CH. 3. SUPERSYMMETRY AND SUPERSPACE

3.0. Introduction

One technically unaesthetic aspect of the grand unified theories in general is the fine tuning problem. As we mentioned earlier, supersymmetry provides a means of solving this hierarchy difficulty due to its improved ultraviolet behavior; there are no quadratic divergences in supersymmetry. Before extending the SU(5) Georgi-Glashow model to make it supersymmetric, let's review the structure of supersymmetric field theories and in particular introduce the methods of superspace.

3.1. Supersymmetry and superfield, superspace techniques,...

Before introducing supersymmetry, let's first recall some properties of the Poincaré transformations and SL(2,C) spinor representations of the Lorentz group. Poincaré transformations consist of a Lorentz transformation Λ^{μ}_{ν} and a space-time translation a^{μ};

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu} + a^{\mu} \tag{3.1.1}$$

where the interval ds^2 is left invariant

$$\begin{aligned} ds^2 &= dx'^{\mu} g_{\mu\nu} dx'^{\nu} = dx^{\alpha} g_{\alpha\beta} dx^{\beta} \\ &= dx^{\alpha} \Lambda^{\mu}_{\alpha} g_{\mu\nu} \Lambda^{\nu}_{\beta} dx^{\beta} \end{aligned} \tag{3.1.2}$$

hence

$$g_{\alpha\beta} = \Lambda^{\mu}_{\alpha} g_{\mu\nu} \Lambda^{\nu}_{\beta} \tag{3.1.3}$$

with the metric

$$g_{\mu\nu} = \begin{bmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}_{\mu\nu} . \tag{3.1.4}$$

The Poincaré transformations form a group with the product of two transformations $(\Lambda_1, a_1)(\Lambda_2, a_2)$ being equal to another transformation (Λ, a) with

$$\begin{aligned} \Lambda^\mu_\nu &= \Lambda^\mu_{1\rho} \Lambda^\rho_{2\nu} \\ a^\mu &= \Lambda^\mu_{1\nu} a^\nu_2 + a^\mu_1 . \end{aligned} \tag{3.1.5}$$

As usual we can define contravariant and covariant tensor fields according to their Lorentz transformation properties

$$T'^{\mu_1 \dots \mu_n}(x') = \Lambda^{\mu_1}_{\nu_1} \dots \Lambda^{\mu_n}_{\nu_n} T^{\nu_1 \dots \nu_n}(x) \tag{3.1.6}$$

for a contravariant rank n-tensor field and

$$T'_{\mu_1 \dots \mu_n}(x') = \Lambda^{-1\nu_1}{}_{\mu_1} \dots \Lambda^{-1\nu_n}{}_{\mu_n} T_{\nu_1 \dots \nu_n}(x) \tag{3.1.7}$$

for a covariant rank n-tensor field and

$$T'^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x') = \Lambda^{\mu_1}_{\alpha_1} \dots \Lambda^{\mu_m}_{\alpha_m} \Lambda^{-1\beta_1}{}_{\nu_1} \dots \Lambda^{-1\beta_n}{}_{\nu_n} T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}(x) \tag{3.1.8}$$

for a mixed (m,n) tensor, where

$$\Lambda^\mu_\nu \Lambda^{-1\nu}{}_\rho = \delta^\mu{}_\rho . \tag{3.1.9}$$

Note that $V'^{\mu}W'_{\mu} = \Lambda^{\mu}_{\nu}\Lambda^{-1\rho}_{\mu}V^{\nu}W_{\rho}$
$= \delta^{\rho}_{\nu}\, V^{\nu}W_{\rho} = V^{\mu}W_{\mu}$

is a lorentz invariant. We can define the contravariant metric tensor $g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}{}_{\rho} \qquad (3.1.10)$$

Note that $g = \Lambda^{T}g\Lambda$ implies that

$$g'_{\mu\nu} = \Lambda^{-1\alpha}_{\mu}\Lambda^{-1\beta}_{\nu}g_{\alpha\beta} = g_{\mu\nu} \qquad (3.1.11)$$

and similarly $g'^{\mu\nu} = g^{\mu\nu}$; the metric tensor is invariant.

Then for every contravariant vector there is an associated covariant vector, and vice versa, obtained by lowering or raising an index with $g_{\mu\nu}$ or $g^{\mu\nu}$. That is if

$V'^{\mu} = \Lambda^{\mu}_{\nu}V^{\nu}$ then

$V_{\mu} \equiv g_{\mu\nu}V^{\nu}$ is covariant since

$$V'_{\mu} = g_{\mu\nu}V'^{\nu} = g_{\mu\nu}\Lambda^{\nu}_{\rho}V^{\rho}$$

but

$$g_{\mu\nu}\Lambda^{\nu}_{\rho} = \Lambda^{-1\beta}_{\mu}g_{\beta\rho}$$

so $V'_{\mu} = \Lambda^{-1\beta}_{\mu}g_{\beta\rho}V^{\rho}$

hence $V'_{\mu} = \Lambda^{-1\beta}_{\mu}V_{\beta}$, the transformation law for covariant vectors.

Using this notation to define $\Lambda_{\mu\nu} = g_{\mu\alpha}\Lambda^{\alpha}_{\ \nu}$ etc. we find

$$\delta^{\alpha}_{\beta} = \Lambda^{\mu\alpha}\Lambda_{\mu\beta} \quad \text{that is} \quad \Lambda^{-1\mu\nu} = \Lambda^{T\mu\nu} = \Lambda^{\nu\mu}. \tag{3.1.12}$$

For infinitesimal Lorentz transformations $\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\ \nu}$ then

$$\begin{aligned} g_{\alpha\beta} &= (\delta^{\mu}_{\alpha}+\omega^{\mu}_{\ \alpha})g_{\mu\nu}(\delta^{\nu}_{\beta}+\omega^{\nu}_{\ \beta}) \quad \text{which implies that} \\ \omega_{\mu\nu} + \omega_{\nu\mu} &= 0 \end{aligned} \tag{3.1.13}$$

that is $\omega_{\mu\nu}$ is anti-symmetric. The fundamental vector representation of the Lorentz group is then given by the coordinate transformations

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega^{\mu}_{\ \nu}x^{\nu} \\ &\equiv x^{\mu} + \frac{\omega^{\beta}_{\ \alpha}}{2}[T^{\alpha}_{\ \beta}]^{\mu}_{\ \nu}x^{\nu} = x^{\mu} + \frac{\omega^{\beta}_{\ \alpha}}{2}[D^{\alpha}_{\ \beta}]^{\mu}_{\ \nu}x^{\nu} \end{aligned} \tag{3.1.14}$$

where here we use $D^{\alpha}_{\ \beta}$ instead of T^{a}_{b} as we had been using previously for the internal symmetry generators. Thus

$$(D^{\alpha}_{\ \beta})^{\mu}_{\ \nu} = \delta^{\mu}_{\beta}\delta^{\alpha}_{\nu} - g^{\mu\alpha}g_{\nu\beta}$$

or

$$(D^{\alpha\beta})_{\mu\nu} = \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} \quad \text{so} \quad D^{\alpha\beta} = -D^{\beta\alpha} .$$

From this vector representation we can obtain the commutation relations for the $D^{\alpha\beta}$ which all representations of the Lorentz group must obey:

$$
\begin{aligned}
[D^{\mu\nu},D^{\rho\sigma}]_{\alpha\beta} &= (D^{\mu\nu})_{\alpha}^{\ \gamma}(D^{\rho\sigma})_{\gamma\beta} - (D^{\rho\sigma})_{\alpha}^{\ \gamma}(D^{\mu\nu})_{\gamma\beta} \\
&= (\delta^{\nu}_{\alpha}g^{\mu\gamma}-\delta^{\mu}_{\alpha}g^{\nu\gamma})(\delta^{\sigma}_{\gamma}\delta^{\rho}_{\beta}-\delta^{\rho}_{\gamma}\delta^{\sigma}_{\beta}) - (D^{\rho\sigma}D^{\mu\nu})_{\alpha\beta} \\
&= g^{\mu\sigma}(\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}-\delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}) - g^{\mu\rho}(\delta^{\nu}_{\alpha}\delta^{\sigma}_{\beta}-\delta^{\sigma}_{\alpha}\delta^{\nu}_{\beta}) - g^{\nu\sigma}(\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}-\delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}) \\
&\quad + g^{\nu\rho}(\delta^{\mu}_{\alpha}\delta^{\sigma}_{\beta}-\delta^{\sigma}_{\alpha}\delta^{\mu}_{\beta}) \\
&= [g^{\mu\sigma}D^{\rho\nu}-g^{\mu\rho}D^{\sigma\nu}-g^{\nu\sigma}D^{\rho\mu}+g^{\nu\rho}D^{\sigma\mu}]_{\alpha\beta} \ .
\end{aligned}
$$

Thus the defining commutation relations are

$$
[D^{\mu\nu},D^{\rho\sigma}] = g^{\mu\rho}D^{\nu\sigma}+g^{\nu\sigma}D^{\mu\rho}-g^{\mu\sigma}D^{\nu\rho}-g^{\nu\rho}D^{\mu\sigma} \quad . \tag{3.1.16}
$$

For a general tensor field we can find its Lorentz representation matrix in a similar way. Consider

$$
\begin{aligned}
T'^{\mu_1\cdots\mu_n}(x') &= (g^{\mu_1\nu_1}+\omega^{\mu_1\nu_1})\cdots(g^{\mu_n\nu_n}+\omega^{\mu_n\nu_n})\ T_{\nu_1\cdots\nu_n}(x) \\
&= T^{\mu_1\cdots\mu_n}(x) + \sum_{i=1}^{n} \omega^{\mu_i\nu_i}T_{\mu_1\cdots\nu_i\cdots\mu_n}(x) \\
&= T^{\mu_1\cdots\mu_n}(x) + \frac{\omega_{\alpha\beta}}{2}\sum_{i=1}^{n}[g^{\alpha\mu_i}g^{\beta\nu_i}-g^{\alpha\nu_i}g^{\beta\mu_i}] \\
&\quad \times g^{\mu_1\nu_1}\cdots\cancel{g^{\mu_i\nu_i}}\cdots g^{\mu_n\nu_n}T_{\nu_1\cdots\nu_n}(x)
\end{aligned} \tag{3.1.17}
$$

Thus using the notation $(\mu) = \mu_1\cdots\mu_n$ we have

$$
T'^{(\mu)}(x') - T^{(\mu)}(x) \equiv \frac{\omega_{\beta\alpha}}{2}(D^{\alpha\beta})^{(\mu)(\nu)}T_{(\nu)}(x) \tag{3.1.18}
$$

where

$$(D^{\alpha\beta})^{(\mu)(\nu)} \equiv \sum_{i=1}^{n} g^{\mu_1\nu_1}\ldots[-g^{\alpha\mu_i}g^{\beta\nu_i}+g^{\alpha\nu_i}g^{\beta\mu_i}]\ldots g^{\mu_n\nu_n} . \qquad (3.1.19)$$

As in the vector case the $(D^{\alpha\beta})^{(\mu)(\nu)}$ obey the Lorentz group commutation relations. In general, the difference $\delta T(x)$;

$$T'(x')-T(x) \equiv \delta T(x) \qquad (3.1.20)$$

is called the total variation of T. The intrinsic variation of T is defined by

$$\begin{aligned}\bar{\delta}T(x) &\equiv T'(x) - T(x) \\ &= [T'(x')-T(x)] - [T'(x')-T'(x)] \\ &= \delta T(x) - [T'(x')-T'(x)]\end{aligned} \qquad (3.1.21)$$

Since these are infinitesimal variations

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} \quad \text{and} \quad T'(x') = T'(x) + \delta x^{\mu}\partial_{\mu}T(x) . \qquad (3.1.22)$$

So

$$\bar{\delta}T(x) = \delta T(x) - \delta x^{\mu}\partial_{\mu}T(x) \qquad (3.1.23)$$

For a Lorentz transformation

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu}x^{\nu} = x^{\mu} + \frac{\omega_{\beta\alpha}}{2}[g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta}]x_{\nu} . \qquad (3.1.24)$$

So for an n tensor $T^{(\mu)}$

$$\bar{\delta}T^{(\mu)} = \frac{\omega_{\beta\alpha}}{2}\{(D^{\alpha\beta})^{(\mu)(\nu)} - g^{(\mu)(\nu)}(x^{\alpha}\partial^{\beta}-x^{\beta}\partial^{\alpha})\}T_{(\nu)} . \qquad (3.1.25)$$

Thus we define the differential (angular momentum) operator $M^{\mu\nu}$ as

$$M^{\mu\nu}T^{(\alpha)} \equiv i[(x^\mu\partial^\nu - x^\nu\partial^\mu)g^{(\alpha)(\beta)} - (D^{\mu\nu})^{(\alpha)(\beta)}]T_{(\beta)}$$

$$= (M^{\mu\nu})^{(\alpha)(\beta)}T_{(\beta)} \quad . \qquad (3.1.26)$$

Again we check that

$$[M^{\mu\nu},M^{\rho\sigma}] = (-i)[g^{\mu\rho}M^{\nu\sigma}+g^{\nu\sigma}M^{\mu\rho}-g^{\mu\sigma}M^{\nu\rho}-g^{\nu\rho}M^{\mu\sigma}] \ . \qquad (3.1.27)$$

Thus we can represent the Lorentz group by finite matrices $D^{\mu\nu}$ and by space-time differential operators acting on tensor fields $T^{(\alpha)}$.

Similarly we can consider infinitesimal space-time translations

$$x^{\mu'} = x^\mu + \varepsilon^\mu \ . \qquad (3.1.28)$$

Then for translationally invariant fields $T'(x') = T(x)$ we have

$$\bar{\delta}T = -\varepsilon^\mu\partial_\mu T \equiv +i\varepsilon^\mu P_\mu T \qquad (3.1.29)$$

that is the momentum operator

$$P_\mu = i\partial_\mu \qquad (3.1.30)$$

represents the generator of translations. Thus for any representation of the Lorentz group $M^{\mu\nu}$ we can calculate the commutator of P^λ with it:

$$[M^{\mu\nu},P^\lambda] = i[P^\mu g^{\nu\lambda}-P^\nu g^{\mu\lambda}] \ . \qquad (3.1.31)$$

Along with $[P_\mu,P_\nu] = 0$ these three sets of commutation relations define the action of the Poincare group on fields $T^{(\mu)}$.

As usual the field obtained after a finite Poincaré transformation is obtained by exponentiation. For translations $x'^{\mu} = x^{\mu} + a^{\mu}$ we have

$$T'(x) = \lim_{n\to\infty} (1 + \frac{ia^{\mu}}{n} P_{\mu})^{n} T(x)$$

$$= e^{+ia^{\mu}P_{\mu}} T(x) \tag{3.1.32}$$

$$= e^{-a^{\mu}\partial_{\mu}} T(x) = T(x-a)$$

(just the definition of a translationally invariant field). Thus for the action $\Gamma = i\int dx L(x)$ to be invariant under the Poincaré group the lagrangian L must be a translationally invariant, scalar function.

For $$x'^{\mu} = x^{\mu} + \omega^{\mu\nu}x_{\nu} + a^{\mu} \tag{3.1.33}$$

$$\delta L = 0 .$$

This symmetry yields the conserved currents $T^{\mu\nu}$ the energy momentum tensor and $M^{\mu\nu\rho}$ the angular momentum tensor.

