

Eight-Dimensional Quantum Hall Effect and “Octonions”

Bogdan A. Bernevig,¹ Jiangping Hu,² Nicolaos Toumbas,³ and Shou-Cheng Zhang¹

¹*Department of Physics, Stanford University, Stanford, California 94305, USA*

²*Department of Astronomy and Physics, UCLA, Los Angeles, California 90095, USA*

³*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 10 June 2003; published 5 December 2003)

We construct a generalization of the quantum Hall effect where particles move in an eight-dimensional space under an $SO(8)$ gauge field. The underlying mathematics of this particle liquid is that of the last normed division algebra, the octonions. Two fundamentally different liquids with distinct configuration spaces can be constructed, depending on whether the particles carry spinor or vector $SO(8)$ quantum numbers. One of the liquids lives on a 20-dimensional manifold with an internal component of $SO(7)$ holonomy, whereas the second liquid lives on a 14-dimensional manifold with an internal component of G_2 holonomy.

DOI: 10.1103/PhysRevLett.91.236803

PACS numbers: 73.43.Cd, 03.70.+k, 11.15.-q, 11.25.-w

The two fundamental mathematical structures (division algebras) a physicist uses in his everyday life are the real \mathbb{R} and the complex \mathbb{C} numbers. As we well know, complex numbers can be treated as pairs of real numbers with a specific multiplication law. One can, however, go even further and build two other sets of numbers, known in mathematics as quaternions \mathbb{Q} and octonions \mathbb{O} . The quaternions, formed as pairs of complex numbers, are noncommutative, whereas the octonions, formed as pairs of quaternionic numbers, are both noncommutative and nonassociative. The four sets of numbers are mathematically known as division algebras. The octonions are the last division algebra, no further generalization being consistent with the laws of mathematics. Strikingly, in physics, some of the division algebras are realized as fundamental structures of the quantum Hall effect (QHE). The complex \mathbb{C} division algebra is realized as the fundamental structure of the two-dimensional QHE [1]. In a generalization of the two-dimensional QHE, two of us constructed a four-dimensional QH liquid whose underlying structure is the quaternionic division algebra \mathbb{Q} [2].

In this Letter, we present the construction of an eight-dimensional quantum Hall liquid whose fundamental structure is the division algebra of octonions. Although this system shares many of the properties of the previous two-dimensional and four-dimensional QHE, its structure is much richer. In particular, our fluid is composed of particles interacting with an $SO(8)$ background gauge field. Depending on whether our particles are in the spinor or vector representation of the $SO(8)$ gauge group, our liquid lives on a configuration space manifold which is either 20 dimensional or 14 dimensional. The total con-

figuration space of the liquid is composed of the initial eight-dimensional base space on which the particles live plus the configuration space of the particle's spin. As such, the internal configuration space also deposits energy density in the same way as S^8 , but the particle's equations of motion involve the real space coordinates on S^8 . In the four-dimensional case, the base space was a four sphere S^4 while the spin space was a two sphere S^2 thereby making the total configuration space six dimensional. The system reported here is likely to be the last QHE with a rich structure, and points to connections with the world of string theory. In the present manuscript, we report only the important results of our analysis, with the full detailed presentation reserved for a future, longer publication.

The existence of only four division algebras— $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$ —is related to the existence of only four (Hopf) fibrations of spheres over sphere spaces: $S^1 \xrightarrow{Z_2} S^1$ for \mathbb{R} , $S^3 \xrightarrow{S^1} S^2$ for \mathbb{C} , $S^7 \xrightarrow{S^3} S^4$ for \mathbb{Q} , and $S^{15} \xrightarrow{S^7} S^8$ for \mathbb{O} [3]. The algebraic structure of the eight-dimensional QHE is the fractionalization of the vector coordinate into two spinor coordinates fundamental in the Hopf map $S^{15} \xrightarrow{S^7} S^8$:

$$x_a = \Psi_\alpha^T \Gamma_{\alpha\alpha'}^a \Psi_{\alpha'}, \quad \Psi_\alpha^T \Psi_\alpha = 1, \quad a = 1, \dots, 9. \quad (1)$$

