

Effective Field Theory Description of the Higher Dimensional Quantum Hall Liquid

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We derive an effective topological field theory model of the four dimensional quantum Hall liquid state recently constructed by Zhang and Hu. Using a generalization of the flux attachment transformation, the effective field theory can be formulated as a $U(1)$ Chern–Simons theory over the total configuration space CP_3 , or as a $SU(2)$ Chern–Simons theory over S^4 . The new quantum Hall liquid supports various types of topological excitations, including the 0-brane (particles), the 2-brane (membranes), and the 4-brane. There is a topological phase interaction among the membranes which generalizes the concept of fractional statistics. © 2002 Elsevier Science (USA)

INTRODUCTION

Recently, a higher dimensional generalization of the quantum Hall effect was constructed by two of us (ZH) [1]. This incompressible liquid was defined on a 4-dimensional spherical surface with a $SU(2)$ monopole at its center [2–4]. The fermionic particles carry a $SU(2)$ quantum number I , which scales with the radius R as $I \sim R^2$. This quantum liquid state shares many properties with its 2-dimensional counterpart [5]. The ground state is separated from all excited states by a finite energy gap, and the density correlation functions decay gaussianly. Fractional quantum Hall states can also be constructed, and they support fractionally charged quasi-particle excitations. The full spectrum of the boundary excitations of this quantum liquid is still not fully understood, but a partial analysis reveals collective excitations with all relativistic helicities [1, 6]. More recently, Fabinger [7] found a realization of this system within string theory. Sparling [8] found a deep connection between this system and the twistor theory. Karabali and Nair [9] investigated the generalization of the quantum Hall effect on the CP_n manifolds. Ho and Ramgoolam [10] obtained the matrix descriptions of even

dimensional fuzzy spherical branes S^{2k} in Matrix theory. Chen *et al.* [11] discussed the relationship between the 4-dimensional quantum Hall liquid and the non-commutative geometry on S^4 .

Emergence of the Chern–Simons (CS) gauge structure in the 2-dimensional fractional quantum Hall effect (FQHE) was an exciting development in condensed matter physics in recent years [12–14]. The Chern–Simons–Landau–Ginzburg (CSLG) theory of the FQHE describes the long wavelength physics of the incompressible quantum Hall liquid in terms of a topological field theory, which to the leading order is independent of the space-time metric. The $1 + 1$ dimensional chiral relativistic dynamics emerges when the CSLG theory is restricted to the edge of the quantum Hall liquid [15, 16]. It would be highly desirable to see how this elegant connection between the microscopic wave function and the topological field theory can also be generalized to higher dimensions.

In this work we focus on the effective topological field theory of the quantum liquid state constructed by ZH. The key step is to generalize the concept of flux attachment transformation [12, 13] to higher dimensions, and to use the concept of non-commutative geometry [17] to relate theories defined in different dimensions. We find two equivalent CS theories, an abelian CS theory in $6 + 1$ dimensions, and a $SU(2)$ non-abelian CS theory in $4 + 1$ dimensions. The quantum liquid constructed by ZH has orbital degrees of freedom scaling with R^4 and internal isospin degree of freedom scaling with R^2 . Therefore, the total configuration space is 6-dimensional [1]. This can also be seen from the second Hopf map $S^7 \rightarrow S^4$, which was the central mathematical construct used by ZH. S^7 describes the combined orbital and isospin degrees of freedom. However, since quantum mechanics is based on $U(1)$ projective representations, the actual configuration space is $S^7/U(1) = CP_3$, or the 3 complex dimensional (6 real dimensional) projective space. Our $6 + 1$ dimensional CS theory is naturally defined over the $CP_3 \times \mathcal{R}$ manifold, where the 6 volume form on CP_3 plays the role of the generalized background flux. Since the CP_3 manifold locally decomposes as $S^4 \times S^2$, we can also arrive at a $4 + 1$ dimensional continuum field theory by treating S^2 as a “fuzzy sphere,” with discrete matrix model degrees of freedom. By this procedure, we arrive at the equivalent $SU(2)$ non-abelian CS gauge field theory in $4 + 1$ dimensions, where particles with $SU(2)$ internal isospin degrees of freedom are attached to the $SU(2)$ instanton density over S^4 .

Besides the quasi-particle like elementary excitations, we show here that the quantum liquid constructed by ZH also supports topologically stable extended objects, namely the membrane (2-brane) and the 4-brane. The topological action for these extended objects can be derived exactly. A particularly interesting aspect of the membranes is that they have non-trivial phase interactions [18, 19] which is a direct generalization of the fractional statistics carried by the Laughlin quasi-particles in the 2D quantum Hall liquid.

SINGLE PARTICLE LAGRANGIAN

The 2D quantum Hall effect arises when a 2-dimensional electron gas is subjected to a strong magnetic field perpendicular to its plane. It can also arise when electrons move on a two sphere S^2 with a $U(1)$ magnetic monopole at the center [20]. The two systems can be related to each other by a conformal transformation, namely, the inverse stereographic projection from S^2 to R^2 , with all points at infinity identified with the south pole. Much of the spherical physics relies on the topological properties of the first Hopf map, $S^3 \rightarrow S^2$. To exhibit the map, we introduce a two component complex spinor u with $\bar{u}u = 1$ and set

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\frac{\phi}{2} + i\frac{\chi}{2}} \\ \sin \frac{\theta}{2} e^{i\frac{\phi}{2} - i\frac{\chi}{2}} \end{pmatrix}, \quad (1)$$

where θ is in $[0, \pi]$; ϕ and χ are in $[0, 2\pi)$. The first Hopf map is defined as

$$\frac{n_i}{r} = \bar{u}\sigma_i u, \quad i = 1, 2, 3, \quad (2)$$

where the σ_i 's are the three Pauli matrices, and $n_i^2 = r^2$ is manifestly satisfied. Here n_i is the coordinate of S^2 with radius r . The Hopf fibration defines the principal $U(1)$ bundle over S^2 . The Dirac quantization condition on the sphere gives $eg = I$, where e and g are the electric and magnetic charges, respectively. I is quantized to be an integer or a half-integer. We are interested in the limit of infinite I and r , such that the ratio I/r^2 is fixed. The many-body electronic system can be thought of as an incompressible liquid on S^2 .

In the strong magnetic field limit, or in the limit of vanishing kinetic mass $m \rightarrow 0$, the single particle Lagrangian contains only a first order time derivative. It is in fact nothing but the Berry's phase of the u spinors. Since $S^3/U(1) = S^2$, the $U(1)$ Berry's phase over the base space S^2 can simply be expressed in the following form

$$L = 2iI\bar{u}\frac{d}{dt}u = iI\text{Tr}(\sigma_3 s^{-1}(x)\dot{s}(x)). \quad (3)$$

Here I is chosen to label the spinor representation of $SU(2)$, and $s(x)$ is an element of $SU(2)$. It can be expressed as

$$s(x) = \begin{pmatrix} u_1 & u_2^* \\ -u_2 & u_1^* \end{pmatrix}. \quad (4)$$

The second form in (3) is due to Balachandran [21]. In the gauge where $\dot{\phi} = -\dot{\chi}$, (3) becomes

$$L = I\epsilon_{3ij}\frac{n_i\dot{n}_j}{r(r+n_3)}. \quad (5)$$

This is the Lagrangian of a particle moving on S^2 , interacting with a $U(1)$ monopole gauge potential. The Dirac string of this monopole potential is located at the south pole in this gauge. This gauge will be used throughout the paper unless stated otherwise.

In a higher dimensional generalization, ZH[1] considered fermions moving on a 4-sphere interacting with an $SU(2)$ magnetic monopole at the center [2–4]. Yang [4] has shown that the $SU(2)$ monopole with ± 1 topological charge is $SO(5)$ invariant. The field strength is self-dual for the topological charge $+1$ and anti-self-dual for the opposite case. The underlying algebraic structure of this system is the second Hopf fibration, $S^7 \rightarrow S^4$. Let Ψ be a 4-component complex spinor with $\bar{\Psi}\Psi = 1$. The second Hopf map is given by

$$\frac{X_a}{R} = \bar{\Psi}\Gamma_a\Psi. \quad (6)$$

Here the Γ_a 's are the five Dirac Gamma matrices of $SO(5)$, satisfying the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. It is easy to see that $X_a^2 = R^2$ follows from the normalization condition $\bar{\Psi}\Psi = 1$. Since Ψ is a 4 component complex spinor, the normalization condition defines a 7-sphere S^7 embedded in 8-dimensional Euclidean space. On the other hand, $X_a^2 = R^2$ defines a 4-sphere S^4 with radius R embedded in 5-dimensional Euclidean space. Therefore, Eq. (6) defines a mapping from S^7 to S^4 . S^7 can be viewed as a principal $SU(2)$ bundle over S^4 . It can also be viewed as a $U(1)$ bundle over $S^7/U(1) = CP_3$.

