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SO(5) superconductors in a Zeeman magnetic field

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The generic symmetry of a system under a uniform Zeeman magnetic field is $U(1)\times U(1)$. However, we show that SO(5) models in the presence of a finite chemical potential and a finite Zeeman magnetic field can have an exact $SU(2)\times U(1)$ symmetry. This principle can be used to test SO(5) symmetry at any doping level.

A fundamental question one can ask in connection with high-T_c superconductors is whether they are in the same universality class of conventional d wave Bardeen-Cooper-Schrieffer theory (BCS) superconductors. While many aspects of high-T_c superconductors are anomalous and quantitatively different from conventional BCS superconductors, no sharp distinction based on symmetry has been made so far. In the absence of an external magnetic field and spin anisotropy, the symmetry of the Hamiltonian is SU(2) $\times$U(1), where the U(1) charge symmetry is spontaneously broken in the superconducting state.

A notable exception is the idea of SO(5) symmetry between antiferromagnetism (AF) and superconductivity (SC) (Ref. 1). This theory predicts a finite temperature bicritical point with an enlarged SO(5) symmetry at the transition point between AF and SC. It also predicts a spin triplet π resonance[2] in the SC state which can be interpreted as the pseudo-Goldstone mode associated with the spontaneous symmetry breaking. However, in the presence of a finite chemical potential, the explicit symmetry of the Hamiltonian is still a direct product of the spin SU(2) and the charge U(1) symmetry, which is not different from that of a conventional BCS system.

In this paper, we point out a remarkable symmetry property of SO(5) symmetric Hamiltonians. In the presence of a finite chemical potential μ and a finite Zeeman magnetic field B, the original SO(5) symmetry is broken to U(1) $\times$U(1). Here the first U(1) group describes the spin rotation symmetry in a plane perpendicular to the applied magnetic field and the second U(1) group is the usual charge symmetry. In fact, any generic spin invariant Hamiltonian in the presence of a finite Zeeman field would have the same $U(1)\times U(1)$ symmetry. From that point of view, SO(5) symmetric models do not seem to be different from any generic models once a chemical potential or a magnetic field is applied. However, we will show that for a special combination where $B=\mu$, the SO(5) symmetric models enjoy an enlarged $SU(2)\times U(1)$ symmetry, which is not shared by generic models. Furthermore, this special $SU(2)\times U(1)$ symmetry at $B=\mu$ is equivalent to the original SO(5) symmetry in the absence of these fields. This gives a powerful new tool to test the SO(5) symmetry *at any doping level*. The original SO(5) symmetry exists only at a particular doping level where the AF to SC transition occurs. This point is very difficult to reach in high-T_c superconductors because of complicated doping chemistry, and has not yet been identified experimentally. Under the current proposal, however, the SO(5) symmetry can be revealed at any doping level, provided one applies a Zeeman magnetic field. This current test can give a sharp symmetry distinction between a SO(5) superconductor and a conventional BCS superconductor, and it can also distinguish various explanations of the π resonance. In particular, the SO(5) theory predicts that the π resonance intensity is proportional to the superconducting order parameter,[2] therefore it should be dramatically reduced in the type II phase in the presence of a c-axis magnetic field.[3] On the other hand, because of the symmetry discussed in this paper, a Zeeman mangnetic field applied in the ab plane should only split the π triplet, without changing its intensity and commensurability at (π,π).

Let us consider the following Hamiltonian

$$H=H_{SO(5)}-\mu Q-BS_z, \tag{1}$$

where $H_{SO(5)}$ is an SO(5) symmetric Hamiltonian which commutes with the ten SO(5) symmetry generators L_{ab}. (for notations and definitions, please see Ref. 1). Q and S_z are members of SO(5) symmetry generators L_{ab}, they generate charge rotation and spin rotation in the xy plane perpendicular to the external Zeeman field. Since SO(5) is a rank two algebra, one can choose Q and S_z as the two mutually commuting generators. For $B=0$, the generic symmetry of H is $SU(2)\times U(1)$, while for nonvanishing values of B, the original SO(5) symmetry of $H_{SO(5)}$ is broken explicitly to $U(1)\times U(1)$, generated by Q and S_z.

