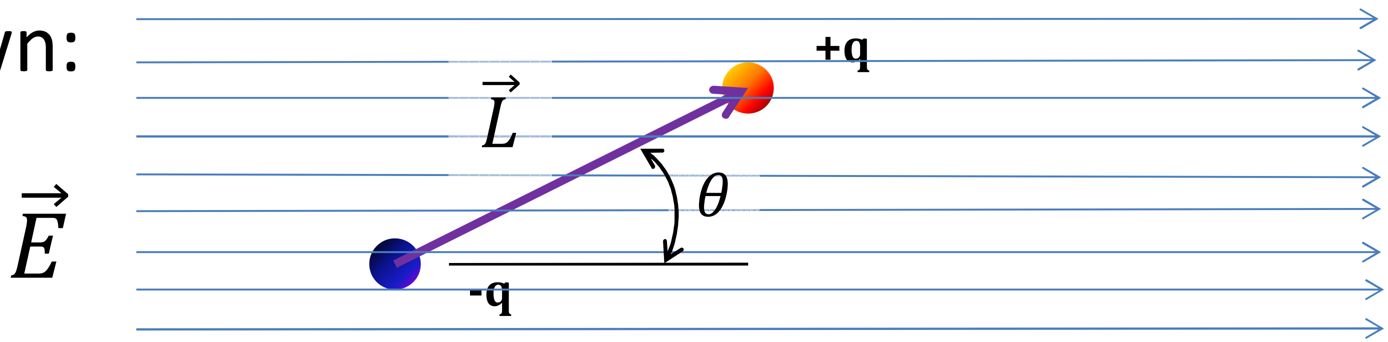
Thursday's Question for Credit

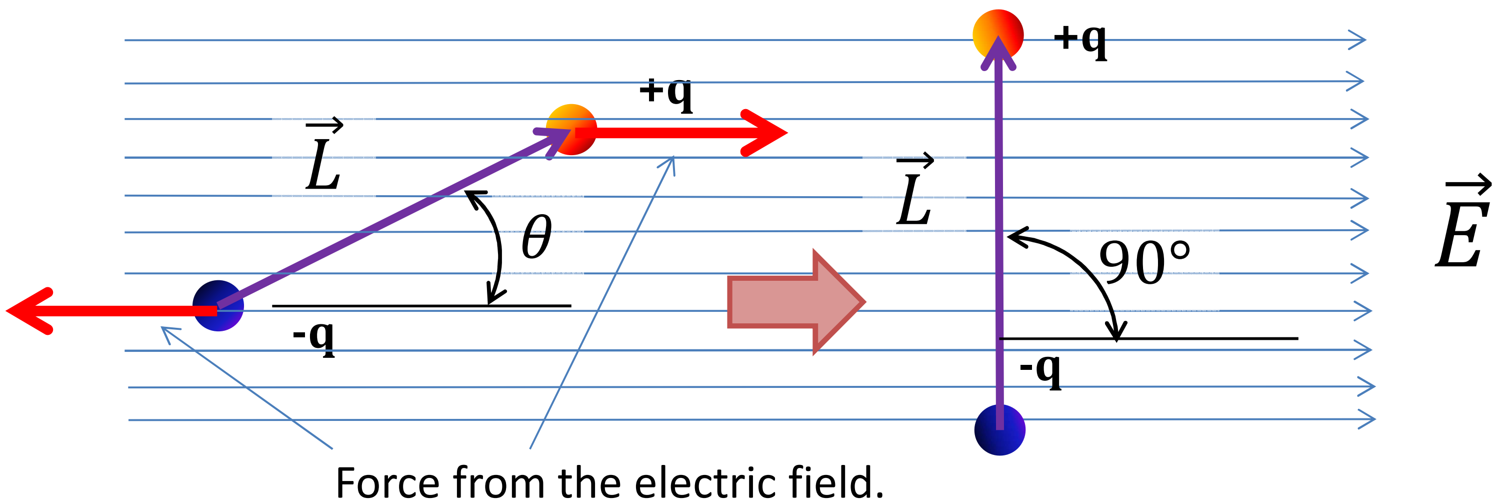
- An electric dipole is placed in an electric field as shown:



- If someone rotates the dipole from this orientation to one where $\theta = 90°$ then…

(a) Work is done ***on*** the electric field
(b) Work is done ***by*** the electric field
(c) The net force is zero, so no work is done
(d) The potential energy of the dipole decreases

Thursday's Question



You need to push against the electric forces to re-orient the dipole. The torque you apply *winds it up*, storing energy.