To generalize (3), we consider Berry's phase for the Ψ spinor. The $SU(2)$ non-abelian Berry's phase over the base space S^4 has been computed by Demler and Zhang in [22]. Here we shall find it useful to compute the $U(1)$ abelian Berry's phase over CP_3 . Since CP_3 can be defined as the space of the Ψ spinors up to an overall $U(1)$ phase factor, the $U(1)$ Berry's phase Lagrangian is simply

given by

$$L = 2iI\bar{\Psi} \frac{d}{dt} \Psi. \tag{7}$$

We take Ψ to be the solution of the Hopf equation, which is given by $(X_a/R)\Gamma_a \Psi = \Psi$. Then Ψ is solved as

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{R+X_5}{2R}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \frac{1}{\sqrt{2R(R+X_5)}} (X_4 - iX_i\sigma^i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}, \tag{8}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{r+n_3}{2r}} \\ \frac{n_1 + in_2}{\sqrt{2r(r+n_3)}} \end{pmatrix}. \tag{9}$$

We can write the action more explicitly as

$$\begin{aligned} S &= 2iI \int dt \bar{\Psi} \partial_t \Psi = - \int dt \left(I \varepsilon_{3ij} \frac{n_i \dot{n}_j}{r(r+n_3)} + \frac{I}{R(R+X_5)} \eta_{\mu\nu}^i \frac{n_i}{r} X_\mu \dot{X}_\nu \right) \\ &= \int dt (A_a(X, n) \partial_t X_a + A_i(X, n) \partial_t n_i) \\ &\equiv \int dt g^{AB} A_A \partial_t X_B, \end{aligned} \tag{10}$$

where $\eta_{\mu\nu}^i = \varepsilon_{i\mu\nu 4} + \delta_{i\mu} \delta_{4\nu} - \delta_{i\nu} \delta_{4\mu}$ is the 't Hooft symbol. X_B denotes the local coordinate of the CP_3 manifold. From the parameterization of Ψ and u given in (8) and (9), we see explicitly that the CP_3 manifold locally decomposes as $S^4 \times S^2$. Index conventions and the explicit form of the metric tensor g_{AB} over CP_3 are given in the Appendix. The quantization of the Lagrangian will be determined later. From (10), we can read out the $U(1)$ gauge connection over the CP_3 manifold:

$$A_\mu = \frac{I}{R(R+X_5)} \eta_{\mu\nu}^i \frac{n_i}{r} X_\nu, \quad A_5 = 0 \tag{11}$$

$$A_i = \frac{I}{r(r+n_3)} \varepsilon_{3ij} n_j. \tag{12}$$

The gauge potential is defined only patch by patch. Here the gauge potential is defined over the north pole region $X_5 \approx R$ and $n_3 \approx r$. A similar gauge potential can be defined over the south pole region. In the overlap of patches, the gauge potentials defined in each patch differ by a $U(1)$ gauge transformation. The gauge potential also satisfies the following transversality conditions

$$X_a A_a = 0, \quad n_i A_i = 0. \tag{13}$$

DERIVATION OF EQUATIONS OF MOTION

The field strength of the $U(1)$ gauge potential, $F_{\lambda\tau}(\lambda, \tau = \{a, i\})$, is given by

$$F_{\lambda\tau} = \partial_\lambda A_\tau - \partial_\tau A_\lambda. \quad (14)$$

Since we are using a redundant set of coordinates here, we need to be careful and differentiate R and r , using $\partial R/\partial X_a = X_a/R$ and $\partial r/\partial n_i = n_i/r$. The resulting field strength is given by

$$\begin{aligned} F_{\mu\nu} &= \frac{-2I}{R(R+X_5)} \eta_{\mu\nu}^i \frac{n_i}{r} + \frac{2R+X_5}{R^2(R+X_5)} (A_\mu X_\nu - A_\nu X_\mu) \\ F_{\mu i} &= \frac{-I}{rR(R+X_5)} \eta_{\mu\nu}^i X_\nu + \frac{I}{rR(R+X_5)} \eta_{\mu\nu}^k \frac{n_k}{r} \frac{n_i}{r} X_\nu \\ F_{5i} &= 0, \quad F_{\mu 5} = \frac{I}{R^3} \eta_{\mu\tau}^i \frac{n_i}{r} X_\tau, \quad F_{ij} = \frac{I}{r^3} \varepsilon_{ijk} n_k, \end{aligned} \quad (15)$$

and the following equations are satisfied,

$$X_a F_{ab} = 0, \quad n_i F_{ij} = 0, \quad F_{ai} n_i = 0, \quad F_{ai} X_a = 0. \quad (16)$$

The equations of motion can be derived from the Lagrangian (10) together with an external potential $V(n, X)$. Using Lagrangian multipliers for the constraints, $X_a^2 = R^2$ and $n_i^2 = r^2$, and taking the variation of X_b and n_k , respectively, we obtain the following two equations of motion¹

$$\begin{aligned} F_{ba} \dot{X}_a + \frac{X_a}{R^2} \frac{\partial V}{\partial X_a} X_b - \frac{\partial V}{\partial X_b} + F_{bi} \dot{n}_i &= 0 \\ F_{ki} \dot{n}_i + F_{ka} \dot{X}_a + \frac{n_i}{r^2} \frac{\partial V}{\partial n_i} n_k - \frac{\partial V}{\partial n_k} &= 0. \end{aligned} \quad (17)$$

The above equations can be simplified to

$$\begin{aligned} (F_{ba} - 4r^2 F_{bi} F_{ik} F_{ka}) \dot{X}_a &= -\frac{X_a}{R^2} \frac{\partial V}{\partial X_a} X_b + \frac{\partial V}{\partial X_b} - 4r^4 F_{bj} F_{jk} \frac{\partial V}{\partial n_k} \\ \dot{n}_j &= 4r^4 F_{jk} \frac{\partial V}{\partial n_k} - 4r^4 F_{jk} F_{k\mu} \dot{X}_\mu. \end{aligned} \quad (18)$$

For most cases we shall consider, the potential $V(n, X)$ does not depend on the isospin coordinates, i.e., $\partial V/\partial n = 0$ on the right hand side of (18). The first term on the right hand side of the second equation in (18) is zero. Recall that the $SU(2)$ matrix valued gauge potential \mathbf{A}_a and field strength $\mathbf{F}_{ab} = \partial_a \mathbf{A}_b - \partial_b \mathbf{A}_a + [\mathbf{A}_a, \mathbf{A}_b]$ are explicitly given by [1]

$$\begin{aligned} \mathbf{A}_\mu &= \frac{-i}{R(R+X_5)} \eta_{\mu\nu}^i \mathbf{I}_i X_\nu, \quad \mathbf{A}_5 = 0 \\ \mathbf{F}_{\mu\nu} &= \frac{1}{R^2} (X_\nu \mathbf{A}_\mu - X_\mu \mathbf{A}_\nu + i \eta_{\mu\nu}^i \mathbf{I}_i) \\ \mathbf{F}_{5\mu} &= \partial_5 \mathbf{A}_\mu = -\frac{R+X_5}{R^2} \mathbf{A}_\mu. \end{aligned} \quad (19)$$

¹ For convenience, we show the results with $I = \frac{1}{2}$. The general expressions follow easily.

The Heisenberg operator equations of motion in the large field ($m \rightarrow 0$) limit are given by

$$\begin{aligned} \dot{X}_b \mathbf{F}_{ab} &= i \left(\frac{\partial V}{\partial X_a} - \frac{X_a X_b}{R^2} \frac{\partial V}{\partial X_b} \right) \\ \dot{\mathbf{I}}_i &= \varepsilon_{ijk} A_\mu^j \dot{X}_\mu \mathbf{I}_k. \end{aligned} \tag{20}$$

Equations (18) and (20) are exactly equivalent to each other. Their equivalence is established by the following mapping between the iso-spin $SU(2)$ generators and S^2 coordinates

$$\frac{n_i}{2r} = \bar{u} \mathbf{I}_i u. \tag{21}$$

By this mapping, we can also show the following important identity

$$\bar{u} [\mathbf{A}_\mu, \mathbf{A}_\nu] u = -2ir^4 F_{\mu i} F_{ik} F_{kv} \tag{22}$$

Note that the $SU(2)$ field strength on S^4 is finite everywhere while the $U(1)$ field strength is singular. For \mathbf{F}_{ab} , it turns out that the singularity in $\partial_a \mathbf{A}_b - \partial_b \mathbf{A}_a$ exactly cancels the singularity in the commutator $[\mathbf{A}_a, \mathbf{A}_b]$. Therefore, the $SU(2)$ field strength \mathbf{F}_{ab} and the $U(1)$ $F_{\lambda\tau}$ have the following correspondence

$$\begin{aligned} \bar{u} \mathbf{F}_{a5} u &= -i F_{a5} \\ \bar{u} \mathbf{F}_{\mu\nu} u &= -i F_{\mu\nu} - 2ir^4 F_{\mu i} F_{ik} F_{kv}. \end{aligned} \tag{23}$$

Using these identities, and sandwiching the matrix equations (20) by \bar{u} and u on both sides, we obtain (18) for the case when $\partial V/\partial n = 0$. This proves the exact equivalence between the CP_3 equations of motion obtained here and the S^4 non-abelian equations of motion obtained in [1]. The deep meaning of these remarkable identities will become clear after we consider the fuzzy S^2 in the $CP_3 \sim S^4 \times S^2$ local decomposition.

