SO(5) superconductors in a Zeeman magnetic field

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The generic symmetry of a system under a uniform Zeeman magnetic field is $U(1) \times U(1)$. However, we show that SO(5) models in the presence of a finite chemical potential and a finite Zeeman magnetic field can have an exact $SU(2) \times U(1)$ symmetry. This principle can be used to test SO(5) symmetry at any doping level.

A fundamental question one can ask in connection with high-$T_c$ superconductors is whether they are in the same universality class of conventional $d$ wave Bardeen-Cooper-Schrieffer theory (BCS) superconductors. While many aspects of high-$T_c$ superconductors are anomalous and quantitatively different from conventional BCS superconductors, no sharp distinction based on symmetry has been made so far. In the absence of an external magnetic field and spin anisotropy, the symmetry of the Hamiltonian is $SU(2) \times U(1)$, where the $U(1)$ charge symmetry is spontaneously broken in the superconducting state.

A notable exception is the idea of SO(5) symmetry between antiferromagnetism (AF) and superconductivity (SC) (Ref. 1). This theory predicts a finite temperature bicritical point with an enlarged SO(5) symmetry at the transition point between AF and SC. It also predicts a spin triplet $\pi$ resonance in the SC state which can be interpreted as the pseudo-Goldstone mode associated with the spontaneous symmetry breaking. However, in the presence of a finite chemical potential, the explicit symmetry of the Hamiltonian is still a direct product of the spin $SU(2)$ and the charge $U(1)$ symmetry, which is not different from that of a conventional BCS system.

In this paper, we point out a remarkable symmetry property of SO(5) symmetric Hamiltonians. In the presence of a finite chemical potential $\mu$ and a finite Zeeman magnetic field $B$, the original SO(5) symmetry is broken to $U(1) \times U(1)$. Here the first $U(1)$ group describes the spin rotation symmetry in a plane perpendicular to the applied magnetic field and the second $U(1)$ group is the usual charge symmetry. In fact, any generic spin invariant Hamiltonian in the presence of a finite Zeeman field would have the same $SU(2) \times U(1)$ symmetry. From that point of view, SO(5) symmetric models do not seem to be different from any generic models once a chemical potential or a magnetic field is applied. However, we will show that for a special combination where $B = \mu$, the SO(5) symmetric models enjoy an enlarged $SU(2) \times U(1)$ symmetry, which is not shared by generic models. Furthermore, this special $SU(2) \times U(1)$ symmetry at $B = \mu$ is equivalent to the original SO(5) symmetry in the absence of these fields. This gives a powerful new tool to test the SO(5) symmetry at any doping level. The original SO(5) symmetry exists only at a particular doping level where the AF to SC transition occurs. This point is very difficult to reach in high-$T_c$ superconductors because of complicated doping chemistry, and has not yet been identified experimentally. Under the current proposal, however, the SO(5) symmetry can be revealed at any doping level, provided one applies a Zeeman magnetic field. This current test can give a sharp symmetry distinction between a SO(5) superconductor and a conventional BCS superconductor, and it can also distinguish various explanations of the $\pi$ resonance. In particular, the SO(5) theory predicts that the $\pi$ resonance intensity is proportional to the superconducting order parameter, therefore it should be dramatically reduced in the type II phase in the presence of a c-axis magnetic field. On the other hand, because of the symmetry discussed in this paper, a Zeeman magnetic field applied in the $ab$ plane should only split the $\pi$ triplet, without changing its intensity and commensurability at $(\pi, \pi)$.

Let us consider the following Hamiltonian

$$H = H_{SO(5)} - \mu Q - B S_z,$$

where $H_{SO(5)}$ is an SO(5) symmetric Hamiltonian which commutes with the ten SO(5) symmetry generators $L_{ab}$. (for notations and definitions, please see Ref. 1). $Q$ and $S_z$ are members of SO(5) symmetry generators $L_{ab}$, they generate charge rotation and spin rotation in the $xy$ plane perpendicular to the external Zeeman field. Since SO(5) is a rank two algebra, one can choose $Q$ and $S_z$ as the two mutually commuting generators. For $B = 0$, the generic symmetry of $H$ is $SU(2) \times U(1)$, while for nonvanishing values of $B$, the original SO(5) symmetry of $H_{SO(5)}$ is broken explicitly to $U(1) \times U(1)$, generated by $Q$ and $S_z$.