- Net force is zero, but torque is non-zero. $\Delta U = -\int_{\theta_0}^{\theta} \tau \, d\theta$
- Potential energy of the dipole increases! It wants to unwind and give back the energy you put into it.
- You don't allow the electric field to move the charges – work is not done by the field.
- Instead, work is done on the electric field – the configuration of charges gains potential energy of some form.

A Quick Poll

- In the examples, which notation do you prefer to use for the unit vectors along the x-, y- and z-axes?

(a) $\hat{\imath}, \hat{\jmath}, \hat{k}$

(b) $\hat{x}, \hat{y}, \hat{z}$

Continuous Charge Distributions

- Electric field due to a point charge Q_1 located at position vector $\vec{x}_1$:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{|\vec{x}-\vec{x}_1|^3}(\vec{x}-\vec{x}_1)$$

Remember, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

- Principle of superposition:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ \dots$$

- In general,

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{|\vec{x}-\vec{x}_i|^3}(\vec{x}-\vec{x}_i)$$

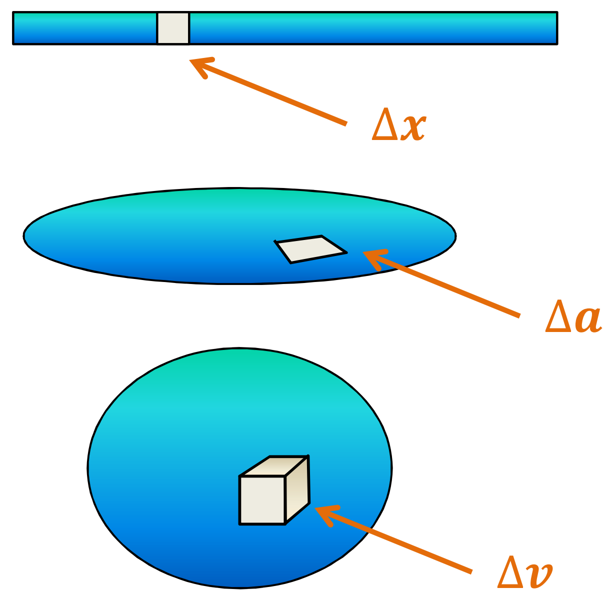
Continuous Charge Distributions

- Instead of discrete charges, Q_i, consider the charge to be continuously distributed…
  - Along a line: $\Delta Q = \lambda \Delta x$

    Units for λ: $C \cdot m^{-1}$
  - On a surface: $\Delta Q = \sigma \Delta a$

    Units for σ: $C \cdot m^{-2}$
  - In a volume: $\Delta Q = \rho \Delta v$

    Units for ρ: $C \cdot m^{-3}$



- If the size of ΔQ is small enough, it becomes equivalent to a point charge…

Question

- Which has the most charge:

A line, 2 m long, with $\lambda = 2\ C \cdot m^{-1}$

A spherical surface with radius 2 m and $\sigma = 2\ C \cdot m^{-2}$

A sphere with radius 2 m and $\rho = 2\ C \cdot m^{-3}$

(a) The line
(b) The spherical surface
(c) The sphere
(d) They all have the same charge

Question

- Which has the most charge:

A line, 2 m long, with $\lambda = 2\ C \cdot m^{-1}$

A spherical surface with radius 2 m and $\sigma = 2\ C \cdot m^{-2}$

A sphere with radius 2 m and $\rho = 2\ C \cdot m^{-3}$

  - The line has total charge

$$Q_{line} = \lambda \mathrm{L} = 4\ \mathrm{C}$$

  - The surface has total charge

$$Q_{surf} = \sigma A = 4\pi\sigma r^2 = 4\pi\ (8\ C) \approx 100\ C$$

  - The sphere has total charge

$$Q_{vol} = \rho V = \frac{4}{3}\pi\rho r^3 = \frac{4\pi}{3}(16\ C) \approx 67\ C$$

  - The ratio is $\frac{Q_{vol}}{Q_{surf}} = \frac{1}{3}\frac{\rho}{\sigma}r$ so the sphere would only have more charge when $r > 3\ \sigma/\rho$.

Continuous Charge Distributions

- Terminology used in the text:
  - Source point, $\vec{x}_s$, where charge $\Delta Q(\vec{x}_s)$ is located.
  - Field point, $\vec{x}_p$, where we want to evaluate $\Delta\vec{E}(\vec{x}_p)$.
  - Vector from $\vec{x}_s$ to $\vec{x}_p$: $\vec{r} = \vec{x}_p - \vec{x}_s$.