SEMICLASSICAL QUANTIZATION AND THE FLAT SPACE LIMIT

Similar to the 2D case, we can carry out a semiclassical quantization of the action (10). In the 2D case, we start with the Lagrangian (3). The action for one orbit is given by

$$S = \oint_\lambda A_i dx^i, \tag{24}$$

where λ is the closed contour on S^2 . The gauge potential in (24), unlike the field strength F , suffers from Dirac string. Therefore, using Stoke's theorem, we can convert the line integral to an area integral

$$S = \frac{1}{2} \int_u F_{ij} dx^i \wedge dx^j, \tag{25}$$

where u denotes the upper hemisphere which is bounded by the closed loop. On the other hand, we can also integrate F over the lower hemisphere. The difference between the two integrals is the integral of F over the whole of S^2 . This integral gives $4\pi I$. Since the two prescriptions should

be equivalent, we must set $4\pi I = 2\pi n$ as a quantization condition. This is the Bohr–Sommerfeld quantization. Thus, we obtain the quantization $2I = n$. I is either an integer or an half integer.

In our CP_3 case, the closed loop should be carefully chosen. Since $H_2(CP_3) = \mathbf{Z}$, we can choose the closed loop on two-cycle CP_1 . $H_2(CP_3)$ is the second homology group of CP_3 . Define the map $\Xi : CP_3 \rightarrow CP_1$ so that the Kähler form on CP_3 maps to the one on CP_1 . We can always do that because CP_1 can be naturally embedded into CP_3 . Under this map, the loop integral of (10) for one orbit becomes

$$S = \oint_{\lambda} A_{\tau} dx^{\tau} = \oint A_i dx^i, \tag{26}$$

where λ is a closed loop on CP_1^2 and the A_i 's are given by the second equation in (12). The A_i 's have the same forms as those in the 2D case. Therefore, we obtain exactly the same quantization condition as that in 2D. I is quantized to be an integer or an half integer.

Just like the case of (3), the single particle Lagrangian can also be obtained by representing CP_3 as a coset space $SU(4)/U(3)$. The Lagrangian has a form similar to (3)

$$L = 2i \text{Tr}(YS^{-1}(x)\dot{S}(x)). \tag{27}$$

Here $S(x)$ is a general group element of $SU(4)$. In order to mod by the stability group $U(3)$, we choose

$$Y = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{28}$$

Similar to the 2D case, it can be shown that (27) is equivalent to (7).

This is a dynamical system with constraints. In order to quantize (27), we follow the Dirac constraint formalism [23] and the methods of [21, 24]. Let $T_k, k = 1, \dots, 15$, be the generators of $SU(4)$, satisfying $\text{Tr}(T_k T_{k'}) = \frac{1}{2} \delta_{kk'}$. We may choose $T_k, k = 1, \dots, 8$, to the generators of a $SU(3)$ subgroup. The hypercharge of $SU(4)$ is chosen to be $Y = (3/\sqrt{6})T_{15}$. A general $SU(4)$ group element $S(\xi)$ takes the form $S(\xi) = e^{iT_k \xi_k}$. ξ_k is used to parameterize the group manifold. The momentum conjugate to ξ_k is given by $\pi_k = \frac{\partial L}{\partial \dot{\xi}_k}$. Let's define $S^{-1}dS = -iT_{k'} E_{k'k} d\xi_k$, where E is a 15×15 matrix and each element in E is a function of ξ_k . π_k can be expressed as

$$\pi_k = 2i \text{Tr}(YT_{k'} E_{k'k}). \tag{29}$$

Since $E_{k'k}$ is non-singular [21], we can introduce $\Lambda_k = -\pi_{k'}(E^{-1})_{k'k}$. Equation (29) is simplified to

$$\Lambda_k = -2i \text{Tr}(YT_k) = -\frac{\sqrt{6}I}{2} \delta_{k,15}. \tag{30}$$

Using the Poisson brackets

$$\{\xi_k, \xi_{k'}\} = \{\pi_k, \pi_{k'}\} = 0, \quad \{\xi_k, \pi_{k'}\} = \delta_{kk'} \tag{31}$$

² We use the same notations here as in Section 2. τ is in the set $\{a, i\}$.

one can easily show that Λ_k 's are the generators of $SU(4)$ which act on the right of $S(\xi)$, i.e.,

$$\{\Lambda_k, S\} = -iST_k, \quad \{\Lambda_k, \Lambda_{k'}\} = f_{kk'k''} \Lambda_{k''}, \tag{32}$$

where the $f_{kk'k''}$ are the structure constants of $SU(4)$. Moreover, since the Lagrangian is of the first order, (30) essentially provides a set of constraints determining the momentum conjugates. They are expressed as

$$\Lambda_k + \frac{\sqrt{6}I}{2} \delta_{k,15} \approx 0. \tag{33}$$

The \approx means that the equalities are satisfied only on the constraint surface. For $k = 1, \dots, 8, 15$ in (33), the constraints are first class constraints since Poisson brackets $\{\Lambda_k, \Lambda_{k'}\}$ weakly vanish on the constraint surface. The rest of the constraints are the second class constraints. We can rearrange the second class constraints to form a complete set of first class constraints. For $2I \geq 0$, the set of new first class constraints is

$$\Phi_{15} = \Lambda_{15} + \frac{\sqrt{6}I}{2} \approx 0, \quad \Phi_k = \Lambda_k \approx 0, \quad k = 1, \dots, 8 \tag{34}$$

$$\Phi_k^- = \Lambda_k - i\Lambda_{k+1} \approx 0, \quad k = 9, 11, 13. \tag{35}$$

For $2I < 0$, we rearrange the second class constraints by $\Phi_k^+ = \Lambda_k + i\Lambda_{k+1} \approx 0, k = 9, 11, 13$. The two sets of constraints map to different irreducible representations of $SU(4)$.

Following the analysis in [24], we can apply these constraints on functions of $SU(4)$. The states satisfying the above constraints are $SU(3)$ singlets. They have the non-vanishing eigenvalue $-\sqrt{6}I/2$ of Λ_{15} which acts on the right. Therefore, $2I$ must be an integer.³ The irreducible representations (irreps) of $SU(4)$ are labelled by (n_1, n_2, n_3) . For fixed $2I$, (34) and (35) select certain irreps. They uniquely determine $(2I, 0, 0)$ for $2I > 0$, and $(-2I, -2I, -2I) = (-2I, 0, 0)^*$ for $2I < 0$. These representations consist of $SU(3)$ singlet states which cannot be raised or lowered any further by (35). This can be understood more easily by looking at the weight diagram of the representations. For example, for $2I > 0$, the conditions select the state at the bottom in the weight diagram of $(2I, 0, 0)$. The constraint (35) serves the role of $SU(2)$ raising or lowering operators. From the dynamical point of view, CP_3 is the homogeneous space $SU(4)/U(3)$. We can view (34) as the conditions which define the coset space and (35) as the force normal to the constraint surface. The force being zero on the surface indicates that particles only move along CP_3 . This completes the quantization procedure. The $(2I, 0, 0)$ and $(2I, 0, 0)^*$ are the symmetrical tensor irreps of $SU(4)$ with the dimension equal to $\frac{1}{6}(|2I| + 1)(|2I| + 2)(|2I| + 3)$. They are also identical to the spinor irreps $(|2I|, 0)$ of $SO(5)$. These irreps are exactly the lowest-Landau-level functions

$$\sum_{m_1+m_2+m_3+m_4=2I} \sqrt{\frac{(2I)!}{m_1!m_2!m_3!m_4!}} \Psi_1^{m_1} \Psi_2^{m_2} \Psi_3^{m_3} \Psi_4^{m_4} \tag{36}$$

found in Eq. (9) of Ref. [1]. The lowest-Landau-level states of the $SO(5)$ symmetric Hamiltonian defined in Ref. [1] indeed have a larger, $SU(4)$ symmetry group. States including higher Landau levels have only $SO(5)$ symmetry.

³ In other words, the states are the eigenstates of Y with eigenvalues $-\frac{3}{4}(2I)$.

The physical picture of this CP_3 model can be visualized more clearly by taking the flat space limit. Take (10), and expand it around $X_5 = R$ and $n_3 = r$. It becomes

$$L = -I\varepsilon_{3ij} \frac{n_i \dot{n}_j}{2r^2} - I\varepsilon_{3ij} \frac{X_i \dot{X}_j}{2R^2} - I\varepsilon_{12\mu\nu} \frac{X_\mu \dot{X}_\nu}{2R^2}. \quad (37)$$

In this limit, the system behaves like three independent 2D QHE. The three independent planes are (n_1, n_2) , (X_1, X_2) , and (X_3, X_4) . The noncommutative algebra is as follows:

$$\left[\frac{n_1}{2r}, \frac{n_2}{2r} \right] = \frac{i}{2I} \quad (38)$$

$$[X_1, X_2] = \frac{2iR^2}{I} \quad (39)$$

$$[X_3, X_4] = \frac{2iR^2}{I}. \quad (40)$$

In (38), $n_i/2r$ plays the role of a classical spin, and its commutation relation vanishes as the classical limit $I \rightarrow \infty$. These quantization equations agree exactly with the non-commutative geometry equation (14) of Ref. [1], when it is expanded around $n_3 = r$.