However, $H$ has an exact enlarged symmetry $SU(2) \times U(1)$ at $B = \mu$. At this point, both the chemical potential and the Zeeman terms can be combined as $-\mu Q_1$, where $Q_1$ and $Q_1$ measure the number of up-spin and down-spin electrons, respectively. Furthermore, we can define a SU(2) subalgebra of the original SO(5) algebra generated by

$$\pi_1 = \sum_k \text{sgn}(\cos k_x - \cos k_y) c_{Q+k,1} c_{-k,1}, \quad \pi_1^+, \quad Q_1.$$

It is easy to see that they form a closed SU(2) algebra,

$$J_1 = \frac{1}{2}(\pi_1 + \pi_1^+), \quad J_2 = \frac{i}{2}(\pi_1 - \pi_1^+),$$

where $J_1$ and $J_2$ are the SU(2) generators. The Hamiltonian can be written as

$$H = \frac{1}{2} J_1^2 + \frac{1}{4} J_2^2 + \mu Q - B S_z,$$

This Hamiltonian has an explicit $U(1) \times U(1)$ symmetry, which is shared with any generic model. However, the proposed Hamiltonian also has an exact $SU(2)$ symmetry, which is not present in any generic model. This symmetry can be used to test SO(5) symmetry at any doping level.
\[ J_3 = \frac{1}{2} Q_1, \quad [J_{\alpha}, J_{\beta}] = i \epsilon_{\alpha\beta\gamma} J_{\gamma}. \]  

(3)

Since the generators of this subalgebra are formed by linear combinations of the original SO(5) generators \( L_{ab} \), they all commute with \( H_{SO(5)} \). Furthermore, since they only involve down-spin electrons, they commute with \(-\mu \hat{Q}_1\). Therefore, we have proven that at \( R = \mu, H \) has a SU(2)×U(1) symmetry generated by the SU(2) algebra defined by Eq. (3) and the U(1) generator \( \hat{Q}_1 \).

Now we proceed to analyze the collective modes associated with this new symmetry. For this purpose, it is useful to first see how the new symmetry emerges in the Lagrangian formalism. The effective Lagrangian with exact SO(5) symmetry can be expressed as:

\[ \mathcal{L} = \chi (\partial n_a)^2 - \rho (\partial n_a)^2 - V(n), \]  

where \( V(n) = - (\delta / 2) n_1^2 + (W/4) |n|^4 \). We can introduce a magnetic field and a chemical potential simultaneously in the above Lagrangian by applying the following transformation:

\[ \partial n_a \rightarrow \partial n_a - i \epsilon_{\alpha\beta\gamma} B \beta n_\gamma, \quad \alpha = 2, 3, 4; \]
\[ \partial n_4 \rightarrow \partial n_4 - i \epsilon_{ij} n_{ij}, \quad i, j = 1, 5. \]

(5)

Choosing \( \hat{B} = (0, 0, -B) \), the Lagrangian becomes

\[ \mathcal{L} = \chi (\partial n_a)^2 - \rho (\partial n_a)^2 - 2i \chi B n_1 \partial n_2 - B n_2 \partial n_3 - \mu n_1 \partial n_5 + \mu n_5 \partial n_1 + \chi [B^2 (n_1^2 + n_2^2) + \mu^2 (n_3^2 + n_5^2)] - V(n). \]

(6)

Denoting \( \hat{M} = (n_1, n_2, n_5, n_3) \), and taking \( B = \mu \), we can rewrite the above equation into the following form:

\[ \mathcal{L} = \chi (\partial n_a)^2 - \rho (\partial n_a)^2 + 2i \chi B \hat{M} \partial \hat{M}^T + \chi \mu^2 \hat{M}^2 - V(n), \]

where \( \hat{M} \) is a four-dimensional matrix,

\[ \hat{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Now we discuss the symmetry of above Lagrangian. Obviously, except for the third term in the above equation, all other terms have an exact SO(4) symmetry in the \( \hat{M} \) space. However not all of rotation will keep the invariance of the third term. If \( \hat{O} \) denotes a rotation matrix in the \( \hat{M} \) space, then it must satisfy

\[ \hat{O}^T \hat{O} = 1; \quad \hat{O}^T \hat{R} \hat{O} = \hat{R} \]

(8)

in order to keep the Lagrangian (7) invariant. Since SO(4) \( \cong SU(2) \times SU(2) \), we immediately find one of the SU(2) subgroups whose generators are defined by the following matrix:

\[ G_1 = \frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad G_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \sigma_z \\ i \sigma_z & 0 \end{pmatrix}, \]
\[ G_3 = \frac{1}{2} \begin{pmatrix} 0 & i \sigma_z \\ -i \sigma_z & 0 \end{pmatrix}, \quad [G_{\alpha}, G_{\beta}] = i \epsilon_{\alpha\beta\gamma} G_{\gamma}. \]