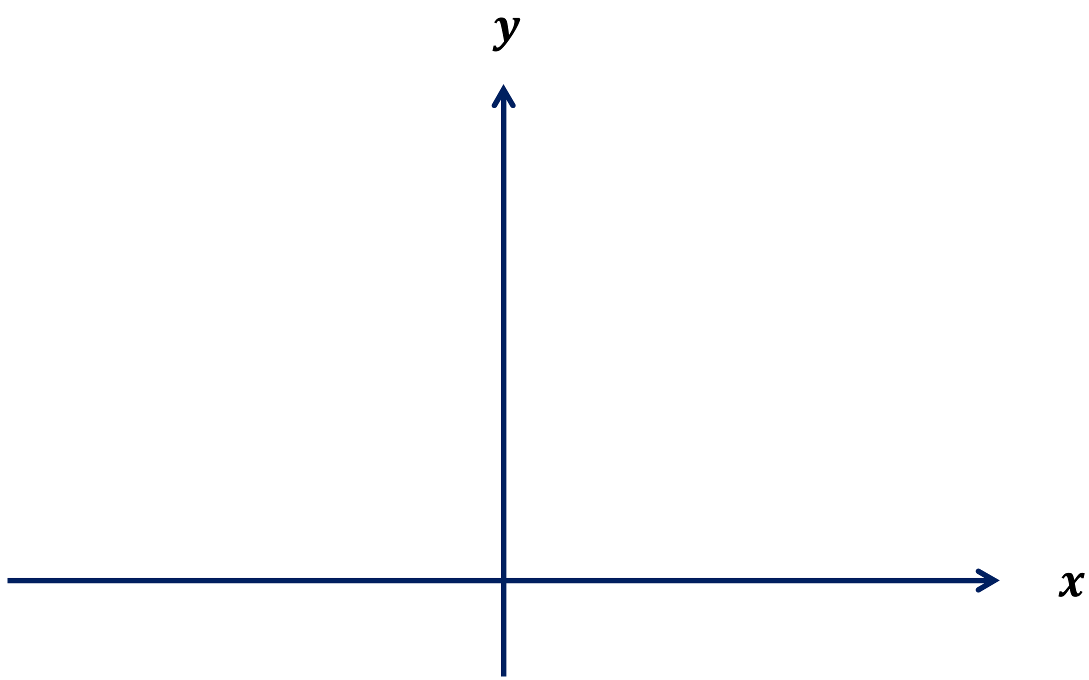
$$\Delta\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\frac{\Delta Q(\vec{x}_s)}{r^2}\hat{r}$$

- Principle of superposition: add up the $\Delta\vec{E}$ created by all elements of charge, $\Delta Q(\vec{x}_s)$.
- Limiting case: replace the sum by an integral over the charge distribution.

$$\vec{E}\left(\vec{x}_p\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{r}}{r^2}\,dQ$$

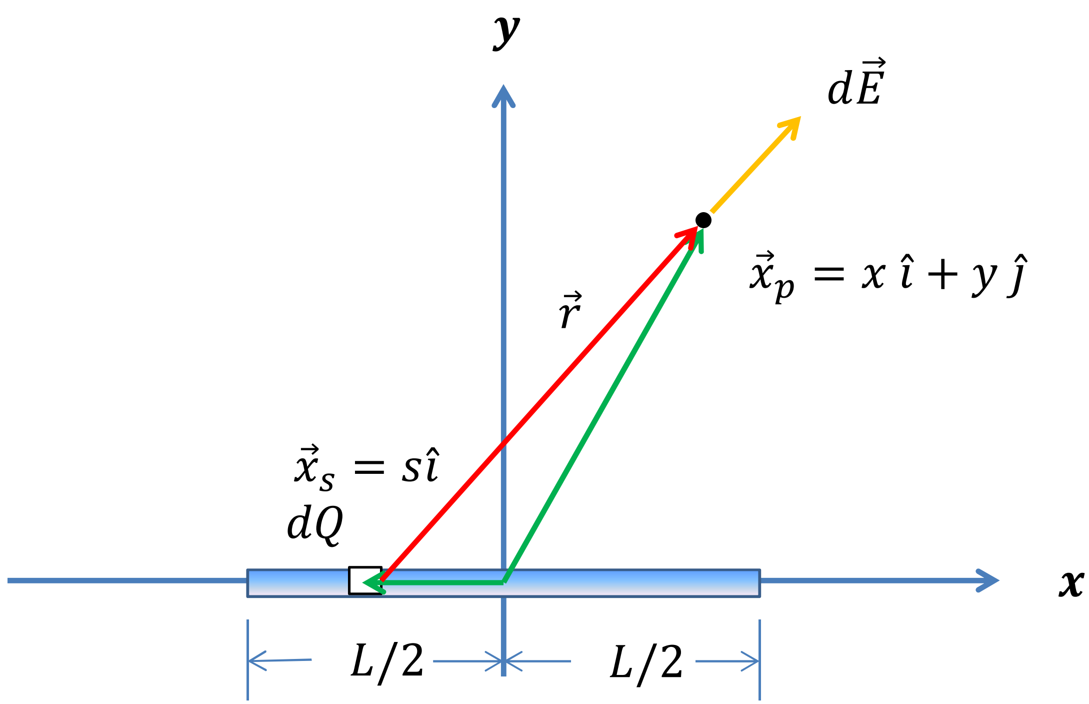
- But we need to re-write this before we can actually evaluate it.
- Some examples should help…