Consider $x'^{\mu} = x^{\mu} + a^{\mu}$, so

$$\bar{\delta} L = -a^{\mu}\partial_{\mu} L \tag{3.1.34}$$

but

$$L = L[\phi, \partial_{\mu}\phi]$$

So

$$\bar{\delta} L = \frac{\partial L}{\partial \phi} \bar{\delta}\phi + \frac{\partial L}{\partial \partial_{\mu}\phi} \bar{\delta}\partial_{\mu}\phi \tag{3.1.35}$$

Now

$$\bar{\delta}\partial_{\mu}\phi = \partial_{\mu}\phi'(x) - \partial_{\mu}\phi(x) = \partial_{\mu}(\phi'(x)-\phi(x)) = \partial_{\mu}\bar{\delta}\phi \quad , \tag{3.1.36}$$

the intrinsic variation commutes with derivatives, while the total variation does not, with

$$\bar{\delta}\phi = -a^{\mu}\partial_{\mu}\phi \quad .$$

So

$$\bar{\delta}L = \frac{\partial L}{\partial \phi}\bar{\delta}\phi + \partial_{\mu}[\frac{\partial L}{\partial \partial_{\mu}\phi}\bar{\delta}\phi] - (\partial_{\mu}\frac{\partial L}{\partial \partial_{\mu}\phi})\bar{\delta}\phi \tag{3.1.37}$$

However the Euler-Lagrange derivative is

$$E(\phi) = \frac{\partial L}{\partial \phi} - \partial_{\mu}\frac{\partial L}{\partial \partial_{\mu}\phi} \tag{3.1.38}$$

giving

$$\bar{\delta}L = \partial_{\mu}[\frac{\partial L}{\partial \partial_{\mu}\phi}\bar{\delta}\phi] + E(\phi)\bar{\delta}\phi \tag{3.1.39}$$

$$= -a^{\mu}\partial_{\mu}L$$

This yields

$$a_{\nu}\partial_{\mu}[\frac{\partial L}{\partial \partial_{\mu}\phi}\partial^{\nu}\phi - g^{\mu\nu}L] = -a_{\nu}E(\phi)\partial^{\nu}\phi \tag{3.1.40}$$

We define the energy momentum tensor as

$$T^{\mu\nu} \equiv \frac{\partial L}{\partial \partial_{\mu}\phi}\partial^{\nu}\phi - g^{\mu\nu}L \tag{3.1.41}$$

and

$$\partial_{\mu}T^{\mu\nu} = -E(\phi)\partial^{\nu}\phi \quad , \tag{3.1.42}$$

upon application of the field equations the Euler derivative vanishes and the energy-momentum tensor is conserved

$$\partial_{\mu}T^{\mu\nu} = 0 \ .$$

To summarize, the Poincaré group is defined by the algebra its operators P_μ, the energy-momentum operator, the generator of space-time translation, and $M_{\mu\nu}$, the angular momentum operator, the generator of Lorentz transformations and space rotations, obey

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\lambda] &= i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}) \ . \end{aligned} \tag{3.1.43}$$

For the tensor representations of the Lorentz group this algebra is realized by the intrinsic variations of the fields as

$$\begin{aligned} P_\mu T^{(\alpha)}(x) &= i\partial_\mu T^{(\alpha)}(x) \\ M_{\mu\nu} T^{(\alpha)}(x) &= i[(x_\mu \partial_\nu - x_\nu \partial_\mu) g^{(\alpha)(\beta)} - (D_{\mu\nu})^{(\alpha)(\beta)}] T_{(\beta)}(x) \end{aligned} \tag{3.1.44}$$

where $(\alpha) = \alpha_1 \ldots \alpha_n$ for an n^{th} rank tensor, $g^{(\alpha)(\beta)} = g^{\alpha_1\beta_1} \ldots g^{\alpha_n\beta_n}$, and

$$(D^{\mu\nu})^{(\alpha)(\beta)} = \sum_{i=1}^{n} g^{\alpha_1\beta_1} \ldots [g^{\mu\beta_i} g^{\nu\alpha_i} - g^{\nu\beta_i} g^{\mu\alpha_i}] \ldots g^{\alpha_n\beta_n} \ , \tag{3.1.45}$$

is the matrix for the finite tensor representation of the Lorentz group where the $D^{\mu\nu}$ obey the algebra

$$[D^{\mu\nu}, D^{\rho\sigma}] = g^{\mu\rho}D^{\nu\sigma} - g^{\mu\sigma}D^{\nu\rho} + g^{\nu\sigma}D^{\mu\rho} - g^{\nu\rho}D^{\mu\sigma} \ . \tag{3.1.46}$$

The transformations induced in $T^{(\alpha)}$ when finite Poincaré transformations are made $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$ are obtained by exponentiating the operators with $\omega_{\mu\nu}(\Lambda)$ the finite angles of rotation defining $\Lambda_{\mu\nu}$

$$T'^{\alpha_1\ldots\alpha_n}(x) = \Lambda^{\alpha_1}_{\beta_1}\ldots\Lambda^{\alpha_n}_{\beta_n}T^{\beta_1\ldots\beta_n}(\Lambda^{-1}(x-a))$$

$$= \left[e^{+ia^\mu P_\mu}\, e^{-\frac{i\omega_{\mu\nu}(\Lambda)}{2}M^{\mu\nu}}\right]^{(\alpha)(\beta)} T_{(\beta)}(x) \quad . \tag{3.1.47}$$

The tensor representations are not the only realizations of the algebra possible--there are also the spinor representations.

Thus we have obtained the tensor representations of the Lorentz group. The 2 x 2 complex matrices with determinant 1 form a group called SL(2,C) and we will represent the Lorentz group by the action of these matrices on two-component complex spinors. To obtain the relation of the Lorentz group to SL(2,C) we must first recall that there exists a 1-1 correspondence between 2 x 2 Hermitian matrices and space-time points.

The Pauli matrices

$$(\sigma^0)_{\alpha\dot\alpha} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{\alpha\dot\alpha} ; \quad (\sigma^1)_{\alpha\dot\alpha} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}_{\alpha\dot\alpha}$$

$$(\sigma^2)_{\alpha\dot\alpha} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}_{\alpha\dot\alpha} ; \quad (\sigma^3)_{\alpha\dot\alpha} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}_{\alpha\dot\alpha} \tag{3.1.48}$$

where $\alpha = 1,2$ and $\dot\alpha = 1,2$ form a basis for 2×2 Hermitian matrices. Let $\chi_{\alpha\dot\alpha}$ be a Hermitian matrix that is

$$\chi^\dagger = \chi \; ; \qquad \chi_{\alpha\dot\alpha} = \chi^*_{\dot\alpha\alpha} \tag{3.1.49}$$

Then it has the most general form

$$\chi_{\alpha\dot{\alpha}} = \begin{bmatrix} \chi_0+\chi_3 & \chi_1-i\chi_2 \\ \chi_1+i\chi_2 & \chi_0-\chi_3 \end{bmatrix} \qquad (3.1.50)$$

$$= \chi_\mu \sigma^\mu_{\alpha\dot{\alpha}} \equiv \chi_{\alpha\dot{\alpha}} \quad \text{for } \chi_\mu \text{ real.}$$

Using the trace relation for the product of 2 Pauli matrices

$$\sigma^\mu_{\alpha\dot{\alpha}} (i\sigma^2)_{\dot{\alpha}\dot{\beta}} \; \overset{T}{\sigma}{}^\nu_{\dot{\beta}\beta}(i\sigma^2)_{\beta\alpha} = -2g^{\mu\nu}$$

or more succinctly

$$\mathrm{Tr}\sigma^\mu(i\sigma^2)\sigma^{T\nu}(i\sigma^2) = -2g^{\mu\nu} \qquad (3,1,51)$$

we have for every Hermitian matrix $\chi_{\alpha\dot{\alpha}}$ an associated 4-vector

$$\chi^\mu = -\frac{1}{2}\,\mathrm{Tr}[\chi(i\sigma^2)\sigma^{T\mu}(i\sigma^2)] \; . \qquad (3.1.52)$$

Simplifying the notation we introduce an antisymmetric tensor $\varepsilon^{\alpha\beta}$, that is $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ with $\varepsilon^{12} = +1$ and with lowered indices

$$\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} (\varepsilon_{12}=-\varepsilon^{12}=-1)$$

$$= -\varepsilon_{\beta\alpha} \quad .$$

[The matrix is the same when we use dotted indices also.] That is

$$\varepsilon^{\alpha\beta} = (i\sigma^2)_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\alpha\beta} \qquad (3.1.53)$$

$$\varepsilon^{\dot{\alpha}\dot{\beta}} = (i\sigma^2)_{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\dot{\alpha}\dot{\beta}}$$

<u>Note:</u>
$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$
$$\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}_{\dot\gamma} \qquad (3.1.54)$$

Then we can define

$$\bar\sigma^{\mu\dot\alpha\alpha} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^{\mu}_{\beta\dot\beta}$$
$$= -(i\sigma^2)_{\dot\alpha\dot\beta}\sigma^{T\mu}_{\dot\beta\beta}(i\sigma^2)_{\beta\alpha} \qquad (3.1.55)$$

so we can write the trace condition as

$$\sigma^{\mu}_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha} = +2g^{\mu\nu} \qquad (3.1.56)$$

and

$$\chi^{\mu} = +\frac{1}{2}\cancel{\chi}_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha} \quad . \qquad (3.1.57)$$

Note

$$(\bar\sigma^{o})^{\dot\alpha\alpha} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{\dot\alpha\alpha} \quad ; \quad (\bar\sigma^{1})^{\dot\alpha\alpha} = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}_{\dot\alpha\alpha}$$
$$= +(\sigma^{o})_{\dot\alpha\alpha} \qquad\qquad = -(\sigma^{1})_{\dot\alpha\alpha} \qquad (3.1.58)$$
$$(\bar\sigma^{2})^{\dot\alpha\alpha} = \begin{bmatrix}0 & i\\-i & 0\end{bmatrix}_{\dot\alpha\alpha} \quad ; \quad (\bar\sigma^{3})^{\dot\alpha\alpha} = \begin{bmatrix}-1 & 0\\0 & +1\end{bmatrix}_{\dot\alpha\alpha}$$
$$= -(\sigma^{2})_{\dot\alpha\alpha} \qquad\qquad = -(\sigma^{3})_{\dot\alpha\alpha} \quad .$$

We can readily derive the completeness properties of σ:

$$\sigma^{\mu}_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha} = +2g^{\mu\nu}$$
$$\sigma^{\mu}_{\alpha\dot\alpha}\bar\sigma_{\mu}^{\dot\beta\beta} = +2\delta^{\beta}_{\alpha}\delta^{\dot\beta}_{\dot\alpha} \quad . \qquad (3.1.59)$$

Further products of two yield

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta} = 2g^{\mu\nu}\delta_{\alpha}^{\ \beta}$$
$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} + \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}_{\alpha\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}} \ . \qquad (3.1.60)$$

If $S \in SL(2,C)$ with matrix elements $S_{\alpha}^{\ \beta}$ (α = rows, β = columns) and $\not{x}$ a Hermitian matrix; then we define the transformed matrix $\not{x}'$ as

$$\not{x}'_{\alpha\dot{\alpha}} = S_{\alpha}^{\ \beta}\not{x}_{\beta\dot{\beta}}S^{*\dot{\beta}}_{\ \dot{\alpha}} \ . \qquad (3.1.61)$$

Since $\det S = S_1^{\ 1}S_2^{\ 2} - S_1^{\ 2}S_2^{\ 1} = 1$

we have

$$\det \not{x}' = \det \not{x} \qquad (3.1.62)$$

but

$$\begin{aligned} \det \not{x} &= (\chi_0+\chi_3)(\chi_0-\chi_3) - (\chi_1-i\chi_2)(\chi_1+i\chi_2) \\ &= (\chi^0)^2 - (\chi^1)^2 - (\chi^2)^2 - (\chi^3)^2 \\ &= \chi_{\mu}\chi^{\mu} \\ &= \det \not{x}' = \chi'_{\mu}\chi'^{\mu} \ . \end{aligned} \qquad (3.1.63)$$

Thus the transformation.

$$\not{x}' = S\not{x}S^{\dagger} \qquad (3.1.64)$$

corresponds to a Lorentz transformation $\Lambda^{\mu\nu}$, that is

$$\begin{aligned} \chi'^{\mu} &= \frac{1}{2}\,\chi'_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \\ &= \frac{1}{2}\, S_{\alpha}^{\ \beta}\chi_{\beta\dot{\beta}} S^{*\dot{\beta}}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \\ &= \frac{1}{2}\, S_{\alpha}^{\ \beta} S^{*\dot{\beta}}_{\dot{\alpha}}\bar{\sigma}^{\nu}_{\ \beta\dot{\beta}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\chi_{\nu} \\ &\equiv \Lambda^{\mu\nu}\chi_{\nu} \end{aligned} \tag{3.1.65}$$

where

$$\Lambda^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr} S\sigma^{\nu} S^{\dagger}\bar{\sigma}^{\mu}$$

that is

$$\begin{aligned} \Lambda^{\mu\nu}\sigma_{\mu\alpha\dot{\alpha}} &= \frac{1}{2}\, S_{\beta}^{\ \gamma} S^{*\dot{\gamma}}_{\dot{\beta}}\sigma^{\nu}_{\ \gamma\dot{\gamma}}\bar{\sigma}^{\mu\dot{\beta}\beta}\sigma_{\mu\alpha\dot{\alpha}} \\ &= S_{\beta}^{\ \gamma} S^{*\dot{\gamma}}_{\dot{\beta}}\sigma^{\nu}_{\ \gamma\dot{\gamma}}\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}} \\ &= S_{\alpha}^{\ \gamma} S^{*\dot{\gamma}}_{\dot{\alpha}}\sigma^{\nu}_{\ \gamma\dot{\gamma}}\ , \text{ or simply } \ \Lambda^{\mu\nu}\sigma_{\mu} = S\sigma^{\nu} S^{\dagger} \end{aligned} \tag{3.1.67}$$

For every element $\pm S$ of SL(2,C) there is an element Λ of the Lorentz group, the mapping SL(2,C) $\to L_{\uparrow}^{+}$ is 2-1 since $\pm S \to \Lambda$.