However, H has an exact enlarged symmetry SU(2) $\times$U(1) at $B=\mu$. At this point, both the chemical potential and the Zeeman terms can be combined as $-\mu Q_\uparrow$, where $Q_\uparrow$ and $Q_\downarrow$ measure the number of up-spin and down-spin electrons, respectively. Furthermore, we can define a SU(2) subalgebra of the original SO(5) algebra generated by

$$\pi_\downarrow=\sum_k \text{sgn}(\cos k_x-\cos k_y)c_{Q+k,\downarrow}c_{-k,\downarrow}, \quad \pi_\downarrow^+, \quad Q_\downarrow. \tag{2}$$

It is easy to see that they form a closed SU(2) algebra,

$$J_1=\frac{1}{2}(\pi_\downarrow+\pi_\downarrow^+), \quad J_2=\frac{i}{2}(\pi_\downarrow-\pi_\downarrow^+),$$

$$J_3=\frac{1}{2}Q_\downarrow, \quad [J_\alpha,J_\beta]=i\epsilon_{\alpha\beta\gamma}J_\gamma. \tag{3}$$

Since the generators of this subalgebra are formed by linear combinations of the original SO(5) generators L_{ab}, they all commute with $H_{SO(5)}$. Furthermore, since they only involve down-spin electrons, they commute with $-\mu Q_\uparrow$. Therefore, we have proven that at $B=\mu$, H has a SU(2)×U(1) symmetry generated by the SU(2) algebra defined by Eq. (3) and the U(1) generator $Q_\uparrow$.

Now we proceed to analyze the collective modes associated with this new symmetry. For this purpose, it is useful to first see how the new symmetry emerges in the Lagrangian formalism. The effective Lagrangian with exact SO(5) symmetry can be expressed as:

$$\mathcal{L}=\chi(\partial_t n_a)^2-\rho(\partial_k n_a)^2-V(n), \tag{4}$$

where $V(n)=-(\delta/2)\,n_a^2+(W/4)\,|n|^4$. We can introduce a magnetic field and a chemical potential simultaneously in the above Lagrangian by applying the following transformation:

$$\partial_t n_\alpha \mapsto \partial_t n_\alpha - i\epsilon_{\alpha\beta\gamma}B_\beta n_\gamma, \alpha=2,3,4;$$

$$\partial_t n_i \mapsto \partial_t n_i - i\epsilon_{ij}\mu n_j, i,j=1,5. \tag{5}$$

Choosing $\hat{B}=(0,0,-B)$, the Lagrangian becomes

$$\begin{aligned}\mathcal{L}=&\chi(\partial_t n_a)^2-\rho(\partial_k n_a)^2-2i\chi(Bn_3\partial_t n_2-Bn_2\partial_t n_3\\&-\mu n_1\partial_t n_5+\mu n_5\partial_t n_1)+\chi[B^2(n_2^2+n_3^2)+\mu^2(n_1^2+n_5^2)]\\&-V(n).\end{aligned} \tag{6}$$

Denoting $\hat{M}=(n_1,n_2,n_5,n_3)$, and taking $B=\mu$, we can rewrite the above equation into the following form:

$$\mathcal{L}=\chi(\partial_t n_a)^2-\rho(\partial_k n_a)^2+2i\chi\mu\hat{M}R\partial_t\hat{M}^T+\chi\mu^2\hat{M}^2-V(n), \tag{7}$$

where R is a four-dimensional matrix,

$$R=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

Now we discuss the symmetry of above Lagrangian. Obviously, except for the third term in the above equation, all other terms have an exact SO(4) symmetry in the $\hat{M}$ space. However not all of rotation will keep the invariance of the third term. If $\hat{O}$ denotes a rotation matrix in the $\hat{M}$ space, then it must satisfy

$$\hat{O}^T\hat{O}=1; \quad \hat{O}^T R\hat{O}=R \tag{8}$$

in order to keep the Lagrangian (7) invariant. Since SO(4) ≅SU(2)×SU(2), we immediately find one of the SU(2) subgroups whose generators are defined by the following matrix:

$$G_1=\frac{1}{2}\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix}, \quad G_2=\frac{1}{2}\begin{pmatrix}0 & -i\sigma_x\\ i\sigma_x & 0\end{pmatrix},$$

$$G_3=\frac{1}{2}\begin{pmatrix}0 & i\sigma_z\\ -i\sigma_z & 0\end{pmatrix}, \quad [G_\alpha,G_\beta]=i\epsilon_{\alpha\beta\gamma}G_\gamma. \tag{9}$$