FIELD THEORY LAGRANGIAN

We go from the single particle Lagrangian (10) to the Lagrangian of many particles

$$\int dt \sum_i A_B(X_i) \partial_t X_i^B = \int dt d^6x \rho(x) A_B \partial_t X^B = \int dt d^6x A_B J^B, \quad (41)$$

where $\rho(x) = \sum_i \delta(x - X^i)$ is the particle density of the liquid, and $J^A = \rho \partial_t X^A$ is the particle current density. They satisfy the equation of continuity

$$\partial_\Gamma J^\Gamma = \partial_t \rho + \partial_A J^A = 0. \quad (42)$$

The key to the Chern–Simons construction in the $2 + 1$ dimensional QHE is the flux attachment transformation [12, 13]. The particle is attached to its dual, the 2-form field strength $J^\mu = v \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$. A conserved current in $6 + 1$ dimensions is equivalent to a 6-form field strength with the continuity equation (42) replaced by the Bianchi identity of the 6-form field strength. Thus the natural generalization in the higher dimensional case is to attach to the particle its dual, the six form field strength, i.e.,

$$\rho = \frac{1}{8} \frac{v}{3!} \epsilon^{ABCDEFG} \mathcal{F}_{AB} \mathcal{F}_{CD} \mathcal{F}_{EG}. \quad (43)$$

From the equation of continuity we also obtain the generalization:

$$J^\Gamma = \frac{v}{3!} \epsilon^{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6} (\partial_{\Gamma_1} \mathcal{A}_{\Gamma_2}) (\partial_{\Gamma_3} \mathcal{A}_{\Gamma_4}) (\partial_{\Gamma_5} \mathcal{A}_{\Gamma_6}). \quad (44)$$

From this representation we see that the equation of continuity (42) is automatically satisfied. The gauge potential and the field strength introduced here have two components, $\mathcal{A} = A + a$ and

$\mathcal{F} = F + f$, where A and F are the background $U(1)$ gauge potential and field strength defined on the CP_3 manifold, which are explicitly given in Eqs. (12) and (15). a and f are the dynamically fluctuating parts of the gauge potential and the field strength; they describe deviations from the equilibrium density and current. In the equilibrium ground state, $a = f = 0$, and from Eq. (43) we see that the uniform ground state particle density is proportional to the background flux density, with the constant of proportionality being the filling factor ν :

$$\bar{\rho} = \frac{1}{8} \frac{\nu}{3!} \epsilon^{ABCDEG} F_{AB} F_{CD} F_{EG}. \tag{45}$$

It can be shown that F is also the Kähler form of the CP_3 manifold [9],⁴ and $F \wedge F \wedge F$ is nothing but the volume form, which is uniform over the CP_3 manifold. Integrating both sides of Eq. (45) we see that $N \sim R^6$, which agrees with the scaling obtained by ZH [1].

The generalized flux attachment equation (44) can be naturally obtained from the functional variation with respect to the following CS action:

$$\int dt d^6x \left(\frac{1}{4} \frac{\nu}{3!} \epsilon^{\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6} \mathcal{A}_\Gamma (\partial_{\Gamma_1} \mathcal{A}_{\Gamma_2}) (\partial_{\Gamma_3} \mathcal{A}_{\Gamma_4}) (\partial_{\Gamma_5} \mathcal{A}_{\Gamma_6}) - J^\Gamma \mathcal{A}_\Gamma \right). \tag{46}$$

The coefficient of the CS term is given by ν , identified with the filling factor of the quantum Hall fluid.

There is a precise way in which the fluid dynamics can be described by maps of CP_3 onto itself. To establish this connection, let us work in a frame ‘‘comoving’’ with the particles in which the velocity fields are taken to be zero and the density fixed in time, and specialize to the $\mathcal{A}_0 = 0$ gauge. We have previously introduced the Ψ spinor as the coordinate of a single particle on the CP_3 manifold. In order to describe the collection of particles, or a continuous fluid, we can promote Ψ into a CP_3 non-linear sigma model field $\Psi(x, t)$ over the base CP_3 manifold with local coordinate x . The four-component complex spinor field $\Psi_\alpha(x)$ is subject to the following identification

$$\bar{\Psi} \Psi = 1, \quad \{\Psi_\alpha\} \sim e^{i\theta} \{\Psi_\alpha\}. \tag{47}$$

It is easy to see that when the gauge function θ is taken to be a local function on CP_3 , the field

$$\mathcal{A}_\Gamma = 2iI \bar{\Psi} \partial_\Gamma \Psi \tag{48}$$

transforms as a 1-form gauge potential. We identify this field with the gauge field appearing in the action (46). For later convenience, we may write the above expression in terms of differential forms: $\mathcal{A} = 2iI \bar{\Psi} d\Psi$. The gauge invariant field strength is given by

$$\mathcal{F} = 2iI d\bar{\Psi} \wedge d\Psi. \tag{49}$$

Substituting this into the expression for the density ρ , we obtain

$$\rho = \frac{\nu I^3}{3!} (2i d\bar{\Psi} \wedge d\Psi) \wedge (2i d\bar{\Psi} \wedge d\Psi) \wedge (2i d\bar{\Psi} \wedge d\Psi). \tag{50}$$

Hence, $\int \rho$ exactly measures the winding number of the $CP_3 \rightarrow CP_3$ mapping. On the other hand, if we substitute (48) into the expression for the Chern–Simons action and make use of the identifications

⁴ See the Appendix.

in (47), we obtain within the $\mathcal{A}_0 = 0$ gauge

$$S = 2iI \int dt d^6x \rho(x) \bar{\Psi} \partial_t \Psi. \quad (51)$$

This agrees exactly with the many particle fluid action (41). The action is invariant under the gauge transformation appearing in (47).

The gauge transformations of the Ψ 's are naturally thought of as volume preserving, time independent coordinate transformations under which the expression for the density (50) remains invariant. Since CP_3 is a Kähler manifold the volume form is given by $dV = (F \wedge F \wedge F)/3!$ where F is the 2-form Kähler metric. Therefore, coordinate transformations that preserve the Kähler form are also volume preserving. These are equivalent to the $U(1)$ gauge transformations in (47). In this picture, the CP_3 manifold of the incompressible quantum liquid can be viewed as a phase space. Liouville's theorem requires that the volume of phase space is conserved and so volume preserving transformations of the Ψ 's are gauged.

Quasiparticles. Maps of CP_3 onto itself are characterized by the topological invariant (or winding number)

$$Q_F = \frac{1}{3!} \int_{CP_3} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = \frac{8I^3}{3!} \int_{CP_3} (id\bar{\Psi} \wedge d\Psi) \wedge (id\bar{\Psi} \wedge d\Psi) \wedge (id\bar{\Psi} \wedge d\Psi). \quad (52)$$

Q_F scales with $I^3 \propto \Omega_{CP_3}$, where Ω_{CP_3} is the volume of the CP_3 manifold. From Eq. (43) we see that the total charge of the fluid is $Q = \nu Q_F$. Since Q_F is a topological winding number, it can only change by an integer value. A quasi-particle or a quasi-hole is created by $\Delta Q_F = \pm 1$. In this case, Eq. (43) implies that the charge of the quasi-particle or quasi-hole is given by

$$\Delta Q = \pm \nu, \quad (53)$$

which confirms the value of the fractional charged obtained in [1]. As in Laughlin's theory, quasi-particles or quasi-holes have one unit of magnetic flux attached to them. The effective field theory does not predict the allowed value of the fractional filling factor, only the microscopic theory restricts $\nu = 1/m^3$ [1].

DIMENSIONAL REDUCTION AND EXTENDED OBJECTS

The $CP_3 \rightarrow CP_3$ maps describe point-like topological excitations. Condensed matter systems also support elementary excitations which are extended objects. The dimension of the extended objects is determined by the homotopy class of the base and the order parameter spaces [25]. For example, a superfluid with $U(1)$ order parameter in D spatial dimension supports $D - 2$ dimensional extended objects, or $(D - 2)$ brane in modern string theory language. We show here that the QH liquid constructed by ZH also supports 2-brane (membrane) and 4-brane excitations, which are associated with the maps of $CP_3 \rightarrow S^4$ and $CP_3 \rightarrow S^2$. In this section we will describe these objects, closely following the field theoretical treatment of Wu and Zee [18] and that of Wilczek and Zee [26].

Thick solitonic membranes can be constructed from $CP_3 \rightarrow S^4$ mappings. They wrap a spherical 2-cycle of CP^3 and are characterized by the topological current

$$J^{\Gamma\Gamma_1\Gamma_2} = \frac{1}{4!} \epsilon^{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6} \epsilon_{abcde} X^a \partial_{\Gamma_3} X^b \partial_{\Gamma_4} X^c \partial_{\Gamma_5} X^d \partial_{\Gamma_6} X^e, \quad (54)$$

where the $X^a(x)$'s are $O(5)$ sigma model-like fields constructed from the embedding fields $\Psi(x)$ by using the second Hopf map: $X^a(x)/R = \tilde{\Psi}(x)\Gamma^a\Psi(x)$. The membranes can be thought of as non-trivial topological configurations built out of $O(5)$ sigma-model fields that wind around a spherical 4-cycle in CP^3 . The non-vanishing of the homotopy group $\pi_4(S^4) = Z$ ensures the stability of such configurations. The conserved topological charges are given by

$$Q^{\Gamma_1\Gamma_2} = \frac{1}{\Omega_4} \int_{S^4} J^{0\Gamma_1\Gamma_2}. \tag{55}$$

There are 15 independent such charges, all integers, and their combinations can be used to classify the maps.

