I. VECTOR SPACE

Consider a two dimensional plane, every point in the place can be mapped to a unique vector. All of the vectors form a linear vector space v with a dimensionality equal to two.
Any vector can be labeled as (x, y) which can be written as

$$(x, y) = x(1, 0) + y(0, 1) \tag{1}$$

The sum of two vectors satisfies,

$$a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2 + ay_1 + by_2) \tag{2}$$

In this vector space, we can only define a dot product,

$$(x_1, y_x) \cdot (x_2, y_2) = (x_1x_2, y_2y_2) \tag{3}$$

Notice that $(1, 0) \cdot (0, 1) = 0$ and $(1, 0) \cdot (1, 0)$=1, which we call the two vectors are orthogonal to each other. Since any vector can be written as a linear combination of the two orthogonal vectors, we call the linear vector space v has dimension two and $(1, 0)$ and $(0, 1)$ form a complete orthonormal basis of V.

II. DIRAC NOTATION

The similar idea can be generalized to a space with any dimensions. We call a general linear vector space V with dimension N if any vector in V can be written as a linear combination of N orthonormal vectors. We will use a ket to denote a vector and the N orthonormal vectors are $|i >, i = 1, ...N$. Therefore any vector,

$$|v >= \sum_{i=1}^{N} a_i |i > \tag{4}$$

We write the vector product $|i > \cdot |j >$ as $< i|j >$. With this notation, we have

$$< i|j >= \delta_{ij} \tag{5}$$

where, δ_{ij} is defined as $\delta_{ii} = 1, i = 1, ...N$, and zero otherwise.

So far we consider the space is real. We can also extend the space to allow the **complex number**, i.e. the coefficient a_i can be a complex number. Since the norm of a vector $< v|v >$ should be a real number, we have to modify the dot product to be an inner product,

$$< v_1|v_2 >= \sum_{i=1}^{N} a_{1i}^* a_{2i} \tag{6}$$

so that

$$< v|v >= \sum_{i=1}^{N} a_i^* a_i \tag{7}$$

is a real number. In general, we have

$$< v_1|v_2 >=< v_2|v_1 >^* \tag{8}$$

and some useful formula:

$$|av >= a|v > \tag{9}$$
$$< av| =< v_1|a^* \tag{10}$$
$$\tag{11}$$

where a is a complex constant.

III. OPERATOR

An operator $\hat{O}$ acting on the vector space takes one vector to other vector, i.e. $\hat{O}|v>$ is still a vector in V space. Therefore, if we know an orthonormal basis $|i>$ of the vector space, we will have

$$\hat{O}|i>=\sum_{j=1}^{N} o_{ij}|j> \tag{12}$$

Thus, we have

$$o_{ij} =< j|\hat{O}|i> \tag{13}$$

When $\hat{o}$ acts on an arbitrary vector $|v>=\sum_{i=1}^{N} a_i|i>$, we have

$$\hat{O}|v>=\sum_{i,j=1}^{N} a_i o_{ij}|j> \tag{14}$$

An operator in a N dimensional vector space can be represented as $N \times N$ matrix.

If a vector $|v>$ satisfies $\hat{O}|v>= o_v|v>$, we call $|v>$ is an eigenvector of operator $|v>$ with eigenvalue equal to o_v. To find all the eigenvectors of $\hat{O}$ is equivalent to diagonalize matrix O_{ij}.

Special operator: The Dirac notation is very convenient. A special operator is $\hat{I} = \sum_{i=1}^{N} |i >< i|$, we can see

$$\hat{I}|v>= |v> \tag{15}$$

i.e, $\hat{I}$ is an identity operator (In fact, this is completeness condition of the orthogonal basis)

IV. δ FUNCTION, ORTHORNORMAL BASIS IN CONTINUOUS SPACE

A vector space can have infinite dimension. In particular, we can have a vector space whose orthonormal basis is labeled by continuous variable $|x>$. In this case, the orthornormal condition, for $x \neq y$

$$< x|y >= 0 \tag{16}$$

Since the space is continuous, we have to replace the sum by integral, namely, for a complete space, we have,

$$\hat{I} = \int_x |x >< x|. \tag{17}$$

Since $\hat{I}|y>= |y>= \int_x |x >< x|y>= |y>$, we obtain $< x|y >= \delta(x-y)$, where the function $\delta(x-y)$ is defined as

$$\int_x \delta(x-y) f(x) = f(y) \tag{18}$$

for any given function $f(x)$. Therefore for a continuous linear space, the orthornormal condition becomes,

$$< x|y >= \delta(x-y) \tag{19}$$