These matrices also have the following properties

$$[G_\alpha,R]=0.$$

Therefore, G_α, together with R, generates a symmetry SU(2)×U(1). The Lagrangain (7) is invariant under above SU(2)×U(1) transformations. By Noether's theorem, each internal symmetry is associated to a conserved charge. From the infinitesimal variations of $\hat{M}$,

$$\delta\hat{M}^T=iG_\alpha\hat{M}^T\delta\phi_\alpha+R\hat{M}\,\delta\phi_R,$$

we obtain the following conserved currents

$$j_t^R=2\chi\partial_t\hat{M}R\hat{M}^T+2\chi\mu\hat{M}\hat{M}^T$$

$$j_k^R=2\rho\partial_k\hat{M}R\hat{M}^T$$

$$j_t^\alpha=2i\chi\partial_t\hat{M}G_\alpha\hat{M}^T-2i\chi\mu\hat{M}RG_\alpha\hat{M}^T$$

$$j_k^\alpha=2i\rho\partial_k\hat{M}G_\alpha\hat{M}^T;$$

$$0=\partial_t j_t^{R,\alpha}+\partial_k j_k^{R,\alpha}. \tag{10}$$

The associated conserved charges can be directly related to the symmetry generators (3) in the Hamiltonian formalism:

$$J_\alpha=\int dx j_t^\alpha; \quad Q_\uparrow=\int dx j_t^R. \tag{11}$$

Since the static potential is explicitly broken from SO(5) to SO(4), one might expect three massless Goldstone modes and one massive mode for this kind of symmetry broking. However, there are two massless modes and two massive modes in this case, because the total Lagrangian (7) has lower SU(2)×U(1) symmetry than the static potential. We can pick a particular direction in $\hat{M}$ space and linearize the mode equation around this direction, say n_1 (superconducting phase):

$$\chi\partial_t^2 n_2=\rho\partial_k^2 n_2-2\mu\chi\partial_t n_3$$

$$\chi\partial_t^2 n_3=\rho\partial_k^2 n_3+2\mu\chi\partial_t n_2$$

$$\chi\partial_t^2 n_5=\rho\partial_k^2 n_5$$

$$\chi\partial_t^2 n_4=\rho\partial_k^2 n_4-\chi\mu^2 n_4. \tag{12}$$

The last equation describes the massive modes with energy $\omega_4=\mu$, which is associated with the explicit symmetry breaking (from SO(5) to SO(4)) of the static potential. The third equation describes the usual Goldstone massless mode (sound mode) of the superconductor with linear dispersion $\omega_5=(\rho/\chi)k$. The first two equations predict new doublet-spin wave modes. One is massless, the other is massive. In the long-wavelength limit, the energies of the modes are $\omega_2=vk^2,\omega_3=2\mu$. Therefore, there are always two gapless modes, one with linear dispersion and the other with quadratic dispersion, independent of the orientation of superspin.

It is also interesting to investigate the case where the SO(5) symmetry is explicitly broken, but a projected SO(5) symmetry defined in Refs. 4 and 5 is present. We can add a term $-g(n_2^2+n_3^2+n_4^2)$ to the SO(5) symmetric potential $V(n)$, and choose $g>0$ so that AF is favored at half-filling where $\mu=0$. In this case, the effective potential in the presence of B and μ is given by

$$V_{eff}(n)=V(n)-g(n_2^2+n_3^2+n_4^2)-\chi[B^2(n_2^2+n_3^2)+\mu^2(n_1^2+n_5^2)]. \quad (13)$$