We can define spinor representations of the Lorentz group by the transformation laws

$$\begin{aligned} \psi'_{\alpha}(x') &= S_{\alpha}^{\ \beta}\psi_{\beta}(x) \\ \psi'^{\alpha}(x') &= S^{-1\alpha}_{\ \ \beta}\psi^{\beta}(x) \end{aligned} \tag{3.1.68}$$

where $S_{\alpha}^{\ \beta} S^{-1\gamma}_{\ \ \beta} = \delta_{\alpha}^{\gamma}$ and ψ_{α} and ψ^{α} are two component complex spinors transforming (as we will see) as $(\frac{1}{2},0)$ representation of the Lorentz group. Similarly we can use S^* and $(S^\dagger)^{-1}$ to define spinors $\bar\psi$ which will transform as $(0,\frac{1}{2})$ representations of the Lorentz group

$$\begin{aligned}\bar\psi'_{\dot\alpha}(x') &= S^{*\dot\beta}_{\dot\alpha}\bar\psi_{\dot\beta}(x) \\ \bar\psi'^{\dot\alpha}(x') &= (S^*)^{-1\dot\alpha}_{\dot\beta}\bar\psi^{\dot\beta}(x) \quad .\end{aligned} \tag{3.1.69}$$

As with tensors higher rank spinors transform just like products of the basics rank 1 spinors, for example

$$\psi'_{\alpha_1\ldots\alpha_n}(x') = S_{\alpha_1}^{\beta_1}\ldots S_{\alpha_n}^{\beta_n}\psi_{\beta_1\ldots\beta_n}(x)$$

or

$$\psi'_{\alpha_1\ldots\alpha_n\,\dot\alpha_1\ldots\dot\alpha_m}(x') = S_{\alpha_1}^{\beta_1}\ldots S_{\alpha_n}^{\beta_m}(S^*)_{\dot\alpha_1}^{\dot\beta_1}\ldots(S^*)_{\dot\alpha_m}^{\dot\beta_m}\,\psi_{\beta_1\ldots\beta_n\,\dot\beta_1\ldots\dot\beta_m}(x) \ .$$

Since S is special, i.e. det S = 1 we have

$$\begin{aligned} S^{-1\beta}_{\ \ \alpha} &= \begin{bmatrix} S_2^2 & -S_1^2 \\ -S_2^1 & S_1^1 \end{bmatrix}_{\alpha\beta} \\ &= -\varepsilon_{\alpha\gamma}\varepsilon^{\beta\delta}S_\delta^{\ \gamma} \end{aligned} \tag{3.1.70}$$

that is $S^{-1} = \varepsilon S^T\varepsilon$ and ε is an invariant 2nd rank spinor, with

$\varepsilon'_{\alpha\beta} = S_\alpha^{\ \gamma} S_\beta^{\ \delta} \varepsilon_{\gamma\delta}$ we find

$$\varepsilon' = S\varepsilon S^T = SS^{-1}\varepsilon = \varepsilon . \tag{3.1.71}$$

(Explicitly $\varepsilon'_{12} = S_1^{\ 1} S_2^{\ 2} \varepsilon_{12} + S_1^{\ 2} S_2^{\ 1} \varepsilon_{21} = -\det S = -1 = \varepsilon_{12}$.

so $\varepsilon'_{\alpha\beta} = \varepsilon_{\alpha\beta}$

Hence we can use ε to lower and raise indices of the spinors

$$\begin{aligned} \psi^\alpha &= \varepsilon^{\alpha\beta}\psi_\beta , & \bar{\psi}^{\dot{\alpha}} &= \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \\ \psi_\alpha &= \varepsilon_{\alpha\beta}\psi^\beta , & \bar{\psi}_{\dot{\alpha}} &= \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \end{aligned} \tag{3.1.72}$$

where for example

$$\begin{aligned} \psi'^\alpha(x') &= \varepsilon^{\alpha\beta}\psi'_\beta(x') = \varepsilon^{\alpha\beta} S_\beta^{\ \gamma}\psi_\gamma(x) \\ &= \varepsilon^{\alpha\beta} S_\beta^{\ \gamma}\varepsilon_{\gamma\delta}\psi^\delta(x) \\ &= -\varepsilon_{\delta\gamma}\varepsilon^{\alpha\beta} S_\beta^{\ \gamma}\psi^\delta(x) \\ &= S_\delta^{-1\alpha}\psi^\delta(x) . \end{aligned} \tag{3.1.73}$$

Thus we have the Lorentz scalars

$$\begin{aligned} \psi'^\alpha(x')\psi'_\alpha(x') &= S_\beta^{-1\alpha} S_\alpha^{\ \gamma}\psi^\beta(x)\psi_\gamma(x) \\ &= \delta_\beta^\gamma\psi^\beta(x)\psi_\gamma(x) = \psi^\alpha(x)\psi_\alpha(x) . \end{aligned} \tag{3.1.74}$$

Similarly for $\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$. Also using the properties of the Pauli matrices we have

$$\psi'^\alpha(x')\sigma^\mu_{\alpha\dot{\alpha}}\partial'_\mu\bar{\psi}'^{\dot{\alpha}}(x') = S_\beta^{-1\alpha}(S^*)_{\dot{\gamma}}^{-1\dot{\alpha}}\Lambda_\mu^{-1\nu}\psi^\beta(x)\sigma^\mu_{\alpha\dot{\alpha}}\partial_\nu\bar{\psi}^{\dot{\gamma}}(x) . \tag{3.1.75}$$

But

$$S_{\beta}^{-1\alpha}\sigma_{\mu\alpha\dot{\alpha}}(S^*)^{-1\dot{\alpha}}_{\dot{\gamma}}\Lambda^{\mu\nu}$$

$$= S_{\beta}^{-1\alpha}\sigma_{\mu\alpha\dot{\alpha}}(S^*)^{-1\dot{\alpha}}_{\dot{\gamma}}(\frac{1}{2}\ \mathrm{Tr}\ S\sigma^{\nu}S^{+}\bar{\sigma}^{\mu})$$

$$= S_{\beta}^{-1\alpha}(S^*)^{-1\dot{\alpha}}_{\dot{\gamma}}[(S\sigma^{\nu}S^{+})_{\delta\dot{\delta}}]\sigma_{\mu\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\delta}\delta}\ \frac{1}{2} \qquad (3,1,76)$$

$$= S_{\beta}^{-1\alpha}(S^*)^{-1\dot{\alpha}}_{\dot{\gamma}}(S\sigma^{\nu}S^{+})_{\delta\dot{\delta}}\delta^{\delta}_{\alpha}\delta^{\dot{\delta}}_{\dot{\alpha}}$$

$$= S_{\beta}^{-1\alpha}(S^*)^{-1\dot{\alpha}}_{\dot{\gamma}}(S\sigma^{\nu}S^{+})_{\alpha\dot{\alpha}}$$

$$= \sigma^{\nu}_{\beta\dot{\gamma}}$$

Thus

$$\psi'^{\alpha}(x')\sigma^{\mu}_{\alpha\dot{\alpha}}\partial'_{\mu}\bar{\psi}'^{\dot{\alpha}}(x') = \psi^{\alpha}(x)\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\bar{\psi}^{\dot{\alpha}}(x) \qquad (3,1,77)$$

that is $\psi\not{\partial}\bar{\psi}$ is also Lorentz invariant.

Finally let's consider infinitesimal Lorentz transformations

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu}x_{\nu} \qquad (3,1,78)$$

where now

$$S_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} + \Sigma_{\alpha}^{\beta}$$

$$S^{*\dot{\beta}}_{\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}} + \Sigma^{*\dot{\beta}}_{\dot{\alpha}} \quad . \qquad (3,1,79)$$

Note

$$\varepsilon_{\alpha\beta} = S^{\gamma}_{\alpha}S^{\delta}_{\beta}\varepsilon_{\gamma\delta}$$

$$= (\delta^{\gamma}_{\alpha}+\Sigma^{\gamma}_{\alpha})(\delta^{\delta}_{\beta}+\Sigma^{\delta}_{\beta})\varepsilon_{\gamma\delta}$$

$$= [\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}+\Sigma^{\gamma}_{\alpha}\delta^{\delta}_{\beta}+\Sigma^{\delta}_{\beta}\delta^{\gamma}_{\alpha}]\varepsilon_{\gamma\delta}$$

$$= \varepsilon_{\alpha\beta} + \varepsilon_{\gamma\beta}\Sigma^{\gamma}_{\alpha} + \varepsilon_{\alpha\gamma}\Sigma^{\gamma}_{\beta} \quad \text{which implies} \tag{3.1.80}$$

$$\Sigma_{\beta\alpha} - \Sigma_{\alpha\beta} = 0 , \qquad \Sigma \text{ is symmetric.}$$

Now given $\omega^{\mu\nu}$ we desire $\Sigma_{\alpha\beta}$; using $\Lambda^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\, S\sigma^{\nu}S^{\dagger}\bar{\sigma}^{\mu}$ we find

$$g^{\mu\nu} + \omega^{\mu\nu} = \frac{1}{2}\,(\delta^{\beta}_{\alpha}+\Sigma^{\beta}_{\alpha})\sigma^{\nu}_{\beta\dot{\beta}}(\delta^{\dot{\beta}}_{\dot{\alpha}}+\Sigma^{*\dot{\beta}}_{\dot{\alpha}})\bar{\sigma}^{\mu\dot{\alpha}\alpha}$$

$$= \frac{1}{2}\,\sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \frac{1}{2}\,\Sigma^{\beta}_{\alpha}\sigma^{\nu}_{\beta\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \frac{1}{2}\,\sigma^{\nu}_{\alpha\dot{\beta}}\Sigma^{*\dot{\beta}}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}$$

$$= g^{\mu\nu} + \frac{1}{2}\,\Sigma^{\beta}_{\alpha}\sigma^{\nu}_{\beta\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \frac{1}{2}\,\Sigma^{*\dot{\beta}}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} \quad . \tag{3.1.81}$$

Thus we must find a solution for

$$\omega^{\mu\nu} = \frac{1}{2}\,\Sigma^{\beta}_{\alpha}\sigma^{\nu}_{\beta\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \frac{1}{2}\,\Sigma^{*\dot{\beta}}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} \quad . \tag{3.1.82}$$

Multiplying by σ_{μ} and $\bar{\sigma}_{\nu}$ we have

$$\sigma_{\mu\gamma\dot{\gamma}}\bar{\sigma}^{\dot{\delta}\delta}_{\nu}\,\omega^{\mu\nu}$$

$$= 2\Sigma^{\delta}_{\gamma}\delta^{\dot{\delta}}_{\dot{\gamma}} + 2\Sigma^{*\dot{\delta}}_{\dot{\gamma}}\delta^{\delta}_{\gamma}$$

$$= \frac{1}{2}\,[\sigma_{\mu\gamma\dot{\gamma}}\bar{\sigma}^{\dot{\delta}\delta}_{\nu} - \sigma_{\nu\gamma\dot{\gamma}}\bar{\sigma}^{\dot{\delta}\delta}_{\mu}]\,\omega^{\mu\nu} \tag{3.1.83}$$

Using $\Sigma_\alpha^\alpha = 0 = \Sigma^{*\dot\alpha}_{\dot\alpha}$ we find

$$\Sigma_\gamma^\delta = \frac{1}{8}[\sigma_{\mu\gamma\dot\gamma}\bar\sigma_\nu^{\dot\gamma\delta} - \sigma_{\nu\gamma\dot\gamma}\bar\sigma_\mu^{\dot\gamma\delta}]\omega^{\mu\nu} \qquad (3.1.84)$$

and similarly

$$\Sigma^{*\dot\delta}_{\dot\gamma} = \frac{1}{8}[\bar\sigma_\nu^{\dot\delta\gamma}\sigma_{\mu\gamma\dot\gamma} - \bar\sigma_\mu^{\dot\delta\gamma}\sigma_{\nu\gamma\dot\gamma}]\omega^{\mu\nu} \quad . \qquad (3.1.85)$$

These commutators of Pauli matrices we define as

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{i}{2}[\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}]$$
$$(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \equiv \frac{i}{2}[\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}] \ . \qquad (3.1.86)$$

Thus

$$\Sigma_\alpha^\beta = \frac{-i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^{\ \beta}$$
$$\Sigma^{*\dot\beta}_{\dot\alpha} = \frac{+i}{4}\omega_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\beta}_{\ \dot\alpha} \qquad (3.1.87)$$

From our definitions of $\sigma^\mu_{\alpha\dot\alpha}$ we see that

$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} = g^{\mu\nu}\delta_\alpha^\beta - i(\sigma^{\mu\nu})_\alpha^{\ \beta}$$
$$\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} = g^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta} - i(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \ . \qquad (3.1,88)$$

The infinitesimal spinor transformations are given by

$$\psi'_\alpha(x') = S_\alpha^{\ \beta}\psi_\beta(x) = \psi_\alpha(x) - \frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x)$$
$$\equiv \psi_\alpha(x) - \frac{\omega_{\mu\nu}}{2}(D^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x) \qquad (3.1.89)$$

and

$$\bar{\psi}'_{\dot{\alpha}}(x') = S^{*\dot{\beta}}_{\dot{\alpha}}\bar{\psi}_{\dot{\beta}}(x) = \bar{\psi}_{\dot{\alpha}}(x) + \frac{i}{4}\,\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\psi}_{\dot{\beta}}(x)$$

$$\equiv \bar{\psi}_{\dot{\alpha}}(x) - \frac{\omega_{\mu\nu}}{2}\,(\bar{D}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\psi}_{\dot{\beta}}(x) \ . \qquad (3.1.90)$$

Thus the spinor representation is given by

$$(D^{\mu\nu})_{\alpha}^{\ \beta} \equiv \frac{i}{2}\,(\sigma^{\mu\nu})_{\alpha}^{\ \beta} \qquad (3.1.91)$$

and

$$(\bar{D}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}} \equiv -\frac{i}{2}\,(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}} \qquad (3.1.92)$$

and we must check that these obey the Lorentz algebra as the tensor representations did

$$[D^{\mu\nu},D^{\rho\sigma}]_{\alpha}^{\ \beta} = (D^{\mu\nu})_{\alpha}^{\ \gamma}(D^{\rho\sigma})_{\gamma}^{\ \beta} - (D^{\rho\sigma})_{\alpha}^{\ \gamma}(D^{\mu\nu})_{\gamma}^{\ \beta} = -\frac{1}{4}\,[\sigma^{\mu\nu},\sigma^{\rho\sigma}]_{\alpha}^{\ \beta} \ . \qquad (3.1.93)$$

Expanding in terms of $\sigma^{\mu}_{\alpha\dot{\alpha}}$ we have

$$[\sigma^{\mu\nu},\sigma^{\rho\sigma}]_{\alpha}^{\ \beta} = \frac{-1}{4}\,[(\sigma^{\mu}\bar{\sigma}^{\nu}-\sigma^{\nu}\bar{\sigma}^{\mu}),(\sigma^{\rho}\bar{\sigma}^{\sigma}-\sigma^{\sigma}\bar{\sigma}^{\rho})]_{\alpha}^{\ \beta}$$

$$= -2i[g^{\mu\rho}\sigma^{\nu\sigma}-g^{\mu\sigma}\sigma^{\nu\rho}+g^{\nu\sigma}\sigma^{\mu\rho}-g^{\nu\rho}\sigma^{\mu\sigma}]_{\alpha}^{\ \beta} \ . \qquad (3.1.94)$$

Thus

$$[D^{\mu\nu},D^{\rho\sigma}]_{\alpha}^{\ \beta} = g^{\mu\rho}(D^{\nu\sigma})_{\alpha}^{\ \beta} - g^{\mu\sigma}(D^{\nu\rho})_{\alpha}^{\ \beta} + g^{\nu\sigma}(D^{\mu\rho})_{\alpha}^{\ \beta} - g^{\nu\rho}(D^{\mu\sigma})_{\alpha}^{\ \beta} \qquad (3.1.95)$$

which is the Lorentz algebra and ψ_{α} is the $(\frac{1}{2},0)$ spinor representation of the Lorentz group. Similarly the complex conjugate dotted spinors are the $(0,\frac{1}{2})$ representation of the Lorentz group with the $(\bar{D}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}$ also obeying the Lorentz algebra.

As with tensors we find the intrinsic variation of a spinor field is given by

$$\bar{\delta}\psi_\alpha = \delta\psi_\alpha - \delta x^\mu \partial_\mu \psi_\alpha$$
$$\bar{\delta}\bar{\psi}_{\dot{\alpha}} = \delta\bar{\psi}_{\dot{\alpha}} - \delta x^\mu \partial_\mu \bar{\psi}_{\dot{\alpha}} . \tag{3.1.96}$$

For Poincaré transformations

$$x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu + \varepsilon^\mu \tag{3.1.97}$$

we find

$$\bar{\delta}\psi_\alpha = + \frac{\omega_{\mu\nu}}{2} [(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta_\alpha^\beta - (D^{\mu\nu})_\alpha^{\ \beta}]\psi_\beta(x) - \varepsilon^\mu\partial_\mu\psi_\alpha(x)$$
$$\equiv - \frac{i\omega_{\mu\nu}}{2} (M^{\mu\nu})_\alpha^{\ \beta}\psi_\beta + i\varepsilon^\mu P_\mu \psi_\alpha \tag{3.1.98}$$

and

$$\bar{\delta}\bar{\psi}_{\dot{\alpha}} = + \frac{\omega_{\mu\nu}}{2} [(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta_{\dot{\alpha}}^{\dot{\beta}} - (\bar{D}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}]\bar{\psi}_{\dot{\beta}}(x) - \varepsilon^\mu\partial_\mu\bar{\psi}_{\dot{\alpha}}(x)$$
$$\equiv - \frac{i\omega_{\mu\nu}}{2} (M^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\psi}_{\dot{\beta}} + i\varepsilon^\mu P_\mu \bar{\psi}_{\dot{\alpha}} . \tag{3.1.99}$$

and as before the P_μ and $M_{\mu\nu}$ obey the defining commutation relations of the Poincaré group.