Here Ψ_α is a 16 component real spinor of $SO(9)$, Γ^a is the nine-dimensional real Clifford algebra of 16×16 matrices $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$, and x_a is a nine component real vector. Because of the normalization condition, Ψ parametrizes an S^{15} . $\Psi_\alpha^T \Psi_\alpha = 1$ also gives $x_a x^a = 1$. Therefore $X_a = R x_a$ parametrizes a point on the S^8 sphere with radius R . An explicit solution of the Hopf map is

$$\Gamma^i = \begin{pmatrix} 0 & \lambda^i \\ -\lambda^i & 0 \end{pmatrix}, \quad i = 1, \dots, 7, \quad \Gamma^8 = \begin{pmatrix} 0 & 1_{8 \times 8} \\ 1_{8 \times 8} & 0 \end{pmatrix},$$

$$\Gamma^9 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix}, \quad \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_8 \end{pmatrix} = \sqrt{\frac{R + X_9}{2R}} \begin{pmatrix} u_1 \\ \vdots \\ u_8 \end{pmatrix}, \quad \begin{pmatrix} \Psi_9 \\ \vdots \\ \Psi_{16} \end{pmatrix} = \frac{1}{\sqrt{2R(R + X_9)}} (X_8 - X_i \lambda^i) \begin{pmatrix} u_1 \\ \vdots \\ u_8 \end{pmatrix}, \quad (2)$$

where λ^i are seven 8×8 real antisymmetric matrices satisfying $\{\lambda^i, \lambda^j\} = -2\delta_{ij}$ which are constructed from the set of structure constants of the algebra of octonions [4]. The fiber S^7 of the Hopf map is parametrized by the arbitrary eight component real spinor of $SO(8)$ (u_1, \dots, u_8) with the normalization condition $u_\alpha u^\alpha = 1$. Any $SO(8)$ rotation on u_α preserves the normalization condition and maps to the same point on the S^8 . Since the manifold S^{15} is described by the real $SO(9)$ spinor Ψ_α , the $U(1)$ geometric connection $\Psi^T d\Psi$ [$U(1)$ Berry phase] is zero; however, the $SO(8)$ connection is nonvanishing $V^T dV = A_a dX_a$, where $V^T = \{\sqrt{(R+X_9)/(2R)}I_{8 \times 8}, [1/\sqrt{2R(R+X_9)}](X_8 - X_i \lambda^i)^T\}$. A_a is the $SO(8)$ gauge field arising from the connection over S^8 and takes the following form:

$$A_\mu = -\frac{1}{2R(R+X_9)} \Sigma^{\mu\nu} X_\nu, \quad A_9 = 0, \quad (3)$$

where $\mu, \nu = 1, \dots, 8$ and where $\Sigma^{\mu\nu}$ are the 28 generators of the $SO(8)$ Lie algebra: $\Sigma^{\mu\nu} = (\Sigma^{ij}, \Sigma^{i8})$, $\Sigma^{ij} = -\frac{1}{2}[\lambda^i, \lambda^j]$, $\Sigma^{i8} = \lambda^i$. Under a conformal transformation from S^8 to Euclidean space R^8 , this gauge potential becomes the sourceless, topologically nontrivial monopole solution of $SO(8)$ Yang-Mills theory of Grossman *et al.* [5]. The matrices $\Sigma^{\mu\nu}$ are the $SO(8)$ spin matrices and the gauge potential in Eq. (3) can be generalized to any arbitrary representation of the $SO(8)$ Lie Algebra. The field strength is $F_{ab} = [D_a, D_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b]$, where $D_a = \partial_a + A_a$ is the covariant derivative on the S^8 manifold. Componentwise, the field strength is $F_{\mu\nu} = (1/R^2)(X_\nu A_\mu - X_\mu A_\nu + \Sigma^{\mu\nu})$ and $F_{9\mu} = -[(R+X_9)/R^2]A_\mu$. Regarding the field strength $F_{\mu\nu}$ as a curvature on S^8 , the Euler number $\int_{S^8} F \wedge F \wedge F$ is 1, attesting to the topological nontriviality of our configuration. The field strength satisfies the generalized self-duality condition $F_{[\mu\nu} F_{\rho\lambda]} = \epsilon_{\mu\nu\rho\lambda\theta\chi\beta\gamma} F_{\theta\chi} F_{\beta\gamma}$, where ϵ_{\dots} is the antisymmetric eight-dimensional epsilon symbol.