Similarly, we can build thick solitonic 4-branes in our fluid. These objects wrap spherical 4-cycles of CP^3 . They are constructed as maps $CP_3 \rightarrow S^2$, and they are characterized by the topological current

$$J^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4} = \frac{1}{2} \epsilon^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6} \epsilon_{ijk} n^i \partial_{\Gamma_5} n^j \partial_{\Gamma_6} n^k. \tag{56}$$

Here, the $n^i(x)$'s are $O(3)$ sigma model-like fields given by $n^i(x)/r = \bar{u}(x)\sigma^i u(x)$, with u given in Eqs. (8) and (9). The 4-branes are non-trivial topological configurations build out of $O(3)$ sigma model fields winding a spherical 2-cycle of CP^3 . The stability of the 4-branes lies in the non-vanishing of the homotopy group $\pi_2(S^2) = Z$. The conserved topological charges are given by

$$Q^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4} = \frac{1}{\Omega_2} \int_{S^2} J^{0\Gamma_1\Gamma_2\Gamma_3\Gamma_4}. \tag{57}$$

As in the membrane case there are 15 such charges, all integers, and their combinations can be used to classify the maps.

The membranes and 4-branes described above couple to p ($p = 3$ and $p = 5$) form gauge fields, through the Lagrangians

$$\int d^7x C_{\Gamma\Gamma_1\Gamma_2} J^{\Gamma\Gamma_1\Gamma_2} \tag{58}$$

and

$$\int d^7x C_{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4} J^{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4}. \tag{59}$$

These Lagrangians are analogs of (41); they can describe a finite density of extended objects.

We may consider the membranes and 4-branes in the thin-brane limit and write the currents as

$$J^{\Gamma\Gamma_1\Gamma_2}(y) = \int d\sigma_0 d\sigma_1 d\sigma_2 \delta^7(y - X(\sigma_0, \sigma_1, \sigma_2)) \frac{\partial(X^\Gamma, X^{\Gamma_1}, X^{\Gamma_2})}{\partial(\sigma_0, \sigma_1, \sigma_2)} \tag{60}$$

and

$$J^{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4}(y) = \int d\sigma_0 d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4 \delta^7(y - X(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)) \frac{\partial(X^\Gamma, X^{\Gamma_1}, X^{\Gamma_2}, X^{\Gamma_3}, X^{\Gamma_4})}{\partial(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)}, \tag{61}$$

where (σ_0, σ_i) are the world-volume coordinates of the extended object and $\partial(\dots)/\partial(\dots)$ denotes a Jacobian.

In the limit of thin objects, the ‘‘single p-brane’’ Lagrangians are a generalization of the single particle Lagrangian (10) and are given by [27, 28]

$$\int d\sigma_0 d\sigma_1 d\sigma_2 C_{\Gamma\Gamma_1\Gamma_2}(X) \epsilon^{\sigma_0\sigma_1\sigma_2} \partial_{\sigma_0} X^\Gamma \partial_{\sigma_1} X^{\Gamma_1} \partial_{\sigma_2} X^{\Gamma_2} \quad (62)$$

and

$$\int d\sigma_0 d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4 C_{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4}(X) \epsilon^{\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4} \partial_{\sigma_0} X^\Gamma \partial_{\sigma_1} X^{\Gamma_1} \partial_{\sigma_2} X^{\Gamma_2} \partial_{\sigma_3} X^{\Gamma_3} \partial_{\sigma_4} X^{\Gamma_4}, \quad (63)$$

where (σ_0, σ_i) are the world-volume coordinates of the extended object. The higher form gauge fields felt by the extended objects can be constructed explicitly out of the 1-form background gauge field A [18], explicitly given in (12)

$$C_3 = A \wedge dA, \quad C_5 = \frac{1}{2} A \wedge dA \wedge dA. \quad (64)$$

From Eqs. (62) (63) we can obtain the equations of motion for membranes and 4-branes in the background p form gauge field.

We start with the membrane. The single membrane Lagrangian in Eq. (62) can be cast in the form

$$\begin{aligned} & \int d\sigma_0 d\sigma_1 d\sigma_2 \epsilon^{\sigma_0\sigma_1\sigma_2} \partial_{\sigma_0} X^\Gamma \partial_{\sigma_1} X^{\Gamma_1} \partial_{\sigma_2} X^{\Gamma_2} C_{\Gamma\Gamma_1\Gamma_2}(X) \\ &= \int d\sigma_0 d\sigma_1 d\sigma_2 \epsilon^{\sigma_0\sigma_1\sigma_2} X^\Gamma \partial_{\sigma_1} X^{\Gamma_1} \partial_{\sigma_2} X^{\Gamma_2} [\partial_{\Gamma_3} C_{\Gamma\Gamma_1\Gamma_2}(X)] \partial_{\sigma_0} X^{\Gamma_3} \\ &= -\frac{1}{4} \int d\sigma_0 d\sigma_1 d\sigma_2 \epsilon^{\sigma_0\sigma_1\sigma_2} F_{\Gamma\Gamma_1\Gamma_2\Gamma_3} X^\Gamma \partial_{\sigma_0} X^{\Gamma_1} \partial_{\sigma_1} X^{\Gamma_2} \partial_{\sigma_2} X^{\Gamma_3}, \end{aligned} \quad (65)$$

where we have defined the field strength of the 3-form C , $F = dC$ by

$$F_{\Gamma\Gamma_1\Gamma_2\Gamma_3} = \partial_\Gamma C_{\Gamma_1\Gamma_2\Gamma_3} + \partial_{\Gamma_1} C_{\Gamma_3\Gamma_2\Gamma} + \partial_{\Gamma_2} C_{\Gamma_3\Gamma\Gamma_1} + \partial_{\Gamma_3} C_{\Gamma_1\Gamma\Gamma_2}. \quad (66)$$

We can therefore now write the Lagrangian density in the 2 + 1 dimensional (σ_0, σ_i) world volume as

$$\mathcal{L} = -\frac{1}{4} \epsilon^{\sigma_0\sigma_1\sigma_2} F_{\Gamma\Gamma_1\Gamma_2\Gamma_3} X^\Gamma \partial_{\sigma_0} X^{\Gamma_1} \partial_{\sigma_1} X^{\Gamma_2} \partial_{\sigma_2} X^{\Gamma_3}. \quad (67)$$

Therefore the equation of motion for the membrane in the presence of the 3-form gauge field C now becomes

$$F_{\Gamma\Gamma_1\Gamma_2\Gamma_3} \epsilon^{\sigma_0\sigma_1\sigma_2} \partial_{\sigma_0} X^{\Gamma_1} \partial_{\sigma_1} X^{\Gamma_2} \partial_{\sigma_2} X^{\Gamma_3} = 0. \quad (68)$$

Going through exactly the same steps, one can obtain the equation of motion for the 4-brane Lagrangian Eq. (63). This reads

$$F_{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5} \epsilon^{\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4} \partial_{\sigma_0} X^{\Gamma_1} \partial_{\sigma_1} X^{\Gamma_2} \partial_{\sigma_2} X^{\Gamma_3} \partial_{\sigma_3} X^{\Gamma_4} \partial_{\sigma_4} X^{\Gamma_5} = 0, \quad (69)$$

where $F_{\Gamma\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5}$ is the field strength associated with the 5-form C .

A particularly nice solution of the equation of motion is that of a membrane extending in the X^5 and X^6 field directions and moving along the other directions perpendicular to the magnetic field. The solution has the form

$$X^0 = \sigma_0, \quad X^a = f_a(\sigma_0), \quad a = 1, \dots, 4 \tag{70}$$

while at the same time

$$X^5 = f(\sigma_1, \sigma_2), \quad X^6 = g(\sigma_1, \sigma_2). \tag{71}$$

Extended membrane solutions stand if the Jacobian $\partial(f, g)/\partial(\sigma_1, \sigma_2)$ is not zero. If it vanishes, we have a particle solution on CP_3 . Plugging (70) and (71) into the equation of motion (68), we have

$$F_{\Gamma\Gamma_156}\epsilon^{\sigma_0\sigma_1\sigma_2}\partial_{\sigma_0}X^{\Gamma_1}\partial_{\sigma_1}f\partial_{\sigma_2}g = 0. \tag{72}$$

This simply says that the membrane has to move in the X^a directions perpendicular to the effective magnetic field: $B_{\Gamma\Gamma_1} = F_{\Gamma\Gamma_156}$.

From this solution we can see that the membrane behaves effectively like a particle in the X^a space. The dynamical degrees of freedom of the membrane are described by a $O(3)$ non-linear σ model on the world volume of the membrane. These degrees of freedom can be discretized by ‘‘fuzzifying’’ the S^2 sphere, in which case they appear as iso-spin quantum numbers of the particle.

DIMENSIONAL REDUCTION USING FUZZY SPHERES

So far, we have treated the extended objects as continuous objects. However, it is also possible to treat them as point-like objects with a large number of internal degrees of freedom. This can be done by fuzzifying the S^2 or S^4 in the $CP^3 \approx S^4 \times S^2$ decomposition.

For the reduction on a fuzzy S^2 , the Cartesian coordinates n_i are replaced by $SU(2)$ matrices \mathbf{I}_i , and the continuous integral over S^2 is replaced by the trace over the $SU(2)$ matrices. Under this procedure, the $U(1)$ Chern–Simons theory over CP_3 is dimensionally reduced to a non-abelian $SU(2)$ Chern–Simons theory over S^4 .