For finite transformations

$$x'^\mu = \Lambda^{\mu\nu} x_\nu + a^\mu \tag{3.1.100}$$

we again exponentiate the operators to obtain

$$\psi'_\alpha(x) = S_\alpha^{\ \beta}\psi_\beta\left(\Lambda^{-1}(x-a)\right)$$

$$= \left[e^{+ia^\mu P_\mu}\, e^{-\frac{i\omega_{\mu\nu}(\Lambda)}{2}M^{\mu\nu}}\right]_\alpha^{\ \beta} \psi_\beta(x) \qquad (3.1.101)$$

and

$$\bar{\psi}'_{\dot\alpha}(x) = S^{*\ \dot\beta}_{\dot\alpha}\bar{\psi}_{\dot\beta}\left(\Lambda^{-1}(x-a)\right)$$

$$= \left[e^{+ia^\mu P_\mu}\, e^{-\frac{i\omega_{\mu\nu}(\Lambda)}{2}M^{\mu\nu}}\right]_{\dot\alpha}^{\ \dot\beta} \bar{\psi}_{\dot\beta}(x) \qquad (3.1.102)$$

and $$\Lambda^{\mu\nu} = \frac{1}{2}\, S_\alpha^{\ \beta}\sigma^\nu_{\beta\dot\beta}S^{*\ \dot\beta}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha} \ . \qquad (3.1.103)$$

Thus we have found all representations of the Poincaré algebra.

Finally let's compare our 2-component spinors with the usual Dirac 4-component spinors. We can realize the algebra defining the Dirac matrices $\gamma^{\mu a}_{\ \ b}$ by using the Pauli matrices--this representation is called the Weyl basis.

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \qquad (3.1.104)$$

that is

$$\gamma^{0a}_{\ \ b} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{ab}$$

$$\gamma^{1a}_{\ \ b} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}_{ab}$$

(3.1.105)

$$\gamma^{2a}_{\ \ b} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}_{ab}$$

$$\gamma^{3a}_{\ \ b} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix}_{ab}$$

Thus the γ^{μ} obey the defining Dirac anti-commutation relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} . \qquad (3.1.106)$$

Also we can define

$$\gamma_5 = +i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (3.1.107)$$

$$= \begin{bmatrix} -\sigma^0 & 0 \\ 0 & +\bar{\sigma}^0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & +1 & \\ 0 & & & +1 \end{bmatrix}$$

In this basis the Dirac spinor ψ_D^a is given in terms of two Weyl spinors;

$$\psi_D^a = \begin{bmatrix} \psi_\alpha \\ \bar{\phi}^{\dot\alpha} \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \bar{\phi}^1 \\ \bar{\phi}^2 \end{bmatrix}^a \quad . \tag{3.1.108}$$

Thus under a Lorentz transformation

$$\psi_D'^a(x') = L^a_b \psi^b_D(x) \tag{3.1.109}$$

where

$$L^a_b = \begin{bmatrix} S_\alpha^\beta & 0 \\ 0 & (S^*)^{-1\dot\alpha}_{\dot\beta} \end{bmatrix}_{ab} , \tag{3.1.110}$$

hence

$$\Lambda^{\mu\nu}\gamma^a_{\mu b} = L^a_c \gamma^{\nu c}_d L^{-1d}_b \tag{3.1.111}$$

as usual for Dirac spinors.

For left and right handed spinors we have

$$\psi^a_{DL} = (\tfrac{1}{2}(1-\gamma_5)\psi_D)^a$$

$$= \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \bar{\phi}^1 \\ \bar{\phi}^2 \end{bmatrix}^a = \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{bmatrix}^a \tag{3.1.112}$$

and

$$\psi_{DR}^{a} = \left(\tfrac{1}{2}(1+\gamma_5)\psi_D\right)^a$$

$$= \begin{bmatrix} 0 \\ 0 \\ \bar{\phi}^1 \\ \bar{\phi}^2 \end{bmatrix}^a \quad . \tag{3.1.113}$$

Thus we have that ψ_{DL} corresponds to our $(\frac{1}{2},0)$ spinor ψ_α while ψ_{DR} corresponds to our $(0,\frac{1}{2})$ spinor $\bar{\phi}^{\dot{\alpha}}$. If the Dirac spinor is a Majorana spinor, ψ_M then

$$\psi_M^C = \psi_M = C\bar{\psi}_M^T \tag{3.1.114}$$

with

$$C = i\gamma^2\gamma^0$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} i\sigma^2 & 0 \\ 0 & i\bar{\sigma}^2 \end{bmatrix} \tag{3.1.115}$$

so

$$(C\bar{\psi}_M^T) = (i\gamma^2\gamma^0\gamma^0\psi^*)$$

$$= \begin{bmatrix} 0 & i\sigma^2_{\alpha\dot{\alpha}} \\ i\bar{\sigma}^{2\dot{\alpha}\alpha} & 0 \end{bmatrix} \begin{bmatrix} \bar{\psi}_\alpha \\ \phi^{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} \phi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{bmatrix} = \psi_M = \begin{bmatrix} \psi_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{bmatrix}, \tag{3.1.116}$$

hence $\bar{\phi}^{\dot{\alpha}} = \bar{\psi}^{\dot{\alpha}}$ and

$$\psi_M = \begin{bmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{bmatrix} \quad . \tag{3.1.117}$$

Wess and Zumino found that the most general symmetry of the S-matrix involves charges which obey both commutation and anti-commutation relations. Such an algebra is called a graded Lie algebra. These algebras are generalizations of the Poincaré algebra. The simplest (N=1) supersymmetry (SUSY) algebra involves the generators of the Poincaré group P^μ, the generators of translations, $M^{\mu\nu}$, the generators of Lorentz transformations, and two, anti-commuting (Fermionic), spinor charges Q_α and $\bar{Q}_{\dot{\alpha}}$, the generators of supersymmetry transformations. The graded Lie algebra of supersymmetry consists of the Poincaré algebra

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\lambda] &= i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma}) \ . \end{aligned} \tag{3.1.118}$$

Plus the anti-commutation relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= +2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \\ \{Q_\alpha, Q_\beta\} &= 0 = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} \end{aligned} \tag{3.1.119}$$

and the fact that the SUSY charges are spinors

$$\begin{aligned} [M^{\mu\nu}, Q_\alpha] &= -\frac{1}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \\ [M^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= +\frac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}} \bar{Q}_{\dot{\beta}} \end{aligned} \tag{3.1.120}$$

and finally the trivial zero commutators

$$[Q_\alpha, P^\mu] = [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \ . \tag{3.1.121}$$

When looking for representations of this algebra we note that $P^2 = P_\mu P^\mu$ still commutes with all the generators P_μ, $M_{\mu\nu}$, Q_α, $\bar{Q}_{\dot\alpha}$. So we expect fields in a super multiplet to have the same mass. However $W^2 = W_\mu W^\mu$ (where $W^\mu = \frac{i}{4}\,\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$ the Pauli-Lubanski covariant spin operator) does not commute with the SUSY generators. Thus the particles in the same super multiplet will have different spins. Fermions and bosons will be combined in the same super multiplet and will have the same mass.

We would like to represent the SUSY algebra by means of linear differential operators as we did for the Poincaré generators P_μ and $M_{\mu\nu}$. Since we now have anti-commuting charges we must extend space-time (x^μ) to include anti-commuting spinor parameters $(\theta_\alpha,\bar\theta_{\dot\alpha})$ to form superspace. A point in superspace is defined by

$$z^M = (x^\mu,\theta_\alpha,\bar\theta^{\dot\alpha})$$

where the $\theta^\alpha,\bar\theta_{\dot\alpha}$ are (two component, complex) Weyl spinors which anti-commute, that is are elements of a Grassmann algebra.

$$\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha \quad \text{and since } \alpha=1,2 \text{ we find}$$

$$\theta^\alpha\theta^\beta\theta^\gamma = 0 \qquad \text{etc.}$$

We can define differentiation with respect to the anti-commuting parameters by the "Taylor" formula

$$\phi(\theta+\delta\theta) = \phi(\theta) + \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\phi(\theta) \tag{3.1.122}$$

$$\phi(\bar\theta+\delta\bar\theta) = \phi(\bar\theta) - \delta\bar\theta_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\phi(\bar\theta)$$

choosing $\phi(\theta) = \theta^{\alpha}$ etc. we find

$$\frac{\partial}{\partial\theta^{\alpha}}\theta^{\beta} = \delta^{\beta}_{\alpha} \qquad \frac{\partial}{\partial\theta_{\alpha}}\theta_{\beta} = -\delta^{\alpha}_{\beta}$$

$$\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}} \quad . \tag{3.1.123}$$

Using these derivatives we can define linear differential operators on functions of $x^{\mu},\theta^{\alpha},\bar{\theta}_{\dot{\alpha}}$. The SUSY algebra generators can then be represented as linear superspace differential operators acting on a super field ϕ. As before space-time translations are given by space-time derivatives, etc.

1) Translations: $P^{\mu}\phi = i\partial^{\mu}\phi$

2) Rotations-Lorentz transformations

$$M_{\mu\nu}\phi = i[x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu} - \frac{i}{2}\,\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta} + \frac{i}{2}\,\bar{\theta}\bar{\sigma}_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}]\,\phi \tag{3.1.124}$$

3) SUSY

$$Q_{\alpha}\phi = i[\frac{\partial}{\partial\theta} + i\not{\partial}\bar{\theta}]_{\alpha}\phi$$

$$\bar{Q}_{\dot{\alpha}}\phi = i[-\frac{\partial}{\partial\bar{\theta}} - i\theta\not{\partial}]_{\dot{\alpha}}\phi$$

where $\phi = \phi(x,\theta,\bar{\theta})$ and we use the summation convention for spinors

$$\psi\chi \equiv \psi^{\alpha}\chi_{\alpha} = -\psi_{\alpha}\chi^{\alpha}$$

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \quad . \tag{3.1.125}$$

So

$$(\not{\partial}\bar{\theta})_{\alpha} = \partial_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$$

for example. Hence we can check the SUSY algebra

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\} = [\frac{\partial}{\partial\theta^\alpha} + i\cancel{\partial}_{\alpha\dot\beta}\bar{\theta}^{\dot\beta}][\frac{\partial}{\partial\bar{\theta}_{\dot\alpha}} + i\theta^\beta\cancel{\partial}_{\beta\dot\alpha}] + \bar{Q}_{\dot\alpha}Q_\alpha \tag{3.1.126}$$

$$= +2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu = +2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

$$[M^{\mu\nu},Q_\alpha] = +\frac{1}{2}(\bar{\theta}\bar{\sigma}^{\mu\nu})_{\dot\beta}\frac{\partial}{\partial\bar{\theta}_{\dot\beta}}\cancel{\partial}_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha} + (\frac{-i}{2})\frac{\partial}{\partial\theta^\alpha}\theta^\beta(\sigma^{\mu\nu}\frac{\partial}{\partial\theta})_\beta$$

$$+ i(\sigma^\lambda\bar{\theta})_\alpha(g^\mu_\lambda\partial^\nu - g^\nu_\lambda\partial^\mu)$$

$$= -\frac{1}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta}[i(\frac{\partial}{\partial\theta^\beta} + i(\cancel{\partial}\bar{\theta})_\beta)] \tag{3.1.127}$$

$$-\frac{1}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta}\cancel{\partial}_{\beta\dot\beta}\bar{\theta}^{\dot\beta} - \frac{1}{2}\bar{\theta}_{\dot\alpha}(\bar{\sigma}^{\mu\nu})^{\dot\alpha\dot\beta}\cancel{\partial}_{\alpha\dot\beta} + i(\sigma^\mu_{\alpha\dot\alpha}g^{\nu\lambda} - \sigma^\nu_{\alpha\dot\alpha}g^{\mu\lambda})\bar{\theta}^{\dot\alpha}\partial_\lambda$$

from our σ^μ algebra we check that

$$(\sigma^{\mu\nu}\sigma^\lambda)_{\alpha\dot\alpha} - (\sigma^\lambda\bar{\sigma}^{\mu\nu})_{\alpha\dot\alpha} = 2i(\sigma^\mu_{\alpha\dot\alpha}g^{\nu\lambda} - \sigma^\nu_{\alpha\dot\alpha}g^{\mu\lambda}) \ . \tag{3.1.128}$$

So $[M^{\mu\nu},Q_\alpha] = -\frac{1}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta$ as required.

This is called the <u>real representation</u> for the SUSY algebra. There are 2 other representations of the algebra that will be quite useful. Instead of

1) $$Q_\alpha = i(\frac{\partial}{\partial\theta} + i\cancel{\partial}\bar{\theta})_\alpha$$

$$\bar{Q}_{\dot\alpha} = -i(\frac{\partial}{\partial\bar{\theta}} + i\theta\cancel{\partial})_{\dot\alpha}$$

called the Real Representation

(3.1.129)

we could choose

$$2)\quad Q_\alpha = i\frac{\partial}{\partial\theta^\alpha}$$

$$\bar{Q}_{\dot\alpha} = -i\left(\frac{\partial}{\partial\bar\theta} + 2i\theta\not{\partial}\right)_{\dot\alpha}$$

called the Chiral Representation (3.1.130)

or

$$3)\quad Q_\alpha = i\left(\frac{\partial}{\partial\theta} + 2i\not{\partial}\bar\theta\right)_\alpha$$

$$\bar{Q}_{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$

called the Anti-Chiral Representation (3.1.131)

as before for each representation

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\} = +2\sigma^\mu_{\alpha\dot\alpha}P_\mu \ . \qquad (3.1.132)$$

We can relate the superfields defined in each of these three representations. Consider

$\phi(x,\theta,\bar\theta)$ defined in the real representation,

$\phi_1(x,\theta,\bar\theta)$ defined in the chiral representation, (subscript "1" denotes the chiral representation)

$\phi_2(x,\theta,\bar\theta)$ defined in the anti-chiral representation, (subscript "2" denotes anti-chiral representation)

then

$$\begin{aligned}\phi(x,\theta,\bar{c}) &= \phi_1(x-i\theta\sigma\bar\theta,\theta,\bar\theta) \\ &= \phi_2(x+i\theta\sigma\bar\theta,\theta,\bar\theta)\end{aligned} \ , \qquad (3.1.133)$$

This can be seen by considering, for example, the SUSY charge in the chiral representation $Q_{1\alpha}$

$$Q_{1\alpha}\phi_1(x,\theta,\bar{\theta}) = i\frac{\partial}{\partial\theta^\alpha}\phi_1(x,\theta,\bar{\theta}),\quad \text{so}$$

$$e^{-i\theta\not{\gamma}\bar{\theta}}\,Q_{1\alpha}\,e^{+i\theta\not{\gamma}\bar{\theta}}\,e^{-i\theta\not{\gamma}\bar{\theta}}\,\phi_1(x,\theta,\bar{\theta}) \qquad (3.1.134)$$

$$\equiv Q_\alpha\,\phi(x,\theta,\bar{\theta}) \quad \text{in the real representation}$$

with $Q_\alpha = e^{-i\theta\not{\gamma}\bar{\theta}}\,Q_{1\alpha}\,e^{+i\theta\not{\gamma}\bar{\theta}} = i\left(\frac{\partial}{\partial\theta} + i\not{\gamma}\bar{\theta}\right)_\alpha$

and $$\phi(x,\theta,\bar{\theta}) = e^{-i\theta\not{\gamma}\bar{\theta}}\,\phi_1(x,\theta,\bar{\theta}) \qquad (3.1.135)$$

$$= \phi_1(x - i\theta\sigma\bar{\theta},\theta,\bar{\theta})$$

We can also ask what the action of the SUSY group is on ϕ; we can find this finite transformation by as usual exponentiating the generators, thus

$$\phi'(x,\theta,\bar{\theta}) = e^{i(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})}\,\phi(x,\theta,\bar{\theta})$$

$$= e^{-[\xi\frac{\partial}{\partial\theta} - \bar{\xi}\frac{\partial}{\partial\bar{\theta}} + i\xi\not{\gamma}\bar{\theta} - i\theta\not{\gamma}\bar{\xi}]}\,\phi(x,\theta,\bar{\theta}) \qquad (3.1.136)$$

and by Taylor's theorem

$$= \phi(x-i(\xi\sigma\bar{\theta}-\theta\sigma\bar{\xi}),\theta-\xi,\bar{\theta}-\bar{\xi}) \ . \qquad (3.1.137)$$

We see that SUSY transformations induce translations in superspace

$$x'^{\mu} = x^{\mu} + i(\xi\sigma^{\mu}\bar{\theta}-\theta\sigma^{\mu}\bar{\xi})$$

$$\theta'^{\alpha} = \theta^{\alpha} + \xi^{\alpha} \qquad (3.1.138)$$

$$\bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \ .$$

Note, $[i(\xi\sigma^{\mu}\bar{\theta}-\theta\sigma^{\mu}\bar{\xi})]^{*} = i(\xi\sigma^{\mu}\bar{\theta}-\theta\sigma^{\mu}\bar{\xi})$

is real; thus $\phi(x,\theta,\bar{\theta})$ can be taken as real $\phi = \phi^{*}$; this as we will see is called a vector superfield, it forms a real representation. In the chiral representation x^{μ} is translated by a pure imaginary vector

$$\phi_1(x,\theta,\bar{\theta}) = \phi(x+i\theta\sigma\bar{\theta},\theta,\bar{\theta})$$

$$(i\theta\sigma\bar{\theta})^{*} = -i\theta\sigma\bar{\theta} \ ,$$

so ϕ_1 and ϕ_2 transform as complex representations.