We now want to introduce a nonrelativistic Hamiltonian for particles moving on the S^8 in the presence of the $SO(8)$ gauge field above. The symmetry group of the S^8 space is $SO(9)$, which is generated by the angular momentum operator $L_{ab}^{(0)} = -i(X_a \partial_b - X_b \partial_a)$. Replacing the usual derivatives with the covariant derivatives which arise due to the presence of the gauge field to get the operators $\Lambda_{ab} = -i(X_a D_b - X_b D_a)$, the single particle Hamiltonian of particles of mass M yields

$$\mathcal{H} = \frac{\hbar^2}{2MR^2} \sum_{a < b} \Lambda_{ab}^2. \quad (4)$$

Similar to the case of the two- and four-dimensional QHE, the Λ_{ab} do not satisfy the $SO(9)$ commutation relations because they are missing the momentum of the monopole gauge field. When this is added, the newly formed operators $L_{ab} = \Lambda_{ab} - iR^2 F_{ab}$ satisfy the $SO(9)$ commutation relations and commute with the

Hamiltonian (4):

$$\begin{aligned} L_{ab} &= (L_{\mu\nu}, L_{\mu 9}), & \mu, \nu &= 1, \dots, 8, \\ L_{\mu\nu} &= L_{\mu\nu}^{(0)} + i\Sigma_{\mu\nu}, & L_{\mu 9} &= L_{\mu 9}^{(0)} - iRA_\mu, \\ [L_{ab}, L_{cd}] &= i[\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc}]. \end{aligned} \quad (5)$$

One can show that $\sum_{a < b} \Lambda_{ab}^2 = \sum_{a < b} L_{ab}^2 - \sum_{\mu < \nu} \Sigma_{\mu\nu}^2$.

The particles carry $SO(8)$ quantum numbers and the generators $\Sigma_{\mu\nu}$ combine nontrivially with the orbital angular momentum $L_{\mu\nu}^{(0)}$ on the equator of S^8 to select only some distinct special representations of the $SO(9)$ L_{ab} which label the particle. Both $SO(9)$ and $SO(8)$ irreducible representations are labeled by four integers (highest weights) $(n_1, n_2, n_3, n_4)_{SO(9)}$ and $(n_1, n_2, n_3, n_4)_{SO(8)}$. The quantum mechanics problem is defined by the choice of the $(n_1, n_2, n_3, n_4)_{SO(8)}$ representation. Once the monopole representation is chosen, the composition of the monopole momentum and the particle orbital momentum on the equator will specify the $SO(9)$ representation of our particles.