(9)

These matrices also have the following properties

\[ [G_{\alpha}, R] = 0. \]

Therefore, \( G_{\alpha} \), together with \( R \), generates a symmetry SU(2)×U(1). The Lagrangian (7) is invariant under above SU(2)×U(1) transformations. By Noether’s theorem, each internal symmetry is associated to a conserved charge. From the infinitesimal variations of \( \hat{M} \),

\[ \delta \hat{M}^T = i G_{\alpha} \hat{M}^T \delta \phi_{\alpha} + R \hat{M} \delta \phi_{R}, \]

we obtain the following conserved currents

\[ j^R_i = 2i \chi \partial \hat{M} \partial \hat{M}^T + 2 \chi \mu \hat{M} \hat{M}^T, \]
\[ j^R_k = 2i \chi \partial \hat{M} \hat{M} \hat{M}^T + 2 \chi \mu \hat{M} \hat{M} \hat{M}^T, \]
\[ j^R_{\hat{M}} = 2i \chi \partial \hat{M} \hat{M} \hat{M}^T + 2 \chi \mu \hat{M} \hat{M} \hat{M}^T, \]
\[ j^R_{\hat{M}} = 2i \chi \partial \hat{M} \hat{M} \hat{M}^T + 2 \chi \mu \hat{M} \hat{M} \hat{M}^T, \]
\[ 0 = \partial j^R_{\alpha} + \partial j^R_{\alpha}. \]

(10)

The associated conserved charges can be directly related to the symmetry generators (3) in the Hamiltonian formalism:

\[ J_{\alpha} = \int dx j^R_{\alpha}; \quad Q_1 = \int dx j^R_1. \]

(11)

Since the static potential is explicitly broken from SO(5) to SO(4), one might expect three massless Goldstone modes and one massive mode for this kind of symmetry breaking. However, there are two massless modes and two massive modes as this case, because the total Lagrangian (7) has lower SU(2)×U(1) symmetry than the static potential. We can pick a particular direction in \( \hat{M} \) space and linearize the mode equation around this direction, say \( n_1 \) (superconducting phase):

\[ \chi \partial_n^2 n_2 = \rho \partial_n^2 n_2 - 2 \chi \partial \partial n_3 \]
\[ \chi \partial_n^2 n_3 = \rho \partial_n^2 n_3 + 2 \chi \partial \partial n_2 \]
\[ \chi \partial_n^2 n_5 = \rho \partial_n^2 n_5 \]
\[ \chi \partial_n^2 n_4 = \rho \partial_n^2 n_4 - \chi \mu^2 n_4. \]

(12)

The last equation describes the massive modes with energy \( \omega_4 = \mu \), which is associated with the explicit symmetry breaking (from SO(5) to SO(4)) of the static potential. The third equation describes the usual Goldstone massless mode (sound mode) of the superconductor with linear dispersion \( \omega_3 = (\rho / \chi) k \). The first two equations predict new doublet-spin wave modes. One is massless, the other is massive. In
the long-wavelength limit, the energies of the modes are
\[ \omega_2 = v k^3, \omega_3 = 2 \mu. \]
Therefore, there are always two gapless modes, one with linear dispersion and the other with quadratic dispersion, independent of the orientation of superspin.

It is also interesting to investigate the case where the SO(5) symmetry is explicitly broken, but a projected SO(5) symmetry defined in Refs. 4 and 5 is present. We can add a term \(-g(n_1^2 + n_2^2 + n_3^2)\) to the SO(5) symmetric potential \( V(n) \), and choose \( g > 0 \) so that AF is favored at half-filling where \( \mu = 0 \). In this case, the effective potential in the presence of \( B \) and \( \mu \) is given by
\[
V_{\text{eff}}(n) = V(n) - g(n_1^2 + n_2^2 + n_3^2) - \chi(B^2(n_1^2 + n_2^2)
+ \mu^2(n_1^2 + n_2^2)).
\] (13)
For \( B = 0 \), there is an AF to SC transition at \( \mu_c = \sqrt{g/\chi} \). For \( \mu > \mu_c \), the system is in a SC state. This SC state has a \( \pi \) resonance mode with frequency
\[
\omega_0 = \sqrt{\mu^2 - \mu_c^2}.
\] (14)
A finite magnetic-field \( B \) causes a triplet Zeeman splitting of this \( \pi \) mode, where the lower mode vanishes at a critical value
\[
B_c = \left( \sqrt{\mu^2 - \mu_c^2} \right)
\] (15)
of the Zeeman field. On the other hand, from Eq. (13), we see that a finite Zeeman magnetic-field \( B \) induces a SC to AF transition when \( B \) exceeds the same critical value \( B_c \) as given by Eq. (15). At \( B = B_c \), the effective potential \( V_{\text{eff}} \) as given in Eq. (13) is exactly SO(4) invariant in the \( \bar{M} = (n_1, n_2, n_3, n_5) \) space. The kinetic terms further break this symmetry to SU(2) × U(1). Summarizing above discussions we conclude that both exact and projected SO(5) symmetric models have a exact quantum SU(2) × U(1) symmetry at a critical value of the Zeeman magnetic field, which is the energy of the \( \pi \) resonance mode measured in the units of the magnetic field.