For $B=0$, there is an AF to SC transition at $\mu_c=\sqrt{g/\chi}$. For $\mu>\mu_c$, the system is in a SC state. This SC state has a π resonance mode with frequency

$$\omega_0=\sqrt{\mu^2-\mu_c^2}. \quad (14)$$

A finite magnetic-field B causes a triplet Zeeman splitting of this π mode, where the lower mode vanishes at a critical value

$$B_c=\sqrt{\mu^2-\mu_c^2} \quad (15)$$

of the Zeeman field. On the other hand, from Eq. (13), we see that a finite Zeeman magnetic-field B induces a SC to AF transition when B exceeds the same critical value B_c as given by Eq. (15). At $B=B_c$, the effective potential V_{eff} as given in Eq. (13) is exactly SO(4) invariant in the $\hat{M}=(n_1,n_2,n_5,n_3)$ space. The kinetic terms further break this symmetry to SU(2)×U(1). Summarizing above discussions we conclude that both exact and projected SO(5) symmetric models have a exact quantum SU(2)×U(1) symmetry at a critical value of the Zeeman magnetic field, which is the energy of the π resonance mode measured in the units of the magnetic field.

From above discussions we see that there are only two remaining massless modes at the $B=\mu$ point. It would be interesting to formulate a low-energy theory where the two other massive modes are explicitly projected out. In the Lagrangian formalism, this can be accomplished by dropping the n_4 degree of freedom, and discarding the second time derivative terms in Eq. (7). This corresponds to an effective low-energy Hamiltonian of the form:

$$\mathcal{H}_{eff}=V(\hat{M}), \quad (16)$$

where $V(\hat{M})$ is a SO(4) symmetric potential which only depends on the magnitude of the $\hat{M}$ vector. This Hamiltonian can be quantized by the following quantization condition:

$$[M^T,M]=\frac{i}{2}R. \quad (17)$$

This formulation gives us yet another way to understand the origin of the SU(2)×U(1) symmetry. $\mathcal{H}_{eff}$ on a single site is nothing but the Hamiltonian for a symmetric two-dimensional harmonic oscillator, where $\hat{M}$ denotes the *phase space* coordinates of a two-dimensional harmonic oscillator, and the quantization condition (17) is nothing but the Heisenberg commutation relation between the coordinates and momenta. A symmetric two-dimensional harmonic oscillator has more than the SO(2) symmetry of the coordinate space, but less than the SO(4) symmetry of the phase space. In fact, it has a U(2)=SU(2)×U(1) symmetry. This discussion carried over straightforwardly to the case of coupled oscillators with a global U(2)=SU(2)×U(1) symmetry.

The observation of the new SU(2)×U(1) symmetry gives us the possibility of testing the SO(5) symmetry of the original model at any doping. Starting from a SC state at zero-magnetic field, the superspin lies in the (n_1,n_5) plane. Within the SO(5) model, the only effect of a applied Zeeman magnetic field is to split the π triplet resonance mode. The intensity and commensurability of each member of the triplet remain the same. At a critical-field B_c, there is a first-order transition from the SC state into the AF state where the superspin lies in the (n_2,n_3) plane. At the same time, one of the π mode softens to zero energy at B_c. The exact coincidence of a mode softening transition and a first-order transition is the signature of the new symmetry. As we shall see, in a generic system, either the first-order transition occurs before the mode softens to zero energy, or the mode softening occurs before the first-order transition, in which case the system will have two separate second-order phase transitions.

All above discussions are based on the assumption where the original model has an exact or projected SO(5) symmetry. In order to see the physical signature of the SO(5) symmetry, it is useful to study the effects of a finite chemical potential and Zeeman magnetic field on models without SO(5) symmetry. A general Landau-Ginzburg potential of an approximate SO(5) model in the presence of a finite Zeeman magnetic field B and chemical potential μ can be expressed as

$$V=-\frac{\delta_c}{2}x-\frac{\delta_s}{2}y-\frac{\delta}{2}z+\frac{W_c}{4}x^2+\frac{W_s}{4}(y+z)^2+\frac{W_0}{2}x(y+z), \quad (18)$$

where $n_1^2+n_5^2=x$, $n_2^2+n_3^2=y$, $n_4^2=z$, $\delta_c=2\chi_c\mu^2+\delta$, and $\delta_s=2\chi_sB^2+\delta$. There are two kinds of generic phase diagrams described by this effective potential. The first type of phase diagram is realized for $W_0^2>W_cW_s$ and is depicted in Fig. 1. In this case, the Zeeman magnetic field induces a