[Complex conjugation changes $\theta_{\alpha} \to \bar{\theta}_{\dot{\alpha}}$ and also interchanges the order of Grassmann spinors, e.g.

$$[\theta^{\alpha}\chi_{\alpha}]^{*} = \bar{\chi}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}]$$

Before introducing SUSY covariant derivatives let's expand $\phi(x,\theta,\bar{\theta})$ in terms of ordinary fields. We can expand ϕ in powers of θ and $\bar{\theta}$. This Taylor expansion terminates after the $\theta^2\bar{\theta}^2$ power due to the anti-commutivity of $\theta,\bar{\theta}$

$$\theta^\alpha\theta^\beta\theta^\gamma = 0 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\bar{\theta}_{\dot{\gamma}} .$$

So we have

$$\phi(x,\theta,\bar{\theta}) = C(x) + \theta^\alpha\chi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} + \frac{1}{2}\theta^2 M(x) + \frac{1}{2}\bar{\theta}^2 M^\dagger(x)$$
$$+ \theta\sigma^\mu\bar{\theta}V_\mu(x) + \frac{1}{2}\theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \frac{1}{2}\bar{\theta}^2\theta^\alpha\lambda_\alpha(x) + \frac{1}{4}\theta^2\bar{\theta}^2 D(x) \tag{3.1.139}$$

where $\phi = \phi^*$ and the Lorentz transformation properties of the ordinary field coefficients are determined by the fact that ϕ is a Lorentz scalar and the Lorentz property of the corresponding power of θ. Also we see that the fields $C,M,M^\dagger,V_\mu,D$ are bosonic and $\chi^\alpha,\bar{\chi}_{\dot{\alpha}},\lambda^\alpha,\bar{\lambda}_{\dot{\alpha}}$ are fermionic spinors. ϕ is called a vector superfield because it contains the ordinary vector field V_μ.

We can now ask how the ordinary space-time component fields transform under SUSY; let

$$Q \equiv \xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} , \tag{3.1.140}$$

then

$$Q\phi = i[\xi\frac{\partial}{\partial\theta} - \bar{\xi}\frac{\partial}{\partial\bar{\theta}} + i(\xi\not{\partial}\bar{\theta}-\theta\not{\partial}\bar{\xi})]\phi$$
$$\equiv QC + \theta^\alpha Q\chi_\alpha + \bar{\theta}_{\dot{\alpha}}Q\bar{\chi}^{\dot{\alpha}} + \ldots + \frac{1}{4}\theta^2\bar{\theta}^2 QD \tag{3.1.141}$$
$$= i\xi\chi + i\xi\theta M + i\xi\sigma^\mu\bar{\theta}V_\mu + \ldots + \frac{1}{2}\theta\not{\partial}\bar{\xi}\bar{\theta}^2\theta\lambda .$$

Thus equating powers of θ and performing tedious θ and σ^μ algebra we find

$$
\begin{aligned}
&1)\quad QC &=&\ i[\xi\chi+\bar{\xi}\bar{\chi}] \\
&2)\quad Q\chi_\alpha &=&\ i\xi_\alpha M + i(\sigma^\mu\bar{\xi})_\alpha[V_\mu - i\partial_\mu C] \\
&3)\quad Q\bar{\chi}^{\dot{\alpha}} &=&\ i\bar{\xi}^{\dot{\alpha}}M^\dagger + i(\xi\sigma^\mu)^{\dot{\alpha}}[V_\mu + i\partial_\mu C] \\
&4)\quad QM &=&\ i\bar{\xi}\bar{\lambda} - \chi\overleftarrow{\not{\partial}}\bar{\xi} \\
&5)\quad QM^\dagger &=&\ i\xi\lambda + \xi\overrightarrow{\not{\partial}}\bar{\chi} \\
&6)\quad QV^\mu &=&\ \frac{i}{2}\lambda\sigma^\mu\bar{\xi} - \frac{i}{2}\xi\sigma^\mu\bar{\lambda} + \frac{1}{2}\partial_\nu\chi\sigma^\mu\bar{\sigma}^\nu\xi + \frac{1}{2}\partial_\nu\bar{\chi}\bar{\sigma}^\mu\sigma^\nu\bar{\xi} \\
&7)\quad Q\bar{\lambda}^{\dot{\alpha}} &=&\ i\bar{\xi}^{\dot{\alpha}}D - (\xi\not{\partial})^{\dot{\alpha}}M + \bar{\xi}^{\dot{\alpha}}\partial_\mu V^\mu + \frac{i}{2}(\bar{\sigma}^{\mu\nu}\bar{\xi})^{\dot{\alpha}}V_{\mu\nu} \\
&8)\quad Q\lambda_\alpha &=&\ i\xi_\alpha D + (\not{\partial}\bar{\xi})_\alpha M^\dagger - \xi_\alpha\partial_\mu V^\mu - \frac{i}{2}(\sigma^{\mu\nu}\xi)_\alpha V_{\mu\nu} \\
&9)\quad QD &=&\ \xi\overrightarrow{\not{\partial}}\lambda - \lambda\overleftarrow{\not{\partial}}\bar{\xi}
\end{aligned}
\qquad (3.1.142)
$$

where

$$V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu \ .$$

Notice the highest weight $\theta^2\bar{\theta}^2$ field D transforms as a total divergence under SUSY. Since the product of superfields ϕ^n is also a super field, i.e. transforms by the same $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ as does ϕ, its $\theta^2\bar{\theta}^2$ term will transform as a total divergence--thus if we integrate $\int d^4x$ the last term (D-term) we have a supersymmetry invariant. We will use this later to build actions which are SUSY invariant.

Next let's introduce susy covariant derivatives. Of course P_μ commutes with Q_α, $\bar{Q}_{\dot\alpha}$ so ∂_μ is a susy covariant derivative in all representations. We can make our spinor derivatives Susy covariant by adding ∂_μ terms with θ or $\bar\theta$. In each representation we will have to add different pieces as we did with Q_α and $\bar{Q}_{\dot\alpha}$ so that

$$\{D_\alpha, Q_\beta\} = 0 = \{D_\alpha, \bar{Q}_{\dot\alpha}\}$$

$$\{\bar{D}_{\dot\alpha}, Q_\alpha\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

where D_α and $\bar{D}_2$ are the susy covariant spinor derivatives:

1) $$\left.\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\not\partial\bar\theta)_\alpha \\ \bar{D}_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\not\partial)_{\dot\alpha} \end{aligned}\right\} \quad \text{real representation} \qquad (3.1.143)$$

2) $$\left.\begin{aligned} D_{1\alpha} &= [\frac{\partial}{\partial\theta^\alpha} - 2i(\not\partial\bar\theta)_\alpha] \\ \bar{D}_{1\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{aligned}\right\} \quad \text{chiral representation} \qquad (3.1.144)$$

3) $$\left.\begin{aligned} D_{2\alpha} &= \frac{\partial}{\partial\theta^\alpha} \\ \bar{D}_{2\dot\alpha} &= [-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i(\theta\not\partial)_{\dot\alpha}] \end{aligned}\right\} \quad \text{anti-chiral representation} \qquad (3.1.145)$$

Note $\{D_\alpha, \bar{D}_{\dot\alpha}\} = +2i\not\partial_{\alpha\dot\alpha}$ in all rep.'s.

Since $D_\alpha\phi$ or $\bar{D}_{\dot\alpha}\phi$ is a superfield if ϕ is a superfield we can use D_α, $\bar{D}_{\dot\alpha}$ to restrict the number of component fields in a superfield - this is a supersymmetric constraint

1) Chiral Superfields $S(x,\theta,\bar{\theta})$ are such that

$$\bar{D}_{\dot{\alpha}}\, S(x,\theta,\bar{\theta}) = 0 \quad . \tag{3.1.146}$$

In the chiral representation $\bar{D}_{1\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$ so this implies

$$S_1(x,\theta,\bar{\theta}) = S_1(x,\theta) \text{ only.}$$ Again

we can expand $S_1(x,\theta)$ is a power series in θ

$$S_1(x,\theta) = A(x) + \theta^{\alpha}\psi_{\alpha}(x) + \theta^2 F(x) \tag{3.1.147}$$

Since S is complex A is a complex scalar as well as F and ψ_{α} a Weyl spinor. We can transform $S_1(x,\theta)$ to the real representation; recall

$$\phi(x,\theta,\bar{\theta}) = \phi_1(x - i\theta\sigma\bar{\theta},\ \theta,\bar{\theta})$$

$$= e^{-i\theta\partial\!\!\!/\bar{\theta}}\ \phi_1(x,\phi,\bar{\phi})$$

so

$$S(x,\theta,\bar{\theta}) = e^{-\theta\partial\!\!\!/\bar{\theta}}\ S_1(x,\theta)$$

that is

$$S(x,\theta,\bar{\theta}) = e^{-i\theta\partial\!\!\!/\bar{\theta}}\ [A + \theta\psi + \theta^2 F] \tag{3.1.148}$$

is the form of the chiral field in the real rep.

We can work out the SUSY transformation for the chiral component fields in the chiral rep.

$$Q_1 = \xi^{\alpha}\ Q_{1\alpha} + \bar{\xi}_{\dot{\alpha}}\bar{Q}_1^{\dot{\alpha}}$$

$$= i[\xi \frac{\partial}{\partial\theta} - \bar{\xi}\frac{\partial}{\partial\bar{\theta}} = -2i\theta\partial\!\!\!/\bar{\xi}] \tag{3.1.149}$$

then

$$QA + \theta^{\alpha}Q\ \psi_{\alpha} + \theta^2 QF \equiv Q_1\ S_1(x,\theta)$$

$$= \quad i[\xi\frac{\partial}{\partial\theta} - \bar{\xi}\frac{\partial}{\partial\bar{\theta}} - 2i\theta\partial\!\!\!/\bar{\xi}][A + \theta\psi + \theta^2 F]$$

$$= \quad i\xi\psi + \theta^{\alpha}\, i[2\xi_{\alpha}F - 2i(\partial\!\!\!/\bar{\xi})_{\alpha}A]$$

$$- \theta^2\ \psi\overleftarrow{\partial\!\!\!/}\bar{\xi} \quad .$$

Thus

$$QA = i\xi\psi$$

$$Q\psi_\alpha = i2\xi_\alpha F + 2(\not\partial\bar{\xi})_\alpha A \qquad (3.1.150)$$

$$QF = -\psi\overleftarrow{\not\partial}\bar{\xi}$$

Again the highest weight field F transforms as a total space-time divergence under SUSY. The product of chiral fields is again chiral (chain rule for $\bar{D}$) so we can make susy invariant terms of the action by integrating the θ^2 component of a product of chiral fields over $\int d^4x$ (F-term).

2) Anti-Chiral Superfields $\bar{S}(x,\theta,\bar{\theta})$ are such that

$$D_\alpha \bar{S}(x,\theta,\bar{\theta}) = 0 \quad . \qquad (3.1.151)$$

Again since $D_{2\alpha} = \dfrac{\partial}{\partial\theta^\alpha}$ in the anti-chiral representation we find

$$\bar{S}_2(x,\theta,\bar{\theta}) = \bar{S}_2(x,\bar{\theta}) \text{ only },$$

expanding in $\bar{\theta}$ we have

$$\bar{S}_2(x,\bar{\theta}) = A^\dagger + \bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + \bar{\theta}^2 F^\dagger \qquad (3.1.152)$$

[often $\bar{A}$ and $\bar{F}$ are written instead of $A^\dagger$ and $F^\dagger$]

(Note: $(\bar{D}_{1\dot{\alpha}}S_1)^\dagger = D_{2\alpha}S_2$ so the components of $\bar{S}$ are the conjugates of S , $\bar{S} = S^\dagger$)

We can transform $\bar{S}_2(x,\bar{\theta})$ to the real representation recall

$$\phi(x,\theta,\bar{\theta}) = \phi_2(x + i\theta\sigma\bar{\theta},\theta,\bar{\theta})$$

$$= e^{+i\theta\not\partial\bar{\theta}}\,\phi_2(x,\theta,\bar{\theta})$$

So

$$\bar{S}(x,\theta,\bar{\theta}) = e^{+i\theta\not\partial\bar{\theta}}\,\bar{S}_2(x,\bar{\theta}) \quad \text{and we find}$$

$$\bar{S}(x,\theta,\bar{\theta}) = e^{+i\theta\not\partial\bar{\theta}}\,[\bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F}] \qquad (3.1.153)$$

for the anti-chiral field in the real rep.

We can find the susy transformation properties of $\bar{A}$, $\bar{\psi}$, $\bar{F}$ by taking the adjoint of those for A, ψ, F or directly in the anti-chiral rep.

$$\begin{aligned} Q_2 &\equiv \xi^{\alpha} Q_{2\alpha} + \bar{\xi}_{\dot{\alpha}} \bar{Q}_2^{\dot{\alpha}} \\ &= i[\xi \frac{\partial}{\partial\theta} - \bar{\xi} \frac{\partial}{\partial\bar{\theta}} + 2i\xi\partial\bar{\theta}] \end{aligned} \tag{3.1.154}$$

then

$$\begin{aligned} Q\bar{A} + \bar{\theta}_{\dot{\alpha}} Q\bar{\psi}^{\dot{\alpha}} + \bar{\theta}^2 Q\bar{F} &\equiv Q_2 \bar{S}_2(x,\bar{\theta}) \\ &= i\bar{\xi}\bar{\psi} + \bar{\theta}_{\dot{\alpha}} [2i\bar{\xi}^{\dot{\alpha}} \bar{F} - 2(\xi\partial)^{\dot{\alpha}}\bar{A}] \\ &\quad + \bar{\theta}^2 \xi\not{\partial} \bar{\psi} \end{aligned}$$

yielding

$$\begin{aligned} Q\bar{A} &= i\bar{\xi}\bar{\psi} \\ Q\bar{\psi}^{\dot{\alpha}} &= 2i\bar{\xi}^{\dot{\alpha}}\bar{F} - 2(\xi\not{\partial})^{\dot{\alpha}}\bar{A} \\ Q\bar{F} &= \xi\not{\partial}\bar{\psi} \end{aligned} \tag{3.1.155}$$

Again susy invariants will be made by integrating the $\bar{\theta}^2$ component of products of anti-chiral superfields over d^4x, (F-term).