Similarly to the four-dimensional QHE, a natural choice is the monopole in the spinor representation of $SO(8)$ $(0, 0, 0, I)_{SO(8)}$, where I is an integer. The eigenstates of the Hamiltonian are specified by the irreducible representations of $SO(9)$, $(n, 0, 0, I)_{SO(9)}$, where n is a non-negative integer specifying the Landau level. The energy eigenvalues are $E(n, I) = [\hbar^2/(2MR^2)](n^2 + nI + 7n + 2I)$ and the degeneracy of each state is equal to the dimension of the $SO(9)$ representation $d(n, I) = \frac{2}{10!} \times (1+n)(2+n)(3+n)(4+n+I)(1+I)(2+I)(3+I)^2 \times (4+I)(5+n+I)(5+I)(6+n+I)(7+2n+I)$. The ground state is the lowest $SO(9)$ level for a given I and is obtained by setting $n=0$, with dimension $d(0, I)$. Therefore the dimension of the $SO(8)$ representation plays the role of the magnetic flux, whereas n plays the role of the Landau level.

The lowest Landau level (III) wave functions are therefore the $SO(9)$ spinors $(0, 0, 0, I)_{SO(9)}$. We can obtain these wave functions from the Hopf spinor Ψ_α by observing it is an eigenstate of the total angular momentum L_{ab} : $L_{ab}\Psi = -\frac{1}{2}\Sigma_{ab}\Psi$. The wave functions can be expanded in the space of the symmetric products of the fundamental spinor, namely, $\text{Sym} \otimes^I (0, 0, 0, 1)_{SO(9)}$. The symmetric product is reducible into $SO(9)$ representations, with the first representation that it reduces into being $(0, 0, 0, I)_{SO(9)}$, which is the only irrep we wish to keep. Therefore the wave function for the system will be the symmetrized product along with two conditions that mod out the extra representations:

$$\Phi = \sum_{\alpha_1, \dots, \alpha_I} f_{\alpha_1, \dots, \alpha_I} \Psi_{\alpha_1} \cdots \Psi_{\alpha_I} \quad (6)$$

where each α can take values from $0, \dots, 16$ and where the coefficients $f_{\alpha_1, \dots, \alpha_I}$ are symmetric in α and subject to the following constraints:

$$\Gamma_{\alpha\beta}^a f_{\alpha\beta\gamma\dots} = 0, \quad f_{\alpha\alpha\beta\dots} = 0. \quad (7)$$

In the thermodynamic limit, both $R, I \rightarrow \infty$. To obtain a finite gap, from the dispersion relation we see that the thermodynamic limit is taken such that $l_0^2 = R^2/I$ is kept constant. l_0 can be considered the magnetic length. The correlation functions can be computed and are Gaussian localized. Our liquid is therefore incompressible.

The degeneracy of each Landau level in the thermodynamic limit $I \rightarrow \infty$ varies as $I^{10} \sim R^{20}$. For each Landau level filled, each degeneracy level can carry a particle, therefore the number of particles in our system varies as R^{20} . Naively, the particle configuration space V is $S^8 \sim R^8$. However, one must keep in mind that the particles carry spin $\Sigma_{\mu\nu}$ in the spinor representation of $SO(8)$. The dimension of the configuration space of the spin is naively equal to the total number of $SO(8)$ generators and varies as R^{28} . However, as said, the spin is an $SO(8)$ spinor and therefore not the full $SO(8)$ manifold will be available as the configuration space. In particular, the dimension of the configuration space available to any representation of a group is the dimension of the group minus the dimension of the stabilizer group of that particular representation. The stabilizer of an $SO(8)$ spinor is $U(4)$; therefore the full configuration space for our particles will be the number of generators of the coset $SO(8)/U(4)$, and will vary as $\sim R^{12}$. The internal configuration space carries the same energy density as the real space S^8 on which the particles live. Hence, the total configuration space is $S^8 \otimes SO(8)/U(4) \approx SO(9)/U(4)$ and the volume scales as $V \sim R^8 \times R^{12} = R^{20}$. Our fluid has finite density $\rho = N/V$ in the thermodynamic limit.