Continuous Line of Charge



1. Pick a coordinate system, label the axes.

Continuous Line of Charge



2. Label the source and field points.
3. Pick variables to express their components.

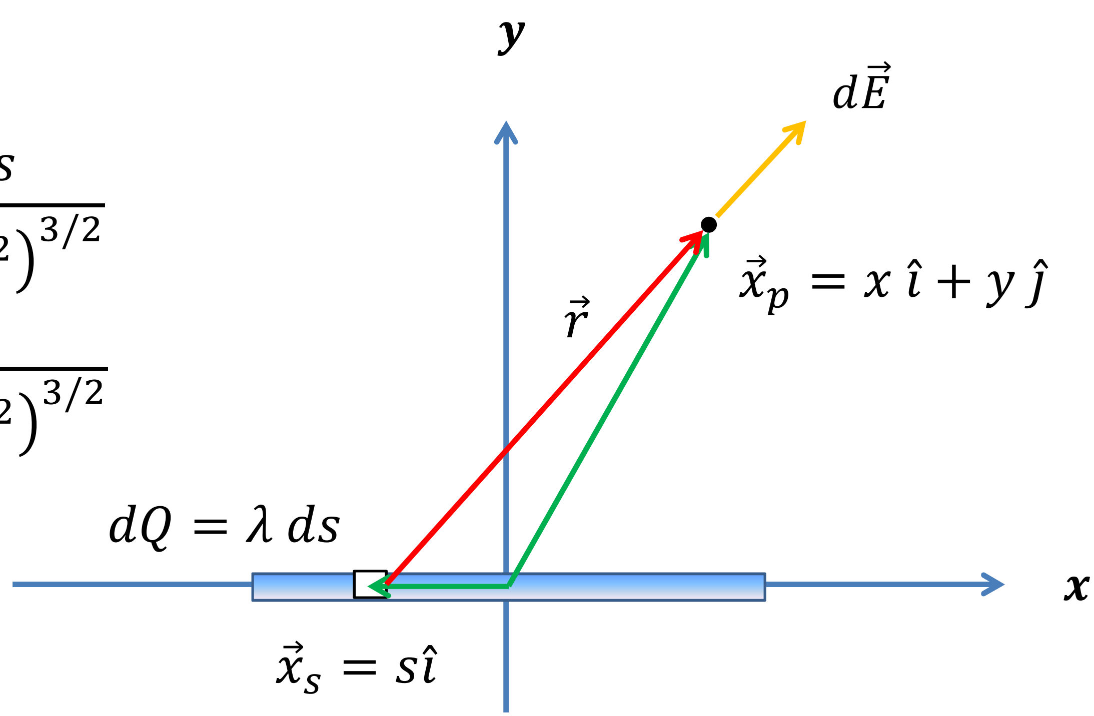
Continuous Line of Charge

$$\vec{r} = \vec{x}_p - \vec{x}_s = (x - s)\hat{\imath} + y\,\hat{\jmath}$$

$$r = \sqrt{(x - s)^2 + y^2}$$

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{\lambda(x - s)ds}{\left((x - s)^2 + y^2\right)^{3/2}}$$

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, y\, ds}{\left((x - s)^2 + y^2\right)^{3/2}}$$



4. Express $\vec{r}$ and r in terms of the components.
5. Write each component of $d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^3}\vec{r}$.
6. Now we can evaluate the integrals.