Before constructing SUSY invariant actions let's introduce the SUSY invariant integration measures and delta functions. Integration of Grassmann variables is defined so that it is translationally invariant

$$\int d\theta \quad f(\theta) = \int d\theta \quad f(\theta + \xi)$$

thus $\int d\theta = 0$ by letting $f = \theta$, and normalizing $\int d\theta\theta = 1$ we have the integrals

$$\int d\theta_{\alpha} \theta^{\beta} = \delta_{\alpha}^{\beta} \qquad \text{and}$$

similarly

$$\int d\bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} . \tag{3.1.156}$$

Thus we see that integration is the same as differentiation for Grassmann variables

$$\int d\theta_\alpha = \frac{\partial}{\partial\theta^\alpha} ,$$
$$\int d\bar{\theta}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} . \tag{3.1.157}$$

We can make susy invariant measures by integrating over $\int d^4x$ and the highest $\theta,\bar{\theta}$ weight:

1) Vector measure (integration)

$$\int dV \equiv \int d^4x\, d^2\theta\, d^2\bar{\theta} \equiv \int d^4x\, d\theta^\alpha\, d\theta_\alpha\, d\bar{\theta}_{\dot{\alpha}}\, d\bar{\theta}^{\dot{\alpha}} \tag{3.1.158}$$
$$= \int d^4x \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} ,$$

we will further assume that total space-time divergences integrate to zero so we can use the covariant derivatives

$$\int dV = \int d^4x\, D^\alpha\, D_\alpha\, \bar{D}_{\dot{\alpha}}\, \bar{D}^{\dot{\alpha}} = \int d^4x\, DD\bar{D}\bar{D} . \tag{3.1.159}$$

2) Chiral measure (integration)

$$\int dS \equiv \int d^4x\, d^2\theta = \int d^4x \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} = \int d^4x\, DD \tag{3.1.160}$$

3) Anti-chiral measure (integration)

$$\int d\bar{S} \equiv \int d^4x\, d^2\bar{\theta} = \int d^4x \frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\bar{\theta}} = \int d^4x\, \bar{D}\bar{D} . \tag{3.1.161}$$

Note that

$$\int d^2\theta\, \theta^2 = -4$$
$$\int d^2\bar{\theta}\, \bar{\theta}^2 = -4 . \tag{3.1.162}$$

So for a

1) Vector superfield ϕ

$$\int dV\phi = 16 \int d^4x\, D \quad \text{and} \quad Q\int dV\phi = 0 , \tag{3.1.163}$$

2) Chiral superfield S; $\bar{D}_{\dot{\alpha}} S = 0$

$$\int dS\ S = -4 \int d^4x\ F \text{ and } Q \int dSS = 0 \quad , \tag{3.1.164}$$

3) Anti-chiral superfield $\bar{S}$; $D_\alpha \bar{S} = 0$

$$\int d\bar{S}\bar{S} = -4 \int d^4x\ \bar{F} \quad \text{and} \quad Q \int d\bar{S}\bar{S} = 0 \quad . \tag{3.1.165}$$

Since a vector x vector superfield = vector

chiral x anti-chiral superfield = vector

(anti-) chiral x vector superfield = vector

chiral x chiral superfield = chiral

anti-chiral x anti-chiral superfield = anti-chiral

we can make susy invariant terms by integrating these products over the appropriate measure e.g.

$S\bar{S}$ = vector superfield it depends on both θ in $\bar{\theta}$ in a non-trivial way hence

$\int dVS\bar{S}$ is susy invariant.

Note, fields must be in the same representation when they are multiplied together if the product is to be a superfield; so either all are in the real, all chiral, or all anti-chiral representations.

SS = chiral superfield

$= e^{-i\theta\not{\partial}\bar{\theta}}\ S_1(x,\theta)\ S_1(x,\theta)$ and

$\int dS\ S^2 = \int dS_1 S_1 S_1$, etc .

Finally when we calculate field equations we will need functional differentiation in superspace and superdelta functions,

1) $\frac{\delta\phi(1)}{\delta\phi(2)} = \delta_V(1,2)$ for a vector superfield (3.1.166)

2) $\frac{\delta S(1)}{\delta S(2)} = \delta_S(1,2)$ for a chiral superfield (3.1.167)

3) $\frac{\delta \bar{S}(1)}{\delta \bar{S}(2)} = \delta_{\bar{S}}(1,2)$ for an anti-chiral superfield, (3.1.168)

where

1) $\delta_V(1,2) \equiv \frac{1}{16}\, \theta_{12}^2 \bar{\theta}_{12}^2 \quad \delta^4(x_1 - x_2)$ (3.1.169)

and

$$\int dV(1)\ \phi(1)\ \delta_V(1,2) = \phi(2) \ ,$$

2) $\delta_S(1,2)\Big|_{\substack{\text{chiral}\\ \text{representation}}} = -\frac{1}{4}\, \theta_{12}^2 \delta^4 (x_1 - x_2)$ and $\delta_s(1,2) = \bar{D}\bar{D}\delta_v(1,2)$

and (3.1.170)

$$\int dS(1)\ S(1)\ \delta_S(1,2) = S(2) \ ,$$

3) $\delta_{\bar{S}}(1,2)\Big|_{\substack{\text{chiral}\\ \text{representation}}} = -\frac{1}{4}\, \bar{\theta}_{12}^2\ \delta^4(x_1 - x_2)$ and $\delta_{\bar{s}}(1,2) = DD\delta_v(1,2)$ (3.1.171)

and

$$\int d\bar{S}(1)\ \bar{S}(1)\ \delta_{\bar{S}}(1,2) = \bar{S}(2)$$

with $\theta_{ij} = \theta_i - \theta_j$; $\bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j$

and arguments (1) and (2) refer to superspace prints $(x_1, \theta_1, \bar{\theta}_1)$ and $(x_2, \theta_2, \bar{\theta}_2)$ respectively.

Note, The Grassmann delta-functions are just given by the square of the variables

$$-\frac{1}{4} \int d^2\theta' \ (\theta' - \theta)^2\ f(\theta') = f(\theta)$$

$$-\frac{1}{4} \int d^2\bar{\theta}' \ (\bar{\theta}' - \bar{\theta})^2\ f(\bar{\theta}') = f(\bar{\theta}) \ .$$

We can transform the (anti-) chiral rep. delta function to the real rep. by using our shifting property

$\delta_S(1,2)$

$$= -\frac{1}{4}\theta_{12}^2\ \delta^4(x_1 - i\theta_1\sigma\bar{\theta}_1 - (x_2 - i\theta_2\sigma\bar{\theta}_2))$$

$$= -\frac{1}{4}\theta_{12}^2\ \delta^4(x_1 - x_2 - i\theta_1\sigma\bar{\theta}_1 + i\theta_2\sigma\bar{\theta}_2)$$

$$= -\frac{1}{4}\theta_{12}^2\ e^{-i(\theta_1\not{\partial}\bar{\theta}_1 - \theta_2\not{\partial}\bar{\theta}_2)}\delta^4(x_1 - x_2)$$

$$= -\frac{1}{4}\theta_{12}^2\, e^{-i\theta_1\not{\partial}_1\bar{\theta}_{12}}\ \delta^4(x_1 - x_2) \quad ,$$

in momentum space this becomes

$$\tilde{\delta}_S(p,1,2) \equiv \int d^4x_{12}\ e^{+ipx_{12}}\ \delta_S(1,2) \tag{3.1.172}$$

$$\tilde{\delta}_S(p,1,2) = -\frac{1}{4}\theta_{12}^2\ e^{-\theta_1\not{p}\bar{\theta}_{12}}$$

and similarly for the anti-chiral delta function in the real rep.

$$\delta_{\bar{S}}(1,2) = -\frac{1}{4}\bar{\theta}_{12}^2\ e^{+i\theta_{12}\not{\partial}\bar{\theta}_1}\ \delta^4(x_1 - x_2)$$

$$\tilde{\delta}_{\bar{S}}(p,1,2) \equiv \int d^4x_{12}\ \ e^{+ipx_{12}}\ \delta_{\bar{S}}(1,2)$$

$$\tilde{\delta}_{\bar{S}}(p,1,2) = -\frac{1}{4}\bar{\theta}_{12}^2\ e^{+\theta_{12}\not{p}\bar{\theta}_1} \quad . \tag{3.1.173}$$

The vector delta function $\delta_V(1,2)$ is the same in all reps.

We are now ready to construct the simplest example of a supersymmetric model, the Wess-Zumino model.

3.2. The Wess-Zumino Model (Chiral Model)

The model consists of one chiral superfield ϕ and its adjoint anti-chiral field $\bar{\phi}$;

$$\bar{D}_{\dot{\alpha}}\phi = 0 = D_\alpha\bar{\phi} \quad .$$

So

$$\phi_1(x,\theta) = A + \theta\psi + \theta^2 F$$
$$\bar{\phi}_2(x,\bar{\theta}) = \bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F} \quad ,$$

in the real rep.

$$\phi(x,\theta,\bar{\theta}) = e^{-i\theta\not{\partial}\bar{\theta}}(A + \theta\psi + \theta^2 F)$$
$$\bar{\phi}(x,\theta,\bar{\theta}) = e^{+i\theta\not{\partial}\bar{\theta}}(A + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F}) \, . \qquad (3.2.1)$$

Since θ, $\bar{\theta}$ have dimension $-\frac{1}{2}$ and $D,\bar{D}$ have dimension $+\frac{1}{2}$, if the scalar field A is to be dimension 1 (the same as ϕ) ψ has dimension 3/2 and F dimension 2. We then make all susy invariants out of ϕ and $\bar{\phi}$ (i.e. products integrated over the appropriate measure) that are also renormalizable , i.e. dimension ≤ 4, the possibilities are

	Monomial	Character	SUSY Invariant
1)	ϕ	chiral	$\int dS\phi$
2)	$\bar{\phi}$	anti-chiral	$\int d\bar{S}\bar{\phi}$
3)	$\phi\bar{\phi}$	vector	$\int dV\phi\bar{\phi}$
4)	ϕ^2	chiral	$\int dS\phi^2$
5)	$\bar{\phi}^2$	anti-chiral	$\int d\bar{S}\bar{\phi}^2$
6)	ϕ^3	chiral	$\int dS\phi^3$
7)	$\bar{\phi}^3$	anti-chiral	$\int d\bar{S}\bar{\phi}^3$.

(3.2.2)

We stop at cubic terms since we demand renormalizability , for example

$$\int dV\phi\bar{\phi} = \int d^4x \underbrace{\underset{\text{dim}=2}{D D \bar{D} \bar{D}} \quad \underset{\text{dim}=2}{\phi\bar{\phi}}}_{\text{dim}=4} \qquad \text{renormalizable}$$

$$\int dS\phi^3 = \int d^4x \underbrace{\underset{\text{dim}=1}{\underbrace{DD}} \quad \underset{\text{dim}=3}{\underbrace{\phi^3}}}_{\text{dim}=4} \qquad \text{renormalizable} \ .$$

Thus our susy invariant action is

$$\begin{aligned}\Gamma = Z\int dV\phi\bar{\phi} &+ 4m\int dS\phi^2 + 4m\int d\bar{S}\bar{\phi}^2 \\ &+ g\int dS\phi^3 + g\int d\bar{S}\bar{\phi}^3 \\ &+ f\int dS\phi + f\int d\bar{S}\bar{\phi}\end{aligned} \tag{3.2.3}$$

where m,g,f are the parameters of the model and Z gives the field (propagator) normalization.

Let's expand the W-Z action in terms of the component fields and find the absolute minimum of the potential and quantize about it. First consider the <u>Kinetic Terms</u>

$$\phi\bar{\phi} = [e^{-i\theta\not{\partial}\bar{\theta}}(A + \theta\psi + \theta^2 F)][e^{+i\theta\not{\partial}\bar{\theta}}(\bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F})] \tag{3.2.4}$$

where both ϕ and $\bar{\phi}$ are in the same (real) representation, expanding the exponential we find after a little algebra

$$e^{\pm i\theta\not{\partial}\bar{\theta}} = 1 \pm i\theta\not{\partial}\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 \ . \tag{3.2.5}$$

Since we are interested in the highest $\theta^2\bar{\theta}^2$ terms, only these will be exhibited so

$$\phi\bar{\phi} = \ldots+\theta^2\bar{\theta}^2[-\frac{1}{4}\bar{A}\,\partial^2 A - \frac{1}{4}\partial^2\bar{A}A + \frac{1}{2}\partial_\lambda\bar{A}\partial^\lambda A + \frac{i}{4}\,\psi\overleftrightarrow{\not\partial}\,\bar{\psi} + \bar{F}F]$$

$$= \ldots+\theta^2\bar{\theta}^2[-\frac{1}{4}\partial^2(A^\dagger A) + \partial_\mu A^\dagger\partial^\mu A + \frac{i}{4}\psi\overleftrightarrow{\not\partial}\bar{\psi} + F^\dagger F] \quad . \tag{3.2.6}$$

Integrating over the vector measure $\int dV = \int d^4x\,\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial\bar{\theta}^2}$ we find

$$\dot{Z}\,dV\phi\bar{\phi} = 16Z\int d^4x\,[\partial_\lambda A^\dagger\partial^\lambda A + \frac{i}{4}\psi\overleftrightarrow{\not\partial}\,\bar{\psi} + F^\dagger F] \tag{3.2.7}$$

where the total space-time divergence $\partial^2(A^+A)$ integrates to zero. Next consider the mass terms, since both fields are chiral we could work directly in the chiral representation

$$\phi_1 = A + \theta\psi + \theta^2 F$$

So

$$\int dS\phi^2 = \int dS(A + \theta\psi + \theta^2F)^2 = \int dS[A^2 + 2\theta^\alpha(A\psi_\alpha) + \theta^2(2AF - \frac{1}{2}\psi\psi)]$$

to yield

$$\int dS\phi^2 = -4\int d^4\, x\ [2AF - \frac{1}{2}\,\psi\psi] \tag{3.2.8}$$

where

$$\begin{aligned}\theta\psi\theta\psi &= \theta^\alpha\psi_\alpha\theta^\beta\psi_\beta \\ &= -\theta^\alpha\theta^\beta\psi_\alpha\psi_\beta \quad \text{but} \quad \theta^\alpha\theta^\beta = -\frac{1}{2}\,\varepsilon^{\alpha\beta}\theta^\gamma\theta_\gamma \\ &= +\frac{1}{2}\,\theta^2\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta = \frac{1}{2}\,\theta^2\psi_\alpha\psi^\alpha \\ &= -\frac{1}{2}\,\theta^2\psi^2\,.\end{aligned} \tag{3.2.9}$$

Similarly we find

$$\int d\bar{S}\bar{\phi}^2 = -4\int d^4x[2A^\dagger F^\dagger - \frac{1}{2}\,\bar{\psi}\bar{\psi}] \tag{3.2.10}$$

(where $\bar{\theta}^{\dot\alpha}\bar{\theta}^{\dot\beta} = +\frac{1}{2}\,\bar{\theta}^2\varepsilon^{\dot\alpha\dot\beta}$ was used) .