We first briefly review the non-abelian theory. The Lagrange density is given by⁵

$$\mathcal{L} = \mu Tr \left(\mathbf{A} \wedge d\mathbf{A} \wedge d\mathbf{A} - \frac{3i}{2} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge d\mathbf{A} - \frac{3}{5} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right). \tag{73}$$

Varying the action with respect to \mathbf{A} , we obtain

$$\delta\mathcal{L}_{CS} = 3\mu Tr(\delta\mathbf{A} \wedge \mathbf{F} \wedge \mathbf{F}) \tag{74}$$

and so, as in the abelian case, the equation of motion can be taken to be $\mathbf{F} \wedge \mathbf{F} = 0$. We may also add a ‘‘background charge’’ density term⁶

$$\delta\mathcal{L} = -Tr(\mathbf{A}_0 dt \wedge \mathbf{J}). \tag{75}$$

⁵ In this section, we use conventions so that the field theory gauge field \mathbf{A} is Hermitian. Its background expectation value is given by $i\mathbf{A}_{back}$, where \mathbf{A}_{back} is given explicitly in Eq. (19). The field strength is given by $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - i[\mathbf{A}_\mu, \mathbf{A}_\nu]$.

⁶ The four-form \mathbf{J} is the Hodge dual of the gauge covariant current \mathbf{J}^μ .

Then, in terms of the four-form \mathbf{J} , the \mathbf{A}_0 equation of motion is replaced by

$$\mathbf{F} \wedge \mathbf{F} = \frac{\mathbf{J}}{3\mu}. \tag{76}$$

We choose the Cartesian coordinates n_i , given by (2) and (9) to parameterize the isospin sphere. In the $CP^3 \approx S^4 \times S^2$ decomposition, the dynamical gauge invariant fields can be taken to be the two sets of sigma-model fields: an $O(5)$ sigma-model field $X_a(x)$ and an $O(3)$ sigma-model field $n_i(x)$. We consider the case where the $O(3)$ sigma-model fields get fixed expectation values

$$\frac{\langle n_i(x) \rangle}{r} = \bar{u} \sigma_i u \tag{77}$$

and consider field configurations $X_a(x)$ that are homogeneous over the internal isospin sphere. Expanding the fluid gauge field introduced in (48), we obtain to the leading order in the fluctuations

$$\frac{\mathcal{A}}{2I} = i\bar{\Psi}d\Psi = i\bar{u}du + \hat{A} = i\bar{u}du + \bar{u}\hat{A}^i\frac{\sigma_i}{2}u = i\bar{u}du + \hat{A}_a^i\frac{n_i}{2r}dx^a. \tag{78}$$

Allowing the fluctuating field $\hat{A}^i\sigma_i/2$ to transform as an $SU(2)$ gauge field, \mathcal{A} is invariant under local $SU(2)$ rotations of the spinor u . Therefore, our theory enjoys a large symmetry consisting of $SU(2)$ gauge transformations. The field strength gets an expectation value

$$\langle \mathcal{F} \rangle = 2iI d\bar{u} \wedge du. \tag{79}$$

The last expression is the magnetic field of a $U(1)$ monopole at the center of the 2-sphere: $F_{ij} = I\epsilon_{ijk}n_k/r^3$.

We now substitute the expansion (78) into the Chern–Simons action (46), and average over the internal isospin sphere. To do this, we first promote the fluctuating gauge field $\hat{A}(n)$ into a non-abelian gauge field as follows. We replace the $O(3)$ sigma-model field with non-commuting isospin operators

$$\frac{In_i}{r} \leftrightarrow \mathbf{I}_i, [\mathbf{I}_i, \mathbf{I}_j] = i\epsilon_{ijk}\mathbf{I}_k. \tag{80}$$

Thus the fields become matrices and the isospin sphere is replaced by a fuzzy two-sphere. Integrals over the internal space can be replaced by

$$\frac{1}{2\pi} \int_{S^2} \langle \mathcal{F} \rangle \dots \leftrightarrow Tr I_d \dots \tag{81}$$

In this way, the dimension of the $SU(2)$ representation is determined by the strength of the monopole charge: $d = 2I + 1$. Derivatives with respect to the isospin coordinates n_i can be replaced by commutators

$$2i\epsilon_{ijk}\partial_j\hat{V}In_k = 2ir^3F_{ij}\partial_j\hat{V} \leftrightarrow [\mathbf{I}_i, \mathbf{V}]. \tag{82}$$

From this identification, we learn that the commutator between any two representation matrices can be obtained from

$$2ir^4\partial_i\hat{V}_1F_{ij}\partial_j\hat{V}_2 \leftrightarrow [\mathbf{V}_1, \mathbf{V}_2]. \tag{83}$$

With these identifications the definition of the covariant derivative and the field strength follows. Finally, we point out a useful identity

$$dn_i \wedge dn_j = 2ir\epsilon_{ijk}n_k d\bar{u} \wedge du. \quad (84)$$

Our matrices satisfy

$$Tr \mathbf{I}_i \mathbf{I}_j = \frac{dC_2}{3}\delta_{ij}, \quad (85)$$

where the numbers $d=2I+1$, $C_2=I(I+1)$ denote the dimension and Casimir of the $SU(2)$ representation. For large I , (85) agrees with the expression of $\int_{S^2}\langle\mathcal{F}\rangle(I^2n_in_j)/2\pi r^2$.

The Lagrange density of the 6+1 dimensional Chern–Simons theory is given by

$$\mathcal{L} = \frac{1}{4} \frac{v}{3!} \mathcal{A} \wedge d\mathcal{A} \wedge d\mathcal{A} \wedge d\mathcal{A} - \mathcal{A}_0 \rho. \quad (86)$$

For simplicity, we may work in the $\mathcal{A}_0=0$ gauge and impose its equation of motion as a constraint

$$\frac{v}{3!} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = \rho. \quad (87)$$

In this gauge, the symmetry of the problem reduces to the group of time independent gauge transformations. Expanding the first term in (86) about $\langle\mathcal{F}\rangle$, we get two types of terms contributing⁷

$$\frac{vI^3}{3}(4(iId\bar{u} \wedge du) \wedge \hat{A} \wedge d\hat{A} \wedge d\hat{A} + I\hat{A} \wedge d\hat{A} \wedge d\hat{A} \wedge d\hat{A}). \quad (88)$$

When written in the $\mathcal{A}_0=0$ gauge, this becomes

$$\frac{vI^3}{3}[4(\hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge d\hat{A} + \hat{A} \wedge d\hat{A} \wedge (\partial_t \hat{A} \wedge dt)) \wedge (iId\bar{u} \wedge du) + 3I\hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge d\hat{A} \wedge d\hat{A}]. \quad (89)$$

In the last term, since $\hat{A} \sim dx$, the operator d has to take derivatives with respect to the 2-dimensional isospin space, in order to satisfy the total antisymmetry. Taking this into account, the last term is given explicitly by

$$I\hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge d_2 \hat{A} \wedge d_2 \hat{A} = \frac{I}{4r^2} \hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge (\hat{A}_\mu^i \hat{A}_\nu^j dx^\mu \wedge dx^\nu) \wedge dn_i \wedge dn_j. \quad (90)$$

Using (84), we may simplify this as

$$\frac{1}{2} \hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge \left(\epsilon_{ijk} \hat{A}_\mu^i \hat{A}_\nu^j \frac{In_k}{r} dx^\mu \wedge dx^\nu \right) \wedge (i d\bar{u} \wedge du). \quad (91)$$

Now, we may replace the term in second parentheses with a commutator using (83). We end up with

$$-i\hat{A} \wedge (\partial_t \hat{A} \wedge dt) \wedge (\mathbf{A} \wedge \mathbf{A}) \wedge (i d\bar{u} \wedge du). \quad (92)$$

⁷ We carried an integration by parts.

Averaging over the isospin sphere and making the replacements mentioned in the previous paragraph gives

$$\langle \mathcal{L} \rangle_{S^2} = \frac{\pi\nu}{6} Tr \left(\mathbf{A} \wedge (\partial_t \mathbf{A} \wedge dt) \wedge d\mathbf{A} + \mathbf{A} \wedge d\mathbf{A} \wedge (\partial_t \mathbf{A} \wedge dt) - \frac{3i}{2} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge (\partial_t \mathbf{A} \wedge dt) \right). \quad (93)$$

This is the Lagrange density obtained from the non-abelian Chern–Simons Lagrangian (73) in the temporal gauge $\mathbf{A}_0 = 0$, with $\mu = \pi\nu/6$. In the temporal gauge, there is a left over symmetry consisting of time independent, infinitesimal $SU(2)$ gauge transformations. As we already mentioned, this symmetry is a symmetry of the full theory. Because of this symmetry, there exist three local conserved quantities, one for each independent $SU(2)$ rotation. It is easily verified, using, for example, the equations of motion by varying (93), that $\mathbf{F} \wedge \mathbf{F}$ is time independent. Therefore, we may consistently set

$$\frac{\pi\nu\mathbf{F}}{2} \wedge \mathbf{F} = \mathbf{J} \quad (94)$$

with \mathbf{J} time independent and treat this as a constraint equation. We normalized the equation to be exactly the same as the \mathbf{A}_0 equation of motion (76). Thus we may introduce the components of \mathbf{A}_0 into the action as a Lagrange multiplier and obtain the full Chern–Simons action (73) together with a background charge density term.