We now focus on the Newtonian equations of motion derived from the Hamiltonian $\mathcal{H} + V(X_a)$, where \mathcal{H} is given by Eq. (4). We can take the infinite mass limit $M \rightarrow \infty$ to project the system to the lowest Landau level. In this limit, the equations of motion are given by

$$\dot{X}_a = \frac{R^4}{I^2} \frac{\partial V}{\partial X_b} F_{ab}. \quad (8)$$

Just as in the III problem, the momentum variables can be fully eliminated, however, at the price of introducing noncommuting coordinates. The projected Hamiltonian in the III is then simply $V(X_a)$ and the commutation relation is

$$[X_a, X_b] = \frac{R^4}{I^2} F_{ab}. \quad (9)$$

We have therefore build a 20-dimensional quantum Hall fluid whose structure in the III is noncommutative and whose particles are spinors of $SO(9)$ and interact with an $SO(8)$ gauge field. The configuration space is $S^8 \otimes SO(8)/U(4) \approx SO(9)/U(4)$. We call this the spinor QH liquid. Equation (9) is very similar to noncommutative relations appearing in the construction of fuzzy spheres [6].

We now obtain a second liquid whose underlying mathematical structure is the octonions. Let us consider the Hamiltonian [3] with the monopole gauge group A_μ in the $(I, 0, 0, 0)_{SO(8)}$ representation. This is realized by picking $\Sigma_{\mu\nu}$ generators of the $SO(8)$ Lie Algebra in the fully symmetric, traceless tensor representation, which we will call from now on (generalized) vector. The monopole field again couples to the orbital $SO(8)$ $L_{\mu\nu}^{(0)}$ (but in a different way) to give the general solution of the Hamiltonian as the $(I + n - m, 2m, 0, 0)_{SO(9)}$ representations of $SO(9)$, where n, m are non-negative integers and $m \leq I$. In order to obtain the vector liquid, we need to consider only the solutions $(I + n, 0, 0, 0)_{SO(9)}$. We therefore need an extra projection that will keep only these representations from the higher space $(I + n - m, 2m, 0, 0)_{SO(9)}$. An elegant way to impose this projection is to look at the decomposition of $(I + n - m, 2m, 0, 0)_{SO(9)}$ into $SO(8)$ representations and impose a physically relevant projection onto the $SO(8)$ space. Attention should be drawn to the fact that this $SO(8)$ is now the full orbital plus monopole $SO(8)$, $L_{\mu\nu}^{(0)} + \Sigma_{\mu\nu}$.

The $SO(8)$ group has the special property that the two (left and right) spinor representations and the vector representation have the same dimensions. This property is called triality. It implies that there are three equivalent $SO(7)$ subgroups of $SO(8)$ which we will call $SO(7), SO^+(7), SO^-(7)$. The cosets of $SO(8)$ and its $SO(7)$ subgroups are the seven spheres S^7, S^{7+}, S^{7-} which differ in the form of their metric [7] and can be written as $SO(8)/SO(7) = S^7, SO(8)/SO^+(7) = S^{7-}$, and $SO(8)/SO^-(7) = S^{7+}$. Under the $SO(8)$ reduction into $SO(7)$, the $SO(8)$ vector splits $8 \rightarrow 1 + 7$ while the two spinors do not split $8_s^\pm \rightarrow 8$. This is the previous structure, which gives us the 20-dimensional quantum Hall spinor liquid. However, under the reduction of $SO(8)$ in $SO^\pm(7)$, the vector of $SO(8)$ gets rotated into the spinor of $SO^\pm(7)$ and does NOT split, while the spinor of $SO(8)$ rotates and splits into the vectors of $SO^\pm(7)$, $8_s^\pm \rightarrow 1 + 7$. The projection of any antisymmetric tensor of $SO(8)$ into the $SO^\pm(7)$ subgroups is achieved by the projection operator:

$$G_{\alpha\beta\mu\nu}^\pm = \frac{3}{8}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \pm \frac{1}{8}\Omega_{\alpha\beta\mu\nu}, \quad (10)$$