So the mass terms are

$$\begin{aligned}&+4m\int dS\phi^2 + 4m\int d\bar{S}\bar{\phi}^2 \\ &= -16m\int d^4x\ [2AF + 2A^\dagger F^\dagger - \frac{1}{2}\,\psi\psi - \frac{1}{2}\,\bar{\psi}\bar{\psi}]\end{aligned} \quad . \tag{3.2.11}$$

The interaction terms are calculated the same way

$$\phi_1^3 = \ldots + \theta^2(3A^2F - \frac{3}{2}\,\psi\psi\ A)$$

$$\bar{\phi}_2^3 = \ldots + \bar{\theta}^2(3A^{\dagger\,2}F^{\,\dagger} - \frac{3}{2}\,\bar{\psi}\bar{\psi}\ A^\dagger)$$

resulting in

$$g\int dS\phi^3 + g\int d\bar{S}\bar{\phi}^3 = -12g\int d^4x[A^2F + A^{\dagger\,2}F^\dagger - \frac{1}{2}\,\psi\psi A - \frac{1}{2}\,\bar{\psi}\bar{\psi}\ A^\dagger\,] \quad . \tag{3.2.12}$$

Finally the linear term is simply

$$f\int dS\phi + f\int d\bar{S}\bar{\phi} = -4f\int d^4x\,(F + F^\dagger) \quad . \qquad (3.2.13)$$

Putting this altogether we have that the Wess-Zumino action is

$$\begin{aligned}\Gamma &= Z\int dV\,\phi\bar{\phi} + 4m[\int dS\phi^2 + \int d\bar{S}\bar{\phi}^2] \\ &\quad + g[\int dS\phi^3 + \int d\bar{S}\bar{\phi}^3] + f[\int dS\phi + \int d\bar{S}\bar{\phi}] \\ &= \int d^4x\,\{16Z[\partial_\lambda A^\dagger\partial^\lambda A + \frac{i}{4}\,\psi\overleftrightarrow{\partial\!\!\!/}\bar{\psi} + F^\dagger F] \\ &\quad - 16m[2AF + 2A^\dagger F^\dagger - \frac{1}{2}\,\psi\psi - \frac{1}{2}\,\bar{\psi}\bar{\psi}] \\ &\quad - 12g[A^2F + A^{\dagger 2}F^\dagger - \frac{1}{2}\,\psi\psi\,A - \frac{1}{2}\,\bar{\psi}\bar{\psi}\,A^\dagger] \\ &\quad -4f(F + F^\dagger)\} \quad .\end{aligned} \qquad (3.2.14)$$

Recall that the component action can be checked to be invariant under the susy transformations for the component fields

$$\begin{aligned} QA &= i\xi\psi & QA^\dagger &= i\bar{\xi}\bar{\psi} \\ Q\psi_\alpha &= 2i\xi_\alpha F + 2(\partial\!\!\!/\bar{\xi})_\alpha A & Q\bar{\psi}^{\dot{\alpha}} &= 2i\,\bar{\xi}^{\dot{\alpha}}F^\dagger - 2(\xi\partial\!\!\!/)^{\dot{\alpha}}A^\dagger \\ QF &= -\psi\overleftarrow{\partial\!\!\!/}\bar{\xi} & QF^\dagger &= +\,\xi\partial\!\!\!/\bar{\psi} \quad . \end{aligned}$$

(Isn't superspace simpler!!)

Since the fields F have no derivatives in Γ they are auxiliary fields; we can use their field equations to eliminate them from the action; the "equations of motion" are

$$\begin{aligned} &1)\quad 16ZF = 4f + 32mA^\dagger + 12gA^{\dagger 2} \\ &2)\quad 16ZF^\dagger = 4f + 32mA + 12gA^2 \quad . \end{aligned} \qquad (3.2.16)$$

Thus we can first write the potential in a simple form

$$\Gamma = \int d^4x \; \{16Z[\partial_\lambda A^\dagger \partial^\lambda A + \frac{i}{4}\, \psi \overleftrightarrow{\not\partial} \bar{\psi}] - 16m[-\frac{1}{2}\,\psi\psi - \frac{1}{2}\,\bar{\psi}\bar{\psi}]$$
$$-12g[-\frac{1}{2}\,\psi\psi\, A - \frac{1}{2}\,\bar{\psi}\bar{\psi}\, A^\dagger] - V\,\} \tag{3.2.17}$$

where

$$\begin{aligned} V &= -\,16ZF^\dagger F + 16m[2AF + 2A^\dagger F^\dagger \\ &\quad + 12g[A^2F + A^{\dagger 2}F^\dagger] + 4f(F + F^\dagger) \\ &= -16ZF^\dagger F + [4f + 32mA + 12gA^2]F \\ &\quad + F^\dagger[4f + 32mA^\dagger + 12g\,A^{\dagger 2}] \\ &= -16ZF^\dagger F + 16ZF^\dagger F + 16ZF^\dagger F \end{aligned} \tag{3.2.18}$$

$$V = 16ZF^\dagger F \quad .$$

For supersymmetric theories the potential is given by $F^\dagger F$ with F and $F^\dagger$ obeying the equations (1) and (2). Thus $\langle F\rangle = 0 = \langle F^\dagger\rangle$ is the absolute minimum of the potential. This implies

$$16Z\langle F\rangle = 0 = 4f + 32m\,\langle A^\dagger\rangle + 12g\,\langle A^\dagger\rangle^2$$
$$16Z\langle F^\dagger\rangle = 0 = 4f + 32m\,\langle A\rangle + 12g\,\langle A\rangle^2 \tag{3.2.19}$$

for the absolute minimum. If $\langle A\rangle = 0$, then $f = 0$; the linear term in ϕ must be excluded from the action in order to quantize about the abs. minimum.

Since a constant $\phi = a$ is a superfield, $Q\phi = 0$ we can always shift our superfields $(A \to A + a)$ by a θ indep. constant and obtain another susy invariant action. So letting $\phi \to \phi + a$, $\bar{\phi} \to \bar{\phi} + a$ we find $\Gamma \to \Gamma'$

$$\Gamma' = Z\int dV\phi\bar{\phi} + [4m + 3ga][\int dS\phi^2 + d\bar{S}\bar{\phi}^2] + g[\int dS\phi^3 + \int d\bar{S}\bar{\phi}^3]$$

$$+ (f + 8ma + 3ga^2)[\int dS\phi + \int d\bar{S}\bar{\phi}] \; . \qquad (3.2.20)$$

If Γ has a minimum at $\langle A\rangle = \alpha$ then Γ' has a minimum at $\langle A\rangle = \alpha + a$. So even if $\langle A\rangle \neq 0$ we can shift to $\langle A\rangle = 0$ by $\alpha = -a$ then $f = 0$. Thus we can always choose $f = 0$ in the W-Z model without loss of generality.

Finally let's use the F equation of motion to write V in terms of the dynamic A field only,

$$\begin{aligned} V &= 16ZF^{\dagger}F \\ &= \frac{1}{Z}[8mA^{\dagger} + 3gA^{\dagger 2}][8mA + 3gA^2] \\ &= \frac{16}{Z}\, 4m^2A^{\dagger}A + \frac{24mg}{Z}\,(A+A^{\dagger})(A^{\dagger}A) \qquad (3.2.21) \\ &+ \frac{9g^2}{Z}\,(A^{\dagger}A)^2 \; . \end{aligned}$$

Hence

$$\begin{aligned} \Gamma = \int d^4x\, \{ & 16Z(\partial_\lambda A^{\dagger}\partial^\lambda A - \frac{4m^2}{Z^2} A^{\dagger}A) \\ & + 16Z(\frac{i}{4}\psi\overleftrightarrow{\partial\!\!\!/}\bar{\psi} + \frac{1}{2}\frac{m}{Z}\psi\psi + \frac{1}{2}\frac{m}{Z}\bar{\psi}\psi) \qquad (3.2.22) \\ & + 6g(\psi\psi A + \bar{\psi}\,\bar{\psi}A^{\dagger}) \\ & - 24g\frac{m}{Z}(A + A^{\dagger})(A^{\dagger}A) - \frac{9g^2}{Z}(A^{\dagger}A)^2 \} \; , \end{aligned}$$

Thus we see that both A and ψ have the same mass $(\frac{2m}{Z})$ even though different spin. This model looks just like a σ-model with fermions--there being a definite relation between masses and between Yukawa and self-interaction terms.

3.3. Gauge Invariance and SUSY

1) Global gauge transformations

We next would like to introduce the idea of gauge symmetry in a supercovariant manner. Let's do this by first studying global gauge transformations. To be specific let the chiral superfield ϕ^a, a = 1,...,L transform according to some L-dimensional representation of SU(N), $(T^a_b)^c_d$, with c,d = 1,...,L a,b = 1,...,N, with gauge parameters λ^a_b that are space-time indep. (and θ indep.) Since a constant commutes with $D,\bar{D},Q,\bar{Q}$ we see that ϕ'^a is still a chiral superfield

$$\phi'^c = [e^{ig(\lambda^b_a T^a_b)}]^c_d\, \phi^d$$

yields

$$\bar{D}_{\dot{\alpha}}\phi'^c = [e^{ig(\underline{\lambda}\cdot\underline{T})}]^c_d\, \bar{D}_{\dot{\alpha}}\phi^d \tag{3.3.1}$$

$$= 0 \text{ and similarly } Q_\alpha\phi' = i(\frac{\partial}{\partial\theta} + i\not{\partial}\bar{\theta})_\alpha\phi' \; .$$

Now as in the Wess-Zumino model we have at most trilinear terms in the chiral fields in our action. If the action is to be gauge invariant, each of these terms must be invariant under SU(N) transformations. The kinetic terms are invariant due to the complex conjugation i.e. if ϕ is a L of SU(N) $\bar{\phi}$ is a $\bar{L}$ of SU(N), so

$$\bar{\phi} = (e^{-ig(\underline{\lambda}\cdot\underline{T})})\bar{\phi}$$

So

$$(\phi^a\bar{\phi}^a)' = \phi^a\bar{\phi}^a \qquad \text{and} \tag{3.3.2}$$

$$\int dV \phi^a\bar{\phi}^a$$

is both susy and (global) SU(N) invariant.

Only if ϕ is in the adjoint rep. (or 2 of SU(2)) can we make SU(N) invariant quadratic and cubic terms $\phi^a_b\phi^b_a$; $\phi^a_b\phi^b_c\phi^c_a$, but in general models will usually consist of several representations of SU(N)

$$\phi^a_A \quad \text{where } a = 1,\ldots,L_A \tag{3.3.3}$$

and A labels the type of rep. we have i.e. $\phi_{\bar{5}a}$; ϕ^{ab}_{15} where the first is a $\bar{5}$ of SU(5) and the second a 15 of SU(5) and a,b, = 1,...,5 as usual. And we can make an invariant $\phi^{ab}_{15}\,\phi_{\bar{5}a}\phi_{\bar{5}b}$ for instance, although no mass term!

Again if $\phi_{\bar{5}}$ then $\overline{(\phi_{\bar{5}})} = (\phi_{\bar{5}})^\dagger = \bar{\phi}_5$ transforms like a 5 and we usually use the same symbol $\bar{\phi}_5$ without confusion.

2) Local gauge transformations

We next would like the gauge parameters to depend on space-time; however they will destroy the superfield character (and chirality) of the field ϕ', if

$$\lambda^a_b = \lambda^a_b(x) \qquad \text{then}$$

$$Q_\alpha\phi' \neq i\left(\frac{\partial}{\partial\theta} + i\not{\partial}\bar{\theta}\right)_\alpha\phi', \text{ and } \bar{D}_{\dot{\alpha}}\cdot\phi'^c \neq 0.$$

Thus we must also let the gauge parameter depend on θ and $\bar{\theta}$. In addition if ϕ is chiral the gauge transformations should also preserve chirality thus the gauge parameter for a chiral field should be a chiral superfield and for an anti-chiral field it should be anti-chiral. Let's call these $\Lambda^a_b(x,\theta,\bar{\theta})$

and $\bar{\Lambda}^a_b(x,\theta,\bar{\theta})$ where

$$\bar{D}_{\dot{\alpha}}\Lambda = 0$$
$$D_\alpha\bar{\Lambda} = 0.$$

And if ϕ^c is an L of SU(N)

$$\phi'^c = [e^{ig\Lambda_a^b T_b^a}]_d^c \ \phi^d \tag{3.3.4}$$

then $\bar{\phi}_c = (\phi^c)^\dagger$ is an $\bar{L}$ of SU(N)

$$\bar{\phi}'_c = [e^{-ig\ \bar{\Lambda}_a^b T_b^a}]_c^d \ \bar{\phi}_d \tag{3.3.5}$$

(that is if $\phi' = U\phi$; $\bar{\phi}' = \bar{\phi}U^\dagger = \bar{\phi}U^{-1}$).

Similarly if ϕ_c is in $\bar{L}$ of SU(N)

$$\phi'_c = [e^{-ig\ \Lambda_a^b T_b^a}]_c^d \ \phi_d \tag{3.3.6}$$

Then $\bar{\phi}^c = (\phi_c)^\dagger$ is an L of SU(N)

$$\bar{\phi}'^c = [e^{+ig\ \bar{\Lambda}_a^b T_b^a}]_d^c \ \phi^d \quad . \tag{3.3.7}$$

So $\bar{D}_{\dot{\alpha}}\phi' = 0 = D_\alpha\bar{\phi}'$ and for example

$$Q\phi' = i(\frac{\partial}{\partial\theta} + i\not{\partial}\bar{\theta})_\alpha\phi' \ ; \quad \text{that is if } \phi \text{ is chiral}$$

ϕ' is chiral; if ϕ is a scalar superfield ϕ' is a scalar superfield etc. .

For local gauge transformations the pure chiral or pure anti-chiral globally gauge invariant quadratic and cubic action terms remain also locally gauge invariant since they transform either all with Λ or all with $\bar{\Lambda}$.

However, the kinetic energy terms are, as usual, not locally gauge invariant for example

$$\bar{\phi}'_c\phi'^c = \bar{\phi}_e[e^{-ig\underline{\bar{\Lambda}}\cdot\underline{T}}]_c^e \ [e^{+ig\underline{\Lambda}\cdot\underline{T}}]_d^c \ \phi^d$$

$$\neq \bar{\phi}_c\phi^c \qquad \text{since } \Lambda \neq \Lambda^\dagger ,$$

$$\text{except for } \Lambda = \text{const.}$$

and real as in the global case.