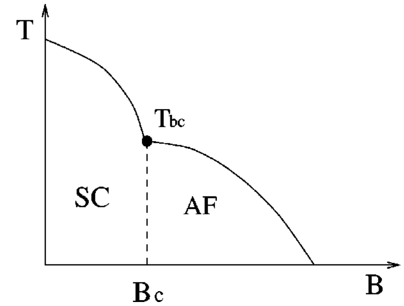

FIG. 1. Generic phase diagram of an approximate SO(5) superconductor in a Zeeman magnetic field. The dashed line describes a direct first-order transition between SC and AF order. At the bicritical point T_{bc}, all static properties have exact SO(4) symmetry and all dynamic properties have exact SU(2)×U(1) symmetry.

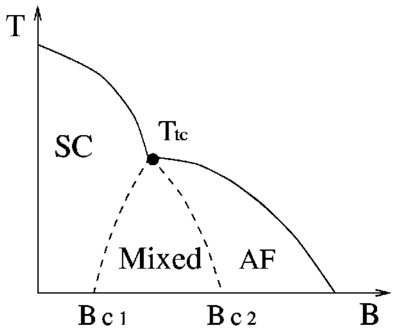

FIG. 2. Generic phase diagram of an approximate SO(5) superconductor in a Zeeman magnetic field. The two dashed lines describe two second-order phase transitions, with an intervening mixed phase. The two second-order lines merge at a tetra-critical-point T_{tc}.

first-order-phase transition from the SC state to the AF state at a critical value of the magnetic-field B_c. However, the π mode is still massive at B_c, which clearly distinguishes this from the SO(5) symmetric case. The first-order line terminates at a bicritical-point T_{bc}, where all static properties have an emergent SO(4) symmetry and all dynamic properties have an SU(2)×U(1) symmetry. The second type of phase diagram is realized for $W_0^2 < W_c W_s$; it describes two second-order-phase transitions, with an intervening mixed phase region where both SC and AF orders coexist, as shown in Fig. 2. The mixed region shrinks to zero at a finite temperature tetra-critical-point T_{tc}. In the mixed phase, there are also two gapless modes and two massive modes. However, there is a major difference for the gapless modes between exact and approximate SO(5) models. The two gapless modes in this approximate SO(5) model both have linear dispersion in mixed phase. In an exact SO(5) symmetry model, as we pointed out before, there is one gapless mode with quadratic dispersion, leading to a system with infinite compressibility at the transition point.[4]

In conclusion we have discovered a symmetry of SO(5) models in the presence of a finite Zeeman magnetic-field B and chemical-potential μ. At the special point $B_c = \mu$, the static potential has an exact SO(4) symmetry and the full Hamiltonian has an exact SU(2)×U(1) symmetry. These considerations also generalize to the projected SO(5) model, where the critical magnetic field is shifted to $B_c = \sqrt{\mu^2 - \mu_c^2}$, as given by Eq. (15). This observation gives the possibility to experimentally test the SO(5) symmetry at any doping level. The Zeeman magnetic field can be experimentally realized by applying a magnetic field in the two dimensional plane,[6,7] so that the orbital effects can be minimized. The critical value of a magnetic field needed for reaching the exact SU(2)×U(1) symmetry point can also be expressed as $B_c = \omega_0 / g\mu_B$, where ω_0 is the neutron resonance energy, g is the electronic g factor, and μ_B is the Bohr magneton. Unfortunately, this value exceeds 100 T for all high-T_c superconductors where neutron resonance has been discovered. Fortunately, some aspects of the new SU(2)×U(1) symmetry can be tested without reaching this high-critical value of the magnetic field. Below the critical-value B_c, our theory predicts that the Zeeman magnetic field will only split the resonance energy, but it does not change the intensity of the π resonance mode. The π mode should also remain commensurate at momentum (π, π). While the critical value of the magnetic field is high for the cuperate superconductors, one can also try to perform the proposed experiments on other materials[8] where the intrinsic energy scales are much lower.

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