Continuous Line of Charge

- Evaluating the integrals can be tedious, but that is a technical issue, not directly related to the physics.
- Suggestions:
  - Simple variable substitution
  - Dig up your calculus text
  - Use tables of integrals (eg. Mathematical handbook)
  - Google "table of integrals"
  - Symbolic math programs (eg. Matlab, Mathematica)
    - Use your judgment, but certainly good for checking another method.

Continuous Line of Charge

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{(x-s)ds}{\left((x-s)^2+y^2\right)^{3/2}}$$

Let $u = (x - s)$
Then $du = -ds$
When $s = L/2$ then $u = x - L/2$
When $s = -L/2$ then $u = x + L/2$

$$E_x = -\frac{\lambda}{4\pi\epsilon_0}\int_{x+L/2}^{x-L/2} \frac{u\,du}{(u^2+y^2)^{3/2}}$$

Let $v = u^2 + y^2$
Then $dv = 2u\,du$, $u\,du = \frac{1}{2}dv$
When $u = x + L/2$ then $v = (x + L/2)^2 + y^2$
When $u = x - L/2$ then $v = (x - L/2)^2 + y^2$

$$E_x = -\frac{\lambda}{8\pi\epsilon_0}\int_{(x+L/2)^2+y^2}^{(x-L/2)^2+y^2} \frac{dv}{v^{3/2}}$$

Recall that $\int x^m dx = \frac{x^{m+1}}{m+1}$
In this case, $m = -3/2$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-L/2)^2+y^2}} - \frac{1}{\sqrt{(x+L/2)^2+y^2}}\right)$$

Continuous Line of Charge

$$E_y = \frac{\lambda\, y}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{y\, ds}{\left((x-s)^2+y^2\right)^{3/2}}$$

Let $u = (x - s)$
Then $du = -ds$
When $s = L/2$ then $u = x - L/2$
When $s = -L/2$ then $u = x + L/2$

$$E_y = -\frac{\lambda\, y}{4\pi\epsilon_0}\int_{x+L/2}^{x-L/2} \frac{du}{(u^2+y^2)^{3/2}}$$

Use a table of integrals…

$$E_y = -\frac{\lambda\, y}{4\pi\epsilon_0}\left(\frac{x-L/2}{y^2\sqrt{(x-L/2)^2+y^2}} - \frac{x+L/2}{y^2\sqrt{(x+L/2)^2+y^2}}\right)$$

- Next, check limiting behavior…

Continuous Line of Charge

- On the x-axis, $y = 0$ and suppose that $x > L/2$
- Then, $E_x = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{x-L/2} - \frac{1}{x+L/2}\right)$
- When $x \gg L/2$, $\frac{1}{x \pm L/2} = \frac{1}{x} \mp \frac{L/2}{x^2} + \cdots$
- So, $E_x \to \frac{\lambda}{4\pi\epsilon_0}\left(\frac{L/2}{x^2} + \frac{L/2}{x^2}\right) = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{x^2}$
- This is the same as the electric field for a point charge $Q = \lambda L$.
- Sometimes you have to watch out for algebraic signs:

$\frac{y}{(y^2)^{3/2}} = \frac{1}{y^2}$ only when $y > 0$. If $y < 0$ then $\frac{y}{(y^2)^{3/2}} = -\frac{1}{y^2}$.

Clicker Question

- What is the limiting form of
  $$E_y = -\frac{\lambda\, y}{4\pi\epsilon_0}\left(\frac{x-L/2}{y^2\sqrt{(x-L/2)^2+y^2}} - \frac{x+L/2}{y^2\sqrt{(x+L/2)^2+y^2}}\right)$$
  on the *positive* y-axis, when $x = 0$ and $y \gg \mathrm{L}/2$?

(a) $E_y = 0$

(b) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{y\sqrt{(L/2)^2+y^2}}$

(c) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{y^2}$

(d) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{x^2}$

Clicker Question

- What is the limiting form of
$$E_y = -\frac{\lambda\, y}{4\pi\epsilon_0}\left(\frac{x-L/2}{y^2\sqrt{(x-L/2)^2+y^2}} - \frac{x+L/2}{y^2\sqrt{(x+L/2)^2+y^2}}\right)$$
on the *positive* y-axis, when $x = 0$ and $y \gg \mathrm{L}/2$?