From above discussions we see that there are only two remaining massless modes at the \( B = \mu \) point. It would be interesting to formulate a low-energy theory where the two other massive modes are explicitly projected out. In the Lagrangian formalism, this can be accomplished by dropping the \( n_4 \) degree of freedom, and discarding the second time derivative terms in Eq. (7). This corresponds to an effective low-energy Hamiltonian of the form:
\[
\mathcal{H}_{\text{eff}} = V(\bar{M}),
\] (16)
where \( V(\bar{M}) \) is a SO(4) symmetric potential which only depends on the magnitude of the \( \bar{M} \) vector. This Hamiltonian can be quantized by the following quantization condition:
\[
[M^T, M] = \frac{i}{2} R.
\] (17)
This formulation gives us yet another way to understand the origin of the SU(2) × U(1) symmetry. \( \mathcal{H}_{\text{eff}} \) on a single site is nothing but the Hamiltonian for a symmetric two-dimensional harmonic oscillator, where \( \bar{M} \) denotes the phase space coordinates of a two-dimensional harmonic oscillator,
first-order-phase transition from the SC state to the AF state at a critical value of the magnetic-field $B_c$. However, the $\pi$ mode is still massive at $B_c$, which clearly distinguishes this from the SO(5) symmetric case. The first-order line terminates at a bicritical-point $T_{bc}$, where all static properties have an emergent SO(4) symmetry and all dynamic properties have an SU(2)$^3U(1)$ symmetry. The second type of phase diagram is realized for $W^2 > W_c$, which describes two second-order-phase transitions, with an intervening mixed phase region where both SC and AF orders coexist, as shown in Fig. 2. The mixed region shrinks to zero at a finite temperature tetra-critical-point $T_{tc}$. The two second-order lines merge at the bicritical-point $T_{bc}$, where there are also two gapless modes and two massive modes. However, there is a major difference for the gapless modes between exact and approximate SO(5) models. The two gapless modes in the approximate SO(5) model both have linear dispersion in mixed phase. In the exact SO(5) symmetry model, as we pointed out before, there is one gapless mode with quadratic dispersion, leading to a system with infinite compressibility at the transition point.\(^5\)

In conclusion we have discovered a symmetry of SO(5) models in the presence of a finite Zeeman magnetic-field $B$ and chemical-potential $\mu$. At the special point $B_c = \mu$, the static potential has an exact SO(4) symmetry and the full Hamiltonian has an exact SU(2)$\times U(1)$ symmetry. These considerations also generalize to the projected SO(5) model, where the critical magnetic field is shifted to $B_c = \sqrt{\mu^2 - \mu_c^2}$, as given by Eq. (15). This observation gives the possibility to experimentally test the SO(5) symmetry at any doping level. The Zeeman magnetic field can be experimentally realized by applying a magnetic field in the two dimensional plane, so that the orbital effects can be minimized. The critical value of a magnetic field needed for reaching the exact SU(2)$\times U(1)$ symmetry point can also be expressed as $B_c = \omega_0/g\mu_B$, where $\omega_0$ is the neutron resonance energy, $g$ is the electronic g factor, and $\mu_B$ is the Bohr magneton. Unfortunately, this value exceeds 100 $T$ for all high-$T_c$ superconductors where neutron resonance has been discovered. Fortunately, some aspects of the new SU(2)$\times U(1)$ symmetry can be tested without reaching this high-critical value of the magnetic field. Below the critical-value $B_c$, our theory predicts that the Zeeman magnetic field will only split the resonance energy, but it does not change the intensity of the $\pi$ resonance mode. The $\pi$ mode should also remain commensurate at momentum ($\pi, \pi$). While the critical value of the magnetic field is high for the cuprate superconductors, one can also try to perform the proposed experiments on other materials where the intrinsic energy scales are much lower.

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