Scattering state and Quantum Tunneling

Phys 460, Fall 2009, JP Hu

Free Particle

- General solution: Particle is in momentum eigenstates:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\hat{H}\varphi(x) = E\varphi(x)$$

$$\hat{P}\varphi(x) = \hbar k\varphi(x)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\varphi(x) = Ae^{ikx}$$

$$\psi_k(x,t) = Ae^{ikx - i\frac{\hbar k^2}{2m}t}$$

- Normalization:

$$\varphi(x) = Ae^{ikx}$$

$$\int \psi^*_k(x,t)\psi_k(x,t)dx = |A|^2 \int dx \to \infty$$

- Wave packet:

$$A = \frac{1}{\sqrt{2\pi}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\alpha(k)e^{ikx - i\frac{\hbar k^2}{2m}t}$$

$$\alpha(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\psi(x,0)e^{-ikx}$$

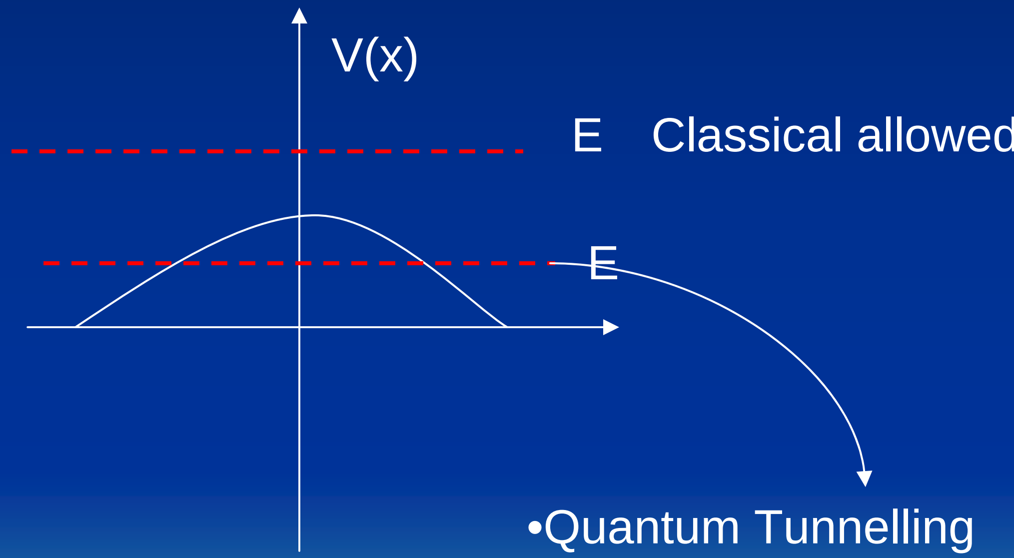
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk \Leftrightarrow F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{-ikx}dx$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk$$

Plancherel's Theorum

Extended state (Scattering state)

- Particle moves from infinite to the other end of infinite:



•Quantum Tunnelling
•Classically, no connection from left to right.

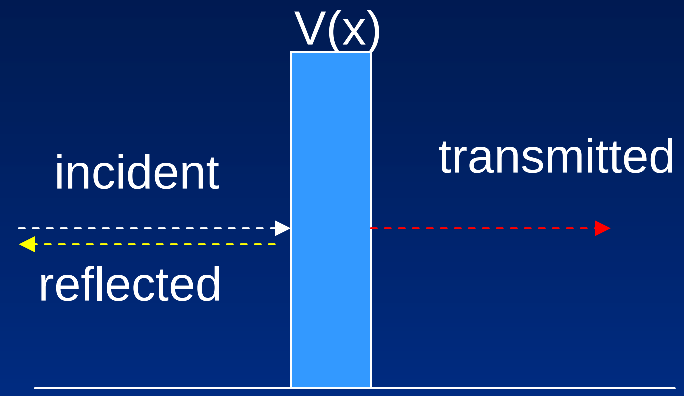
Quantum tunneling

- Square barrier:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{H}\varphi(x) = E\varphi(x)$$