(a) $E_y = 0$

(b) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{y\sqrt{(L/2)^2+y^2}}$

(c) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{y^2}$ ⟵ $Q = \lambda\, L,\ E_y \propto {}^{1}/_{y^2}$

(d) $E_y = \frac{\lambda\, L}{4\pi\epsilon_0}\frac{1}{x^2}$

Continuous Charge Distributions

- Linear distribution:

$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \lambda(\vec{x}_s)\frac{\hat{r}}{r^2}ds$$

- Surface distribution:

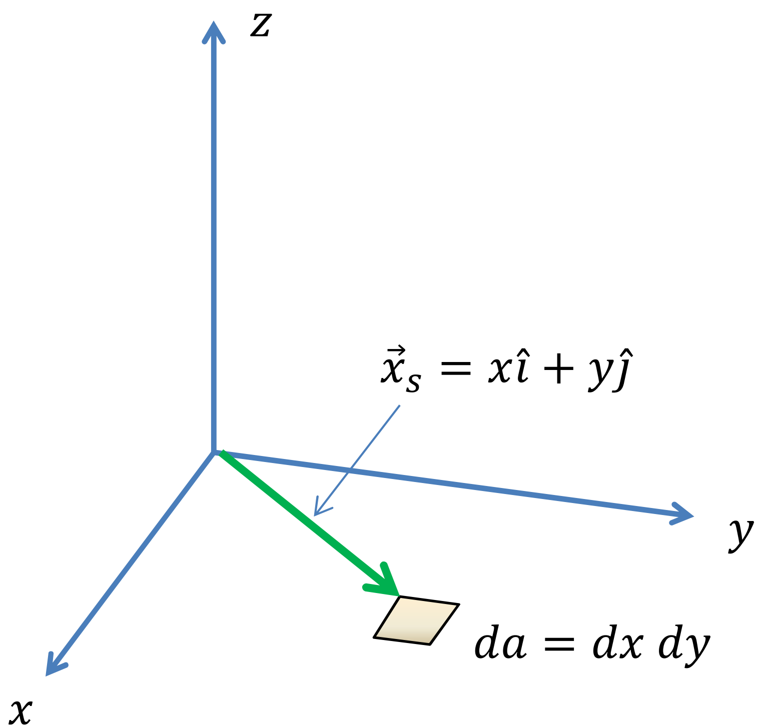
$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \sigma(\vec{x}_s)\frac{\hat{r}}{r^2}da$$

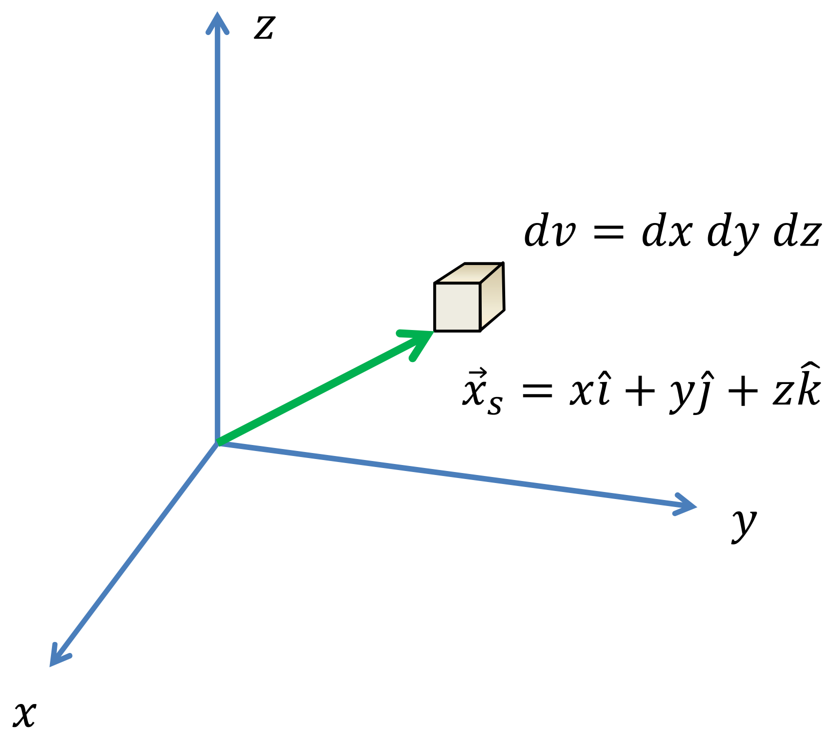
- Volume distribution:

$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \rho\,(\vec{x}_s)\frac{\hat{r}}{r^2}dv$$