There is a constraint on $Tr\mathbf{J}$ from (87). Expanding the density ρ about $\langle \mathcal{F} \rangle$ gives

$$\rho = 4\nu I^3 [(i d\bar{u} \wedge du) \wedge d\hat{A} \wedge d\hat{A} + d_4\hat{A} \wedge d_2\hat{A} \wedge d_2\hat{A}]. \quad (95)$$

The last term can be worked out as before to give

$$\rho = \nu(I^2 d\hat{A} \wedge d\hat{A} - iI(\mathbf{A} \wedge \mathbf{A}) \wedge d\hat{A}) \wedge (iI d\bar{u} \wedge du). \quad (96)$$

Averaging over the two-sphere, we get

$$\langle \rho \rangle_{S^2} = \pi\nu Tr\mathbf{F} \wedge \mathbf{F} = 2Tr\mathbf{J}. \quad (97)$$

This last expression shows that the particles are attached to the “instanton density,” which is topologically conserved.⁸

The vortex free fluid is characterized by the unit instanton number density

$$\int_{S^4} Tr\mathbf{F} \wedge \mathbf{F} = \frac{16\pi^2(2I+1)I(I+1)}{3}. \quad (98)$$

Vortices in the fluid can change the instanton number by an integer. These vortices are localized on the 4-sphere. From the 6-dimensional point of view they are membranes wrapped on the internal isospin sphere. We can also think of them as point-like particles with many internal degrees of freedom.

Therefore, this section fully establishes the exact equivalence between the CS theory in 6 + 1 dimensions, defined over the CP_3 spatial manifold, and the CS theory in 4 + 1 dimensions, defined

⁸ This can also be written locally as $Tr d(\mathbf{A} \wedge d\mathbf{A} - 2i/3\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A})$ but not always globally. Topologically non-trivial gauge configurations on S^4 are classified by the homotopy group $\pi_4(SU(2)) = Z_2$. The integral of (97) over the four-sphere is generally non-vanishing. This is because the integral of the Chern–Simons 3-form over an equator three-sphere may change under gauge transformations. This change is proportional to some integer n . The instantons are classified by the homotopy group $\pi_3(SU(2)) = Z$.

over the S^4 spatial manifold. In a previous section, we also showed that the single particle equations of motion derived from both formulations agree exactly with each other.

We conclude this section with some speculations. In the thermodynamic limit (large I limit), the $SU(2)$ representation matrices are large. This limit is similar to the large N limit of QCD [29] in which the leading contributions arise from planar Feynman diagrams. The filling factor ν behaves as a 't Hooft coupling constant. In this limit, the theory is strongly coupled and should exhibit confinement, with strings or magnetic flux tubes joining instanton-like vortices. Perhaps our theory is dual to a topological string theory. Such a connection between supersymmetric large N CS theory in $2+1$ and topological string theory was studied recently in [30, 31]. The confining phase may be thought of as a superconducting phase of the liquid in the sense similar to the CSLG theory [12]. If this is the case, the QCD-like strings can be thought of as the magnetic partners of the topological membranes we found in the previous section. In $6+1$ dimensions the 3-form coupling electrically to membranes is dual to a 2-form that naturally couples to strings. We would like to speculate that the worldvolume dynamics of these objects may lead to a "spin-gap" in the spectrum of boundary excitations [6]. The finite tension for the strings could arise from higher order corrections at the magnetic length scale I/R^2 .

FRACTIONAL STATISTICS OF EXTENDED OBJECTS

In this section, we describe the statistics of the membranes constructed in a previous section. First, however, let us recall the basic setup in the usual $2+1$ dimensional case.

In $2+1$ dimensions, particles interacting with velocity dependent forces can become anyons with fractional statistics. In $2+1$ dimensions we can obtain a topological current out of an $O(3)$ sigma-model field n_a by virtue of

$$J^\mu = \epsilon^{\mu\nu\lambda} \epsilon_{abc} n^a \partial_\nu n^b \partial_\lambda n^c. \quad (99)$$

We exhibit the first Hopf ($S^3 \rightarrow S^2$) map by $n^a/r = \bar{u}\sigma^a u$ where u is given by (1). The conservation of J^μ allows us to obtain a gauge potential by the equation

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \quad (100)$$

The field A is defined up to the usual abelian gauge freedom. Then the homotopy invariant associated to the $S^3 \rightarrow S^2$ map is called the first Hopf invariant and is given by [26]

$$H_1 = - \int d^3x A_\mu J^\mu. \quad (101)$$

The proof that H_1 is a homotopic invariant of the first Hopf map is given in [26]. Therefore, the action for the solitonic object accepts added, in general, a topological term $S = \theta H_1$ where we take $\theta = 2\pi\nu$, with ν being the filling factor. The factor of θ reflects the fractional charge of the soliton. Rotating a soliton adiabatically (through 2π) over the time T , the wave-function acquires a phase factor $\exp(iS_{total})$ where S_{total} is the action corresponding to the adiabatic rotation [26]. Typically, all other terms but the Hopf invariant will be of order $1/T \rightarrow 0$ as $T \rightarrow \infty$. The phase is given by

$$\exp(i\Delta\phi) = \exp(iS) = \exp(2i\pi\nu H_1) = \exp(2i\pi\nu) \quad (102)$$

with the Lagrangian and current given in (101) and (100).

There is a deep theorem which equates the Hopf invariant to the linking number between two curves in R^3 . The fractional nature of the soliton can be visualized by using the linking number theorem [32]. In $2 + 1$ dimensions, the world-lines described by the particles can link and the particles can have fractional spin and statistics. The relevant mathematics which allows this is the homotopy $\pi_3(S^2) = Z$.

Parallel to the above discussion, in the $6 + 1$ dimensional case, $J_{\Gamma_1\Gamma_2}$, the conserved current defined in the previous section, manufactures a 3-form field $C_{\Gamma_1\Gamma_2}$ through the curl equation

$$J^{\Gamma_1\Gamma_2} = \epsilon^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6} \partial_{\Gamma_3} C_{\Gamma_4\Gamma_5\Gamma_6}. \tag{103}$$

We exhibit the second Hopf map $S^7 \rightarrow S^4$ by $X^a/R = \bar{\Psi} \Gamma^a \Psi$ where the Γ^a satisfy the Clifford algebra. Then the homotopy invariant associated to the second Hopf map is called the second Hopf invariant and is given by

$$H_2 = - \int d^7x C_{\Gamma_1\Gamma_2} J^{\Gamma_1\Gamma_2}. \tag{104}$$

This is associated with the homotopy group $\pi_7(S^4) = Z \oplus Z_{12}$, where Z_{12} is the torsion part of the group. The action for the solitonic membranes accepts, in general, a term $S = \theta H_2$ where $\theta = 2\pi \nu$, with ν being the filling factor. A generalization of the theorem which equates the Hopf invariant to the linking number between two curves in R^3 guarantees the equivalence of H_2 with the linking integral between two thin-membrane world-surfaces. The phase acquired by taking one such (solitonic) membrane around the other is

$$\exp(i \Delta\phi) = \exp(iS) = \exp(i2\pi \nu H_2) = \exp(i2\pi \nu) \tag{105}$$

with the Lagrangian and current as above. Our membranes therefore acquire fractional statistics in $6 + 1$ dimensions by way of the Hopf invariant of the second Hopf map. We also mention that H_1 and H_2 with the 1-form and 3-form gauge fields manufactured out of the conserved currents via (100) and (103) are precisely the terms in the action which one gets by having a particle/membrane with fractional charge interaction with a Chern–Simons gauge field.

Let us now give a simple example of how this interaction materializes in our theory. Consider a thin static membrane oriented along the 5–6 plane, located at the origin in the other 4 directions (see Fig. 1). In this case, $J^{056} = \delta^4(y)$ and

$$F_{1234} = -\delta^4(y). \tag{106}$$

Now consider a second thin membrane, perpendicular to the one we already have, oriented along

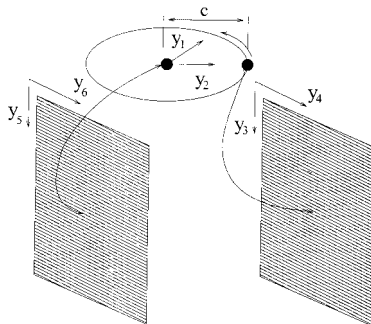


FIG. 1. The thin membrane on the 3–4 directions, originally at a distance c along the 2-axis from the thin static membrane on the 5–6 directions is rotated around the latter along a circle in the 1–2 plane. After rotation, it picks up a fractional phase.