where $\Omega_{\alpha\beta\mu\nu}$ is a totally antisymmetric self-dual tensor in eight dimensions which can be constructed from the octonionic structure constants [7]. The $(I + n - m, 2m, 0, 0)_{SO(9)}$ decompose into $\sum_{k_1=0}^{I+n-m} \sum_{k_2=0}^{2m} (I + n - m + k_2 - k_1, 2m - k_2, 0, 0)_{SO(8)}$. In order to maintain only the vector $SO(9)$ $(I + n, 0, 0, 0)_{SO(9)}$, we need to maintain only the $\sum_{k_1=0}^{I+n} (I + n - k_1, 0, 0, 0)_{SO(8)}$ representations of $SO(8)$. These are the vectors of $SO(8)$. The $SO(8)$ vector is the only $SO(8)$ representation that transforms in the same way under projection to both $SO^+(7)$ and $SO^-(7)$. Therefore, the condition that the projections of a certain representation of $SO(8)$ into $SO^+(7)$ and $SO^-(7)$ is equivalent guarantees that we pick up only

the vector of $SO(8)$. Projecting the unwanted representations from the decomposition of $(I + n - m, 2m, 0, 0)_{SO(9)}$ into $SO(8)$ then takes the form of a condition posed on the wave function: $(G_{\alpha\beta\mu\nu}^+ L_{\mu\nu})^2 \Phi = (G_{\alpha\beta\mu\nu}^- L_{\mu\nu})^2 \Phi$ which reduces to

$$\Omega_{\mu\nu\alpha\beta} L_{\mu\nu} L_{\alpha\beta} \Phi = 0. \quad (11)$$

The $G_{\alpha\beta\mu\nu}^\pm L_{\mu\nu}$ is the projection of the $SO(8)$ $L_{\mu\nu}$ into the $SO^\pm(7)$. By requiring these projections to be equal, it means that they will be invariant under the maximal subgroup of both $SO^+(7)$ and $SO^-(7)$. Because $SO^\pm(7) = S^{7^\pm} \otimes G_2$, this subgroup is no other than the automorphism group of octonions, G_2 . This can also be seen from the decomposition $SO(8) = S^{7^+} \otimes S^{7^-} \otimes G_2$. The discussion above is the physical interpretation of the mathematical statement that a vector in eight dimensions is invariant under G_2 . The vector here is the octonion, and its automorphism group is G_2 .

The spectrum of the Hamiltonian with eigenstates $(I + n, 0, 0, 0)_{SO(9)}$ is given by $E(I, n) = [\hbar^2/(2MR^2)](n^2 + 2nI + 7n + I)$ and the degeneracy of each state is equal to the dimension of the $SO(9)$ representation $d(n, I) = \frac{1}{7!}(1 + n + I)(2 + n + I)(3 + n + I)(4 + n + I)(5 + n + I)(6 + n + I)(7 + 2n + 2I)$. The ground state (lowest Landau level) is the lowest $SO(9)$ level for a given I and is obtained by setting $n = 0$, with dimension $d(0, I)$. Higher Landau levels can be obtained by increasing n . In the thermodynamic limit $R, I \rightarrow \infty$, with a constant magnetic length $l_0^2 = R^2/I$, there is a finite constant energy gap between the Landau levels. The degeneracy of each Landau level in the thermodynamic limit $I \rightarrow \infty$ level varies as $I^7 \sim R^{14}$, which is also the scaling of the number of particles in the level. Hence, this vector liquid is entirely different liquid from the 20-dimensional spinor fluid obtained in the case when we pick the monopole field in the spinor representation of $SO(8)$. The particle configuration space is S^8 multiplied by the configuration space of the spin $\Sigma_{\mu\nu}$. The spin lives in the vector of $SO(8)$ and its stabilizer is therefore $SO(7) \otimes U(1)$. The spin configuration space is therefore $SO(8)/SO(7) \otimes U(1)$, six dimensional, varying as R^6 . The full particle configuration space is therefore $S^8 \otimes SO(8)/SO(7) \otimes U(1) \approx SO(9)/SO(7) \otimes U(1) \approx S^{15}/U(1) = CP^7 \sim R^{14}$. Therefore $V \sim R^{14}$ and our fluid has finite density $\rho = N/V$.