Common Coordinate Systems

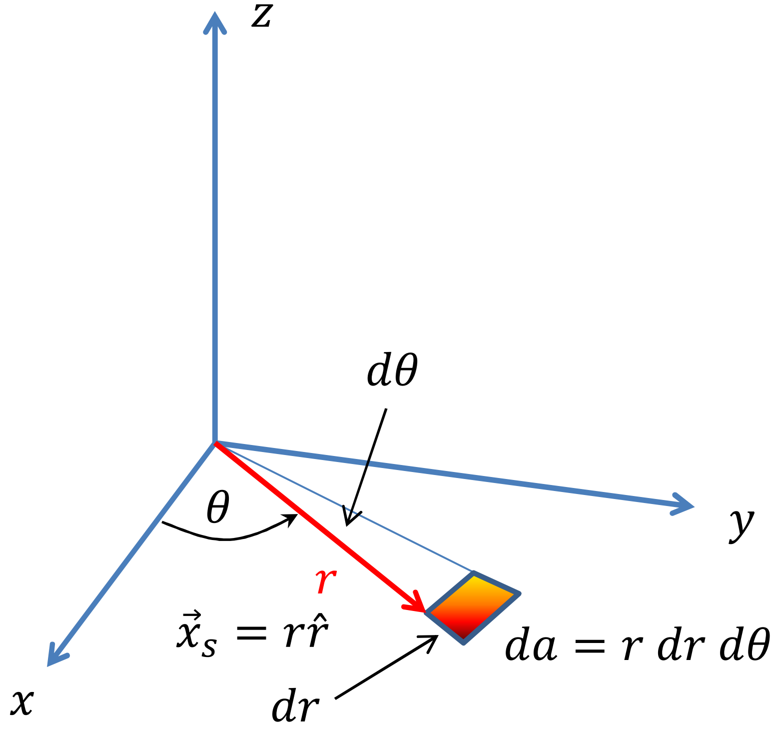
- Cartesian coordinates:



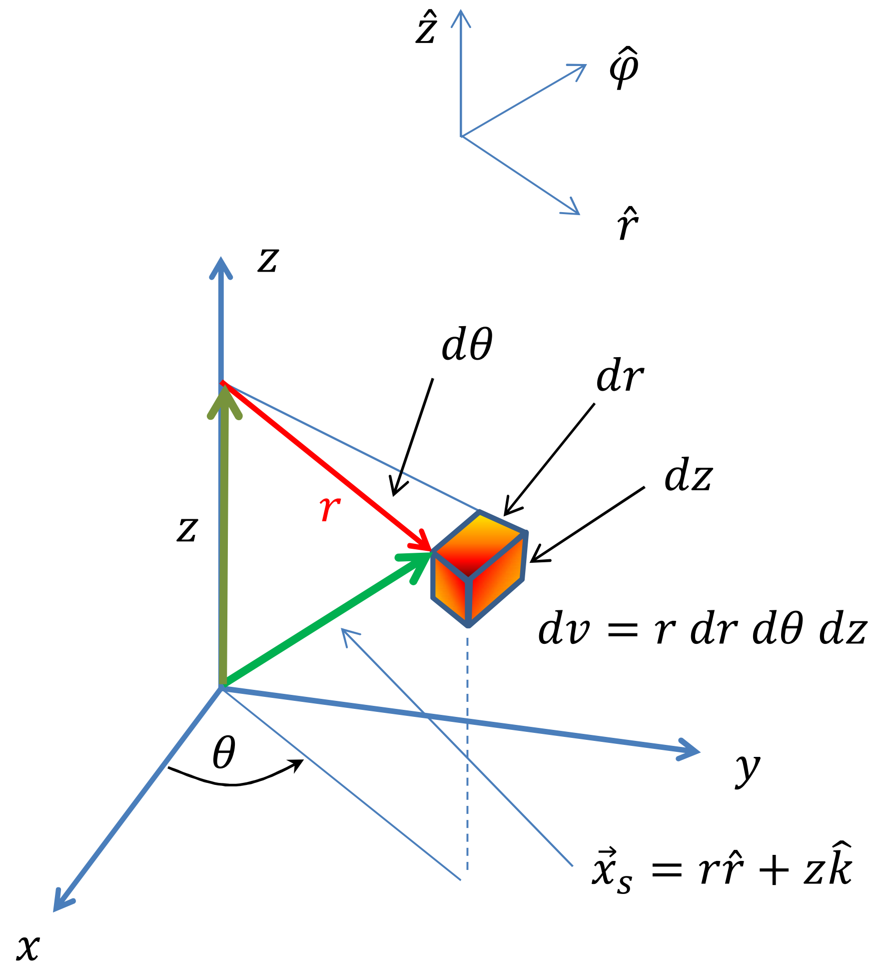


Common Coordinate Systems

- Polar or cylindrical coordinates:

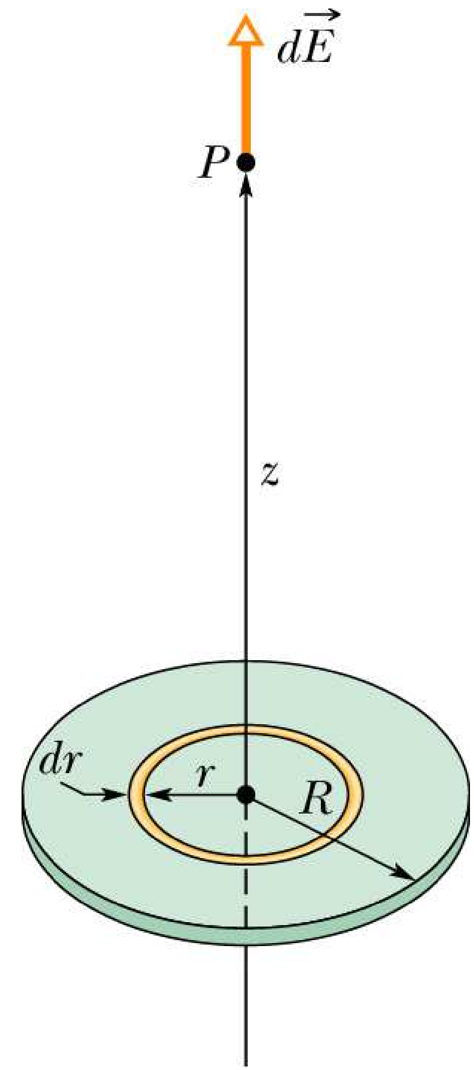


$$\vec{r} = r\ \cos\theta\ \hat{\imath} + r\ \sin\theta\,\hat{\jmath}$$



Another Example

- Calculate $\vec{E}$ at a point P on the z-axis due to a disk of radius R, with uniform surface charge density, σ, as shown…
- From symmetry, we expect that $E_r = E_\varphi = 0$.
- We just need to calculate E_z…

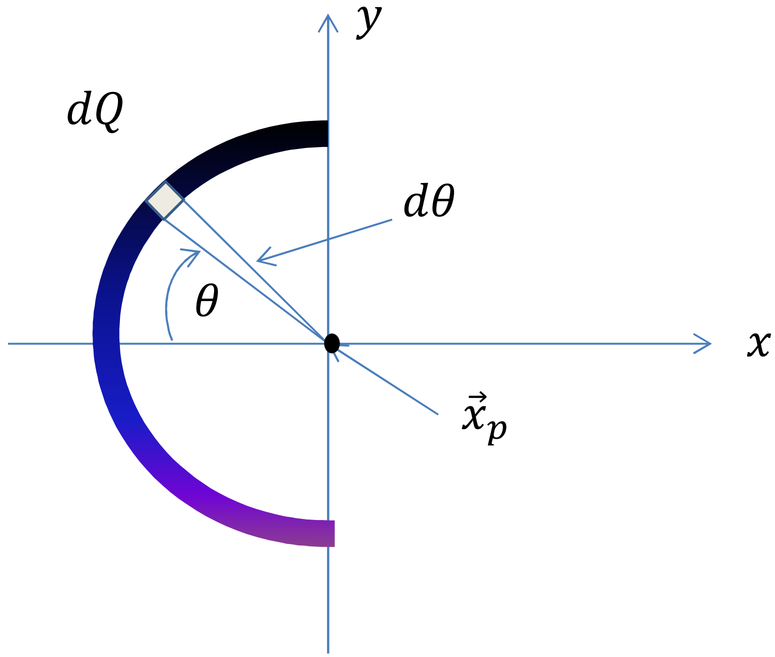


Final Clicker Question

- Charge is uniformly distributed on a semicircular ring with radius a.
- The linear charge density is λ.
- We know that

$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{r^2}\hat{r}$$

- What is $\frac{dQ}{r^2}$ when $\vec{x}_p$ is at the origin?



(a) $\frac{\lambda}{a}d\theta$

(b) $\lambda\, a\, d\theta$

(c) $\frac{\lambda\, d\theta}{a^2}$

(d) $\pi\, a\, \lambda$