$$V(x) = V_0, 0 < x < L$$



Reflection and Transmission coefficients

$$(II)\varphi(x) = a\exp(k'x) + b\exp(-k'x)$$

$$-E + V_0 = \frac{\hbar^2 k'^2}{2m}$$

$$(I, III)$$

$$\varphi^1(x) = A\exp(ikx) + r\exp(-ikx)$$

$$\varphi^2(x) = t\exp(ikx)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

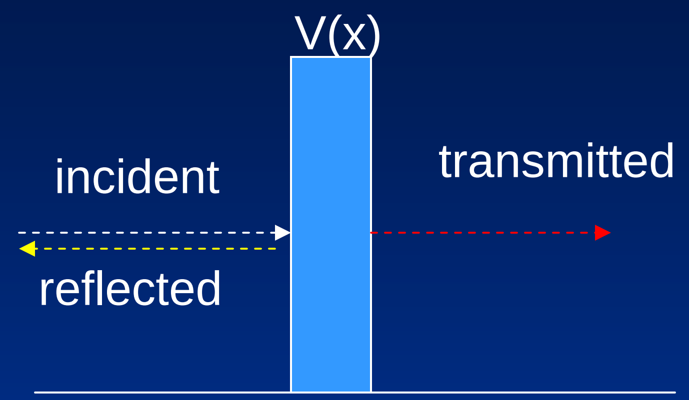
$$R = \frac{|r|^2}{|A|^2}$$

$$T = \frac{|t|^2}{|A|^2}$$

$$R + T = 1$$

Quantum tunneling

- Square barrier:



Result:

$$R = \frac{|r|^2}{|A|^2} = \frac{(k^2 + k'^2)^2 sh^2(k'L)}{(k^2 + k'^2)^2 sh^2(k'L) + 4kk'^2}$$

$$T = \frac{|t|^2}{|A|^2} = 1 - R$$

$$R + T = 1$$

$$t = \frac{-2ik/k'}{(1-(k/k')^2)sh(k'L) - 2ikch(k'L)/k'} e^{ikL} A$$

$$a = \frac{t}{2}[1 + \frac{ik}{k'}]e^{ikL - k'L}$$

$$b = \frac{t}{2}[1 - \frac{ik}{k'}]e^{ikL + k'L}$$

In the limit of k'L>>1,

$$T \propto \exp(-2k'L)$$

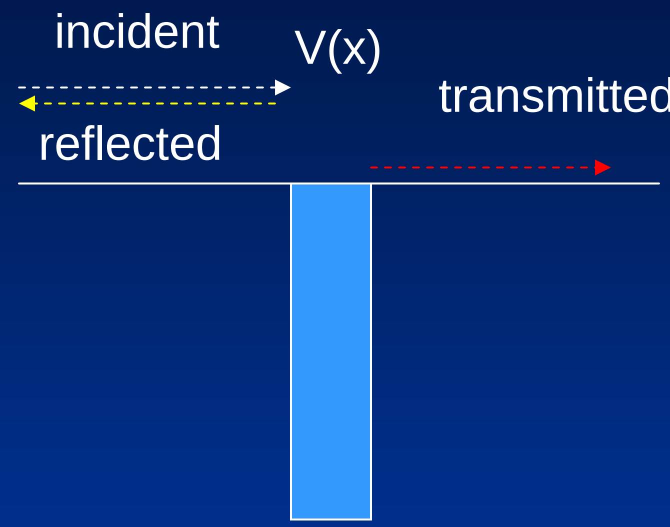
Scattering

- Square barrier:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{H}\varphi(x) = E\varphi(x)$$

$$V(x) = -V_0, 0 < x < L$$



Reflection and Transmission coefficients

$$(II)\varphi(x) = a\exp(ik'x) + b\exp(-ik'x)$$

$$E + V_0 = \frac{\hbar^2 k'^2}{2m}$$

$$(I, III)$$

$$\varphi^1(x) = A\exp(ikx) + r\exp(-ikx)$$

$$\varphi^2(x) = t\exp(ikx)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$R = \frac{|r|^2}{|A|^2}$$