Thus we would like to introduce a super Yang-Mills field V^a_b in the (global) adjoint representation such that

$$\bar{\phi}' f(\underline{V}'\cdot\underline{T})\phi' = \bar{\phi}\, e^{-ig\underline{\bar{\Lambda}}\cdot\underline{T}}\, f(\underline{V}'\cdot\underline{T})\, e^{+ig\underline{\Lambda}\cdot\underline{T}}\,\phi \tag{3.3.8}$$

$$= \bar{\phi}\, f(\underline{V}\cdot\underline{T})\phi$$

Thus

$$e^{-ig\underline{\bar{\Lambda}}\cdot\underline{T}}\, f(\underline{V}'\cdot\underline{T}) e^{+ig\underline{\Lambda}\cdot\underline{T}} = f(\underline{V}\cdot\underline{T}) \tag{3.3.9}$$

this is true if

$$f(\underline{V}'\cdot\underline{T}) = e^{+ig\underline{\bar{\Lambda}}\cdot\underline{T}}\, f(\underline{V}\cdot\underline{T}) e^{-ig\underline{\Lambda}\cdot\underline{T}} \quad . \tag{3.3.10}$$

Of course this is independent of the particular representation we use since only commutators of T appear on the RHS due to the Baker-Hansdorff formula. Since we desire V (and its component A_μ) to transform with an inhomogenous piece we have

$$f = e^{gV^b_a T^a_b}$$

thus

$$e^{g\underline{V}'\cdot\underline{T}} = e^{+ig\underline{\bar{\Lambda}}\cdot\underline{T}}\; e^{g\underline{V}\cdot\underline{T}}\; e^{-ig\underline{\Lambda}\cdot\underline{T}} \tag{3.3.11}$$

So if we start expanding the exponentials we find

$$V' = V + i(\bar{\Lambda} - \Lambda) + \ldots$$

the inhomogeneous term is present. And thus

$$\bar{\phi}\, e^{g\underline{V}\cdot\underline{T}}\ \phi \text{ is gauge invariant} \tag{3.3.12}$$

We can further evaluate the gauge transformation for infinitesimal parameters by means of the C - B - H formula to find

$$V' - V = -ig\mathcal{L}_{\frac{V}{2}}\ [(\Lambda + \bar{\Lambda}) + \coth\ (\mathcal{L}_{\frac{V}{2}})\ (\Lambda - \bar{\Lambda})]$$

where $\mathcal{L}_{\frac{V}{2}}\ (\Lambda) = [\frac{V}{2}\ ,\ \Lambda]$

and

$$(L_{\frac{V}{2}})^n(\Lambda) = [\frac{V}{2}, [\frac{V}{2}, [\ldots, [\frac{V}{2}, \Lambda]\ldots]]]$$

$$\text{n-factors of } \frac{V}{2} \quad .$$

Introducing our vector notation, $\sum_{i=1}^{N^2-1} T^i V^i = \frac{1}{\sqrt{2}} \sum_{a,b=N} T^a_b V^b_a$, for $V, \Lambda, \bar{\Lambda}$ and anti-symmetric structure constants f_{ijk} we find for infinitesimal $\Lambda, \bar{\Lambda}$

$$\begin{aligned} \delta V^i &= V^{i'} - V^i \\ &= \frac{-g}{2} (\Lambda^j + \bar{\Lambda}^j) f_{ijk} V^k \\ &- \frac{ig}{2} (\Lambda^j - \bar{\Lambda}^j) [\frac{g}{2} V \coth \frac{g}{2} V]_{ji} \ . \end{aligned} \tag{3.3.13}$$

with

$$V_{ij} \equiv -i f_{ijk} V^k \ .$$

In order to find the pure super Yang-Mills Lagrangian consider the supersymmetric covariant field strength chiral spinor W^α

$$W_\alpha \equiv \bar{D}\bar{D}[e^{-g\underline{V}\cdot\underline{T}} D_\alpha \, e^{+g\underline{V}\cdot\underline{T}}]$$

where the T are the generators in the adjoint representation and $\bar{D}_{\dot{\alpha}} W_\alpha = 0$.

Using

$$e^{g\underline{V}'\cdot\underline{T}} \; e^{-g\underline{V}'\cdot\underline{T}} = 1$$

we find

$$e^{ig\bar{\underline{\Lambda}}\cdot\underline{T}} \; e^{g\underline{V}\cdot\underline{T}} \; e^{-ig\underline{\Lambda}\cdot\underline{T}} \; e^{-g\underline{V}'\cdot\underline{T}} = 1$$

so that

$$e^{-g\underline{V}'\cdot\underline{T}} = e^{+ig\underline{\Lambda}\cdot\underline{T}} \; e^{-g\underline{V}\cdot\underline{T}} \; e^{-ig\bar{\underline{\Lambda}}\cdot\underline{T}} \quad \text{, hence}$$

we have

$$\begin{aligned} W'_\alpha &= \bar{D}\bar{D}[e^{-g\underline{V}'\cdot\underline{T}} D_\alpha \, e^{+g\underline{V}'\cdot\underline{T}}] \\ &= \bar{D}\bar{D}[e^{+ig\underline{\Lambda}\cdot\underline{T}} \; e^{-g\underline{V}\cdot\underline{T}} \; e^{-ig\bar{\underline{\Lambda}}\cdot\underline{T}} D_\alpha \, e^{+ig\bar{\underline{\Lambda}}\cdot\underline{T}} \\ &\qquad e^{+g\underline{V}\cdot\underline{T}} \, e^{-ig\underline{\Lambda}\cdot\underline{T}}] \ . \end{aligned} \tag{3.3.14a}$$

Since $D_\alpha \bar{\Lambda} = 0$ and $\bar{D}_{\dot\alpha} \Lambda = 0$ we can simplify this to yield

$$W'_\alpha = e^{+ig\underline{\Lambda}\cdot\underline{T}}\, \bar{D}\bar{D}[e^{-g\underline{V}\cdot\underline{T}}\, D_\alpha(e^{+g\underline{V}\cdot\underline{T}}\, e^{-ig\underline{\Lambda}\cdot\underline{T}})]$$

$$= e^{+ig\underline{\Lambda}\cdot\underline{T}}\, \{\bar{D}\bar{D}[e^{-g\underline{V}\cdot\underline{T}}\, D_\alpha\, e^{+g\underline{V}\cdot\underline{T}}]\}\, e^{-ig\underline{\Lambda}\cdot\underline{T}} \tag{3.3.14b}$$

$$+ e^{+ig\underline{\Lambda}\cdot\underline{T}}\, \bar{D}\bar{D}\, D_\alpha\, e^{-ig\underline{\Lambda}\cdot\underline{T}} \quad .$$

But
$$[\bar{D}\bar{D}, D_\alpha] = \bar{D}_{\dot\alpha}\{\bar{D}^{\dot\alpha}, D_\alpha\} - \{D_\alpha, \bar{D}_{\dot\alpha}\}\bar{D}^{\dot\alpha}$$
$$= -4i\, \not{\partial}_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} = -4i(\not{\partial}\bar{D})_\alpha \tag{3.3.15}$$

thus $\bar{D}\bar{D}\, D_\alpha e^{-ig\underline{\Lambda}\cdot\underline{T}} = D_\alpha \bar{D}\bar{D}\, e^{-ig\underline{\Lambda}\cdot\underline{T}} - 4i(\not{\partial}\bar{D})_\alpha e^{-ig\underline{\Lambda}\cdot\underline{T}}$

$$= 0 \quad \text{since } \bar{D}_{\dot\alpha}\Lambda = 0,$$

hence

$$W'_\alpha = e^{+ig\underline{\Lambda}\cdot\underline{T}}\quad W_\alpha\, e^{-ig\underline{\Lambda}\cdot\underline{T}} \quad . \tag{3.3.16}$$

Thus the field strength transforms homogeneously and we can form a gauge invariant quantity by

$Tr[W^\alpha W_\alpha]$ where the trace is over the T-matrices so

$$Tr[W'^\alpha W'_\alpha] = Tr[e^{+ig\underline{\Lambda}\cdot\underline{T}}\, W^\alpha\, e^{-ig\underline{\Lambda}\cdot\underline{T}}\quad e^{+ig\underline{\Lambda}\cdot\underline{T}}\quad W_\alpha e^{-ig\underline{\Lambda}\cdot\underline{T}}]$$

$$= Tr[W^\alpha W_\alpha] \quad . \tag{3.3.17}$$

Since W_α is a chiral superfield the action is made by integrating over the chiral measure

$\frac{1}{g^2} \int dS\, Tr[W^\alpha W_\alpha]$ where the $\frac{1}{g^2}$ factor cancels the g^2 from the W's.

In a similar way we can derive the analogeous formulas for the complex conjugate anti-chiral field strength spinor $\bar{W}_{\dot\alpha}$

$$\bar{W}_{\dot\alpha} \equiv DD[e^{+\, g\underline{V}\cdot\underline{T}}\, \bar{D}_{\dot\alpha}\, e^{-g\underline{V}\cdot\underline{T}}]$$

with
$$D_\alpha \bar{W}_{\dot\alpha} = 0 \quad \text{since} \quad D_\alpha D_\beta D_\gamma = 0 \quad . \tag{3.3.18}$$

Then

$$\bar{W}'_{\dot\alpha} = e^{+ig\bar{\underline{\Lambda}}\cdot\underline{T}}\, \bar{W}_{\dot\alpha}\, e^{-ig\bar{\underline{\Lambda}}\cdot\underline{T}}$$

where

$$[DD, \bar{D}_{\dot\alpha}] = + 4i(D\not{\partial})_{\dot\alpha}$$

was used. Hence

$$\mathrm{Tr}[\bar{W}'_{\dot{\alpha}}\bar{W}'^{\dot{\alpha}}] = \mathrm{Tr}[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}] \tag{3.3.19}$$

and the susy invariant term is made by integrating over the anti-chiral measure

$$\frac{1}{g^2}\int d\bar{S}\ \mathrm{Tr}[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}] \quad .$$

In general the gauge invariant supersymmetric action is of the form

$$\Gamma = \Gamma_{ym} + \Gamma_K + \Gamma_\phi \tag{3.3.20}$$

with the pure super Yang-Mills action

$$\Gamma_{ym} = \frac{1}{g^2}\int dS\ \mathrm{Tr}[W^\alpha W_\alpha] + \frac{1}{g^2}\int d\bar{S}\ \mathrm{Tr}[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}] \quad , \tag{3.3.21}$$

the invariant kinetic energy terms of the form

$$\Gamma_K = \int dV\ \bar{\phi}\ e^{g\underline{V}\cdot\underline{T}}\ \phi \quad , \tag{3.3.22}$$

and the gauge invariant pure chiral or pure anti-chiral self-interactions, for example for ϕ_L an L & $\psi_{\bar{L}}$ a $\bar{L}$ of SU(N)

$$\Gamma_\phi = m\int dS\ \phi_L\psi_{\bar{L}} + m\int d\bar{S}\ \overline{\psi}_L\bar{\phi}_{\bar{L}} \tag{3.3.23}$$

or if ϕ^a_b is in the adjoint rep. of SU(N)

$$\begin{aligned}\Gamma_\phi &= m\int dS\ \mathrm{Tr}\phi^2 + m\int d\bar{S}\ \mathrm{Tr}\bar{\phi}^2 \\ &\quad + g\int dS\ \mathrm{Tr}\phi^3 + g^*\int d\bar{S}\ \mathrm{Tr}\bar{\phi}^3 \quad .\end{aligned} \tag{3.3.24}$$

3.4. Supersymmetric QED

Finally let's study the supersymmetric extension of QED more closely. The matter fields will be two chiral fields with electric charge ±1, denoted $\phi_\pm$. These will transform under the U(1) gauge transformations as

$$\phi'_\pm = e^{\pm ig\Lambda}\phi_\pm \tag{3.4.1}$$

where Λ is a chiral superfield, $\bar{D}_{\dot{\alpha}}\Lambda = 0$ and g will be our charge.

The conjugate anti-chiral fields transform as (leaving the $\pm$ in tact on ϕ)

$$\bar{\phi}'_{\pm} = e^{\mp ig\bar{\Lambda}}\,\bar{\phi}_{\pm} \tag{3.4.2}$$

with $D_\alpha \bar{\Lambda} = 0$. Only a gauge invariant mass term can be made from these (i.e. charge zero monomial)

$$\Gamma_\phi = 4m \int dS\, \phi_+ \phi_- + 4m \int d\bar{S}\, \bar{\phi}_+ \bar{\phi}_- \quad . \tag{3.4.3}$$

The vector super gauge field V has simple transformation properties in this Abelian case

$$\begin{aligned} &\bar{\phi}'_+ e^{gV'} \phi'_+ = \bar{\phi}_+ e^{-ig\bar{\Lambda}} e^{gV'} e^{+ig\Lambda} \phi_+ \\ \text{so} \\ &e^{gV'} = e^{+ig\bar{\Lambda}} e^{gV} e^{-ig\Lambda} \end{aligned} \tag{3.4.4}$$

since these are not matrices we can simply combine the exponentials to find $V' = V + i(\bar{\Lambda} - \Lambda)$ as usual for the abelian case we have just an inhomogeneous term. Similarly

$$\bar{\phi}_- e^{-gV} \phi_- \text{ is gauge invariant}$$

These are the only possibilities (i.e. $\bar{\phi}_+ \ \phi_-$ cannot be made invariant) thus

$$\Gamma_K = \int dV[Z_+ \bar{\phi}_+ e^{+gV} \phi_+ + Z_- \bar{\phi}_- e^{-gV} \phi_-]$$

for the kinetic energy action with $Z_{\pm}$=a convenient normalization factor. Finally the field strength s are given by

$$\begin{aligned} W_\alpha &= \bar{D}\bar{D}[e^{-gV} D_\alpha e^{+gV}] \\ \bar{W}_{\dot{\alpha}} &= DD[e^{+gV} \bar{D}_{\dot{\alpha}} e^{-gV}] \quad . \end{aligned} \tag{3.4.6}$$

Again the abelian character allows us to take the derivatives and cancel the exponentials trivially

$$W_\alpha = g\,\bar{D}\bar{D}\,D_\alpha V$$
$$\bar{W}_{\dot\alpha} = -g\,DD\,\bar{D}_{\dot\alpha}V \tag{3.4.7}$$

and $W'_\alpha = W_\alpha$, $\bar{W}'_{\dot\alpha} = \bar{W}_{\dot\alpha}$ immediately. So

$$\Gamma_{ym} = \frac{Z}{g^2}\int dS\, W^\alpha W_\alpha + \frac{Z}{g^2}\int d\bar{S}\,\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$$
$$= Z\int dS\,\bar{D}\bar{D}\,D^\alpha V\,\bar{D}\bar{D}\,D_\alpha V$$
$$+ Z\int d\bar{S}\,DD\,\bar{D}_{\dot\alpha}V\,DD\,\bar{D}^{\dot\alpha}V$$
$$= Z\int dS\,\bar{D}\bar{D}[D^\alpha V\,\bar{D}\bar{D}\,D_\alpha V]$$
$$+ Z\int d\bar{S}\,DD[\bar{D}_{\dot\alpha}V\,DD\,\bar{D}^{\dot\alpha}V] \tag{3.4.8}$$
$$= -Z\int dV\,\{VD\,\bar{D}\bar{D}\,DV + V\bar{D}\,DD\,\bar{D}V\}$$

with Z again a convenient normalization factor. Using our D commutators we find

$$D\bar{D}\bar{D}D = \bar{D}DD\bar{D}$$

thus

$$\Gamma_{ym} = -Z2\int dV\,V\,D\bar{D}\bar{D}DV \quad . \tag{3.4.9}$$

So we have our super QED action

$$\Gamma^{SQED} = -2Z\int dV\,V\,D\bar{D}\bar{D}\,DV$$
$$+ \int dV[Z_+\,\bar{\phi}_+ e^{+gV}\,\phi_+ + Z_-\bar{\phi}_- e^{-gV}\phi_-] \tag{3.4.10}$$
$$+ 4m\int dS\,\phi_+\phi_- + 4m\int d\bar{S}\,\bar{\phi}_+\bar{\phi}_-$$

Before expanding their action in terms of the component fields let's look at the gauge transformations a bit more closely in terms of components and show how to gauge away many fields. This simplified almost physical gauge is known as the Wess-Zumino gauge. In terms of superfields we had

$$V' = V + i(\bar{\Lambda} - \Lambda)$$

and

$$\phi'_{\pm} = e^{\pm ig\Lambda}\,\phi_{\pm} \tag{3.4.11}$$

$$\bar{\phi}'_{\pm} = e^{\mp ig\bar{\Lambda}}\,\phi_{\pm}\,.$$

Since $\Lambda,\bar{\Lambda}$ are chiral we can expand them

$$\Lambda(x,\theta,\bar{\theta}) = e^{-i\theta