The number of particles to fill a Landau level varies as R^{14} , and is globally the manifold $S^{15}/U(1)$ which is the space CP^7 . In the thermodynamic limit, the wave functions of our system are obtained from the CP^7 picture, and can be constructed from Ψ_α by using only half of them. This is the equivalent of the fact that a complete description of the III in the QHE needs use of either the holomorphic or antiholomorphic functions but not of both. As a coset, $CP^7 = SU(8)/U(7)$; therefore the wave

functions will be symmetric tensor products of $SU(8)$ fundamentals $\Phi = \Phi_1^{m_1} \cdots \Phi_8^{m_8}$ with the constraint $m_1 + \cdots + m_8 = I$, $\Phi_i = \Psi_i + i\Psi_{i+4}$ if $i = 1, \dots, 4$, and $\Phi_i = \Psi_{i+4} + i\Psi_{i+8}$ if $i = 5, \dots, 8$. $SU(8)$ will act on Φ while keeping the subspace $U(7)$ invariant.

We have therefore constructed two eight-dimensional quantum hall fluids, whose dimensions vary as R^{20} and R^{14} . Because of the invariance under different subgroups of the wave function space, we can, say, that the 20-dimensional fluid lives on a space with $SO(7)$ holonomy, whereas the 14-dimensional fluid lives on a space with G_2 holonomy.

We now wish to draw an analogy with M theory: M theory lives in 11 dimensions, but we are interested in compactifications of the theory down to four dimensions that preserve $N = 1$ supersymmetry. For this reason, we compactify the theory on a seven-dimensional compact manifold with G_2 holonomy [8]. The topology of this manifold determines much of the structure and content of the effective four-dimensional low energy theory; for example, the amount of supersymmetry and chiral fermions that arise from singularities. Similarly in our case, one of the liquids lives on a higher 14 (spatial)-dimensional manifold, which is $CP^7 = S^{15}/U(1)$. This is locally isomorphic to $(S^8 \times S^7)/U(1)$. Locally, we can view this as a compactification from 14 dimensions down to eight dimensions and ask the structure of the effective $(8 + 1)$ -dimensional topological field theory describing the liquid. The compact space is $CP_3 = S^7/U(1)$ and it involves a seven sphere with torsion. Because of the torsion, the holonomy group of the seven sphere reduces to G_2 . Much of the structure of the resulting $(8 + 1)$ -dimensional topological effective theory should be determined by the topology of the compact 7D manifold of G_2 holonomy in direct analogy with M -theory compactifications.

We wish to thank D. Bump, S. Thomas, S. Shenker, L. Smolin, B. Laughlin, S. Kachru, J. Franklin, G. Chapline, Y. Chong, D. Oprea, C. H. Chern, I. Bena, and J. Hsu for helpful comments. This work is supported by the NSF under Grant No. DMR-9814289. B. A. B. is also supported by the SGF.

-
- [1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
 - [2] S.-C. Zhang and J.-P. Hu, Science **294**, 823 (2001).
 - [3] J. C. Baez, math.ra/0105155.
 - [4] B. A. Bernevig *et al.* (to be published).
 - [5] B. Grossman *et al.*, Commun. Math. Phys. **96**, 431 (1984).
 - [6] P.-M. Ho and S. Ramgoolam, Nucl. Phys. **B627**, 266 (2002).
 - [7] B. De Wit and H. Nicolai, Nucl. Phys. **B231**, 506 (1984).
 - [8] M. Atiyah and E. Witten, Adv. Theor. Math. Phys. **6**, 1 (2003).