$$T = \frac{|t|^2}{|A|^2}$$

$$R + T = 1$$

Scattering

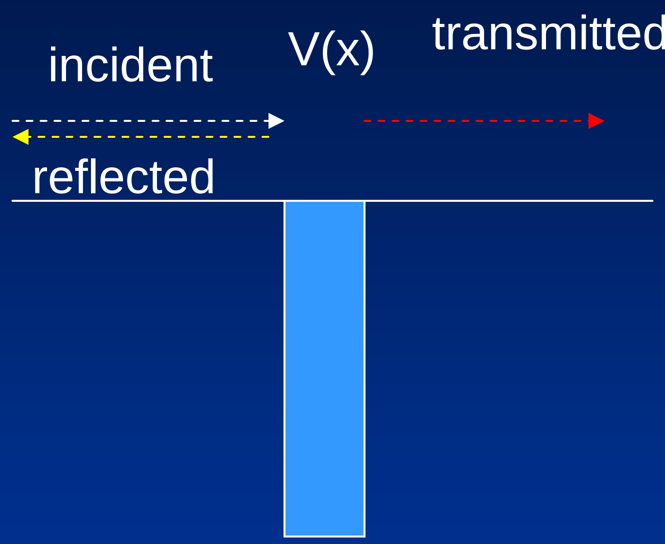
- Square barrier:

Result:

$$R = \frac{|r|^2}{|A|^2} = 1 - T$$

$$T = \frac{|t|^2}{|A|^2} = [1 + \frac{V_0^{\ 2}}{4E(E+V_0)}\sin^2(2L\sqrt{2m(E+V_0)}/\hbar)]^{-1}$$

$$R + T = 1$$



Condition for perfect transmission:

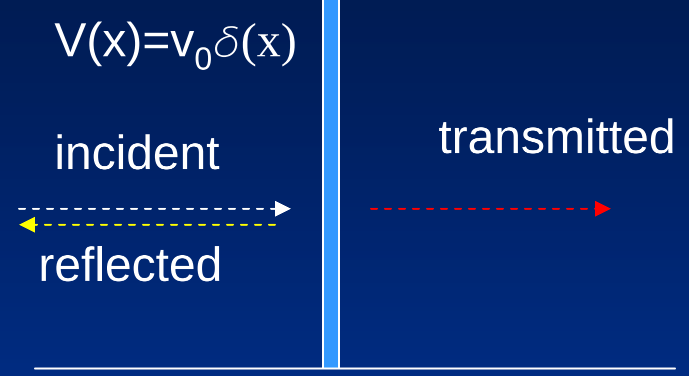
$$k' = \frac{n\pi}{2L}$$

$$t = \frac{-2ik/k'}{-i(k^2+k'^2)\sin(2k'L)/(2kk') - \cos(2k'L)} e^{-i2kL} A$$

Wave length match the boundary!!

Quantum tunneling (δ-function)

- Square barrier:



$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{H}\varphi(x) = E\varphi(x)$$

$$(I, II)$$

$$\varphi^1(x) = A\exp(ikx) + r\exp(-ikx)$$

$$\varphi^2(x) = t\exp(ikx)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

Reflection and Transmission coefficients

$$R = \frac{1}{1 + \frac{2\hbar^2 E}{m v_0^2}}; T = 1 - R$$

It does not depend on the sign of V_0

Scattering Matrix

- For given energy E, we have momentum k, -k states.

  A: incident particles with momentum k

  D: incident particles with momentum –k

  B: reflected (k) + transimitted (-k)

  C: reflected (-k)+ transmitted (k)

$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

$$\Psi(x) = Ae^{ikx} + Be^{-ikx} - (I)$$

$$\Psi(x) = Ce^{ikx} + De^{-ikx} - (II)$$