the 3–4 plane at $y^1 = 0$ and $y^2 = c$, where c is an arbitrary distance from the origin along the 2-axis. Move this membrane around the first one along a circle in the 1–2 plane (see Fig. 1). As it orbits once, it describes a cylinder, $R_{3-4}^2 \times S^1$. This is the boundary of a four-dimensional surface, $R_{3-4}^2 \times D_2$ where D_2 is a disk in the 1–2 plane. This membrane interacts with the field in Eq. (106) produced by the first. This interaction causes it to pick up a phase. Write down the Hopf invariant

$$\begin{aligned} \exp(i\delta\phi) &= \exp\left(-2i\pi\nu \int d^7x J \cdot C\right) = \exp\left(-2i\pi\nu \int_{R_{3-4}^2 \times S^1} C\right) \\ &= \exp\left(-2i\pi\nu \int_{R_{3-4}^2 \times D_2} F\right) = \exp(2\pi i\nu), \end{aligned} \quad (107)$$

where C is the 3-form gauge field defined in Eq. (103). This phase corresponds to an anyonic exchange phase of

$$\delta\phi = \pi\nu. \quad (108)$$

CONCLUSIONS

From this work we see that the precise connection between the microscopic wave function and the CSLG topological field theory description of the 2D quantum Hall effect can be directly generalized to the new quantum liquid constructed by ZH. The abelian $U(1)$ CS theory in $6+1$ dimensions and the $SU(2)$ non-abelian CS theory in $4+1$ dimensions can be constructed directly by a proper generalization of the flux attachment transformation. This effective field theory model can be used to investigate many long wave length properties. In particular, it would be interesting to study the new quantum Hall liquid on different topological backgrounds. It would also be interesting to apply this topological field theory to study the boundary excitations. The new quantum liquid also supports topologically stable extended objects, including the membrane and the 4-brane, whose dynamics is completely determined by the topological action. The 2 branes can intersect each other in a non-trivial way to give rise to the fractional statistics in higher dimensions. The precise microscopic connection should help us to develop a fully regularized quantum theory of these extended objects. The world volume dynamics of these extended objects, especially the membrane, could lead to gaps to higher helicity states. In this case, low energy isospin degrees would be finite, and a truly four dimensional theory could be obtained.

APPENDIX

Notations and Conventions

$A, B = 1, 2, 3, 4, 5, 6$ denote the local CP_3 coordinates.

$\Gamma, \Sigma = 0, 1, 2, 3, 4, 5, 6$ denote space-time coordinates on CP_3 .

$a = 1, 2, 3, 4, 5$ denote the S^4 coordinates.

$i = 1, 2, 3$ denote the S^2 coordinates.

$\mathbf{A}_a, \mathbf{F}_{ab}$, and \mathbf{I}_i denote $SU(2)$ matrix valued quantities.

\mathcal{A}_B and \mathcal{F}_{AB} denote the total gauge potential and field strength.

A_B and F_{AB} denote the background gauge potential and field strength.

a_B and f_{AB} denote the fluctuating part of the gauge potential and field strength.

Local Metric on $CP_3 \sim S^4 \times S^2$

We can parameterize the 2-sphere and the 4-sphere as follows. For S^2 , we define

$$\begin{aligned} n_1 &= r \sin \alpha \cos \beta \\ n_2 &= r \sin \alpha \sin \beta \\ n_3 &= r \cos \alpha. \end{aligned}$$

For S^4 , we define

$$\begin{aligned} X_1 &= R \sin \theta_1 \sin \frac{\theta_2}{2} \sin(\phi_1 - \phi_2) \\ X_2 &= -R \sin \theta_1 \sin \frac{\theta_2}{2} \cos(\phi_1 - \phi_2) \\ X_3 &= -R \sin \theta_1 \cos \frac{\theta_2}{2} \sin(\phi_1 + \phi_2) \\ X_4 &= R \sin \theta_1 \cos \frac{\theta_2}{2} \cos(\phi_1 + \phi_2) \\ X_5 &= R \cos \theta_1. \end{aligned}$$

r and R are kept constant. They are the spherical coordinates for the 2-sphere and 4-sphere, respectively. The metric on these spheres is given by

$$ds^2 = r^2 d\omega^2, \quad d\omega^2 = d\alpha^2 + \sin^2 \alpha d\beta^2 \tag{109}$$

$$dS^2 = R^2 d\Omega^2, \quad d\Omega^2 = d\theta_1^2 + \frac{\sin^2 \theta_1}{4} d\theta_2^2 + \sin^2 \theta_1 d\phi_1^2 + \sin^2 \theta_1 d\phi_2^2. \tag{110}$$

Kähler Form on CP_n

We will follow closely the treatment of Greene [33]. CP_n is defined by introducing $n + 1$ complex coordinates z_1, \dots, z_{n+1} , not all of them simultaneously zero, with an equivalence relation identifying z_1, \dots, z_{n+1} with $\lambda z_1, \dots, \lambda z_{n+1}$ for any complex number λ other than zero. With a suitable choice of $|\lambda|$ we can always choose the representatives of such an equivalence class to satisfy

$$\sum_i \bar{z}_i z_i = 1. \tag{111}$$

This condition fixes only part of the equivalence relation defining CP_n . We still have to impose identifications associated with the phase of λ

$$(z_1, \dots, z_{n+1}) \sim e^{i\theta} (z_1, \dots, z_{n+1}). \tag{112}$$

We introduce local coordinates as follows. Define the j th patch X_j by the condition that $z_j \neq 0$ and set $z_j = s_j e^{i\alpha_j}$ with s_j real. Then define

$$(s_j, u_{(j)}^1, \dots, u_{(j)}^n) = (s_j, e^{-i\alpha_j} z_1, \dots, e^{-i\alpha_j} z_{j-1}, e^{-i\alpha_j} z_{j+1}, \dots, e^{-i\alpha_j} z_{n+1}) \tag{113}$$

to be local coordinates on the patch. Using (111) one can solve for s_j in terms of the $u_{(j)}$'s and work with n independent complex coordinates. We define the Kähler scalar potential to be $K_j = 2 \sum_1^n |u_{(j)}^i|^2$. Then the 2-form

$$J = \partial\bar{\partial}K_j = dA_j, \quad A_j = 2s_j ds_j + 2 \sum_{i=1}^n \bar{u}_{(j)}^i du_{(j)}^i \quad (114)$$

is a *globally* defined closed 2-form class on CP_n and equivalently it defines a Kähler metric. To see that J is globally defined, all we must check is that in two overlapping patches X_{j_1}, X_{j_2} the vector potentials A_{j_1} and A_{j_2} are related by a gauge transformation. But by construction $A_j = 2 \sum_1^{n+1} \bar{z}^i dz^i - 2d\alpha_j$ and so $A_{j_1} = A_{j_2} + 2d(\alpha_{j_2} - \alpha_{j_1})$.

We now consider the two examples of interest to us, namely, the sphere CP_1 and CP_3 and establish a *local* equivalence of the background field strength F with J . For simplicity we demonstrate the case of CP_1 explicitly. For CP_1 , let us work in a patch such that $z_1 \neq 0$ and use Eq. (1). We take u_1 to be real as in (113) and parameterize the u 's using cartesian coordinates n_i as in (9). The south pole $n_3 = -r$ is excluded from the patch. Then, *locally*

$$J = 2d\bar{u} \wedge du = -\frac{i}{2r^3} \epsilon_{ijk} n_k dn_i \wedge dn_j = -iF, \quad (115)$$

where F is the field strength of a $U(1)$ magnetic monopole at the center. Similarly for CP_3 , we can work in the patch $z_1 \neq 0$ and set (Ψ_1, \dots, Ψ_4) to be given by (8) in terms of the X_a, n_i . The 2-sphere $X_5 = -R$ together with the 4-sphere $n_3 = -r$ is excluded from the patch. Then *locally*

$$J = 2d\bar{\Psi} \wedge d\Psi = -iF \quad (116)$$

with F given explicitly by (15). This last equation is equivalent to the Berry's phase computation appearing in (10).

Useful Identities of A and F

In this section, we summarize some careful results for deriving the single particle equation of motion⁹ (18):

$$A_a A_a = A_\mu A_\mu = -\frac{R - X_5}{4R^2(R + X_5)} \quad (117)$$

$$F_{5a} F_{a5} = \frac{R^2 - X_5^2}{4R^6} \quad (118)$$

$$F_{5\mu} F_{\mu\nu} = \frac{X_\nu X_5}{4R^6} \quad (119)$$

$$F_{\mu i} F_{i k} = \frac{-1}{4r^4 R(R + X_5)} \epsilon_{ikl} \eta_{\mu\nu}^i n_l X_\nu \quad (120)$$

$$F_{\mu\tau} F_{\tau\nu} = \frac{\delta_{\mu\nu}}{R^2(R + X_5)^2} + \frac{(2R + X_5)(2R + 3X_5)}{R^2(R + X_5)^2} A_\mu A_\nu - \frac{(2R + X_5)^2}{4R^6(R + X_5)^2} X_\mu X_\nu \quad (121)$$

$$F_{ij} F_{jk} = \frac{1}{4r^4} \left(\delta_{ik} - \frac{n_i n_k}{r^2} \right) \quad (122)$$

⁹ We also set $I = \frac{1}{2}$ here.

$$\begin{aligned} & \sum_{a \neq b} F_{ab} F_{ab} + 2 \sum_{a,i} F_{ai} F_{ai} + \sum_{i \neq j} F_{ij} F_{ij} \\ &= -\frac{3R^4 + R^2 X_5^2 + 4R X_5^3 - 2X_5 R^3 + 2X_5^4}{2R^6(R + X_5)^2} - \frac{R - X_5}{r^2 R^2(R + X_5)} - \frac{1}{2r^4}. \end{aligned} \quad (123)$$

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