Homework 7 Phys660

- Problem 1 (Projection operator)
  Consider two spin 1 particles. Define $\hat{S} = \hat{S}_1 + \hat{S}_2$ which is the total spin operator. Let's consider $|s, m>$ be the eigenstates of $\hat{S}^2, \hat{S}_z$. What are the values of s? Consider an operator $\hat{J}$, it has the following properties: $\hat{J}|s, m>= J_s|s, m>$ where $J_s = 1$ if s takes the maximum value and otherwise, $J_s = 0$. Write down the operator $\hat{J}$ in terms of $\hat{S}_1$ and $\hat{S}_2$.

- Problem 2
  a. What is the time reversed state corresponding to $\hat{D}(R)|S, m>$?
  b. Using the properties of the time reversal and rotations , Prove

$$D^{s*}_{m,m'}(R) = (-1)^{m'-m} D^{s}_{-m,-m'}(R) \tag{1}$$

- Problem 3
  a. Evaluate

$$\sum_{m=-s}^{s} |d^s_{mm'}(\beta)|^2 m \tag{2}$$

  for any s; then check your answer for $s = 1/2$.
  b. Prove, for any s,

$$\sum_{m=-s}^{s} |d^s_{mm'}(\beta)|^2 m^2 = \frac{1}{2}[s(s+1)sin^2\beta + m'^2(3cos^2\beta - 1)] \tag{3}$$

  [hint: Examine the rotational properties of J_z^2]

- Problem 4
  Consider $\hat{A}, \hat{B}$ are two vector operators. $\hat{\sigma}$ is Pauli matrix, Prove

$$(\hat{A} \cdot \hat{\sigma})(\hat{B} \cdot \hat{\sigma}) = \hat{A} \cdot \hat{B} + i\hat{\sigma} \cdot (\hat{A} \times \hat{B}) \tag{4}$$

- Problem 5
  Consider $\hat{A} = (A_x, A_y, A_z), \hat{B} = (B_x, B_y, B_z)$ are two vector operators. Using the product of two operators to construct the spherical tensor T^p_q. Explicitly write $T^1_q, q = \pm 1, 0$ and $T^2_q, q = \pm 2, \pm, 1, 0$.

- Problem 6
  Calculate all C-G coefficients for a system with two spin-one independent spins.

- Problem 7:
  A system is in the total angular momentum eigenstate labeled by $|jm>$. The system is suddenly perturbed by an operator $\hat{U}$, namely, the new

state becomes $\hat{U}|jm>$. It is known that $\hat{U}$ satisfies the following relation with Angular momentum operator,

$$[\hat{U}, \hat{J}_z] = \frac{1}{2}\hat{U}, \tag{5}$$

$$[[\hat{U}, \hat{J}^2], \hat{J}^2] = \frac{1}{2}(\hat{U}\hat{J}^2 + \hat{J}^2\hat{U}) + \frac{3}{16}\hat{U} \tag{6}$$

What is the selection rules in this process in angular momentum base? (namely, $< j'm'|\hat{U}|jm > \neq 0$)

- Problem 8:
(a) Write $\hat{x}\hat{y}$, $\hat{x}\hat{z}$, $(\hat{x}^2 - \hat{y}^2)$ as components of a spherical tensor operator of rank 2.
(b) The expectation value

$$Q = 2 < n, J, J|3\hat{z}^2 - \hat{r}^2|n, J, J > \tag{7}$$

is known as the quadrupole moment where $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$. Evaluate,

$$< n, J, m|\hat{x}^2 - \hat{y}^2|n, J, J > \tag{8}$$

- Problem 9:
In Schwinger's boson picture for angular momentum, let's define

$$\hat{K} = \hat{a}\hat{b} \tag{9}$$

What is the physical significance of the operators $\hat{K}$ and $\hat{K}^+$. Give the nonvanishing matrix elements.

- Problem 10 :
Consider a particle with spin 1/2 and orbital angular momentum, l. The total angular momentum is given by $\hat{J} = \hat{l} + \frac{\hbar}{2}\hat{\sigma}$. Assume that the particle is in an eigenstate of the total angular momentum and $\hat{J}_z$. The eigenstate is labeled as $|ljm>$. Namely, $\hat{l}^2|ljm> = \hbar^2 l(l+1)|ljm>$, $\hat{J}^2|ljm> = \hbar^2 j(j+1)|ljm>$, $\hat{J}_z|ljm> = \hbar m|ljm>$.
a. Prove the following relation which shows that the expectation value of the spin operator is proportional to the expectation value of the total angular momentum,

$$< \sigma_i > = < J_i > \frac{j(j+1) - l(l+1) + \frac{3}{4}}{j(j+1)} \tag{10}$$

b. If we consider the particle is an electron, the magnetic moment of the electron is given by $\hat{\mu} = -\frac{e}{2m_e c}(\hat{l} + \hat{\sigma})$. Calculate the expectation value of μ_z. (the result defines the Lande g factor).

- Problem 11:
  Consider a spin system in a 2×2 square lattice which has four lattice sites labeled as $1, 2, 3, 4$ clockwisely. There is a 1/2 spin at each lattice site, given by $\hat{S}_1, \hat{S}_2, \hat{S}_3, \hat{S}_4$. The hamiltonian of the system is given by

$$H = J(\hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_4 + \hat{S}_4 \cdot \hat{S}_1) \tag{11}$$

  Determine the energy eigenvalues of the system and their degeneracy of the related eigenstates. (hint, consider the state in the following base, the total spin, $\hat{S}_2 + \hat{S}_4$ and $\hat{S}_1 + \hat{S}_3$)

- Problem 12:
  A particle with mass m in a sphere quantum well,

$$V(r) = 0, r < a; V_0 > 0, r \geq a \tag{12}$$

  Consider the bound state of the system ($E < V_0$).
  (a) What is the minimum value of $V_0 a^2$ when the system has the first bound state.
  (b) For each angular momentum l, find condition when a new bound state appears.
  (c) Estimate the number of bound states when V_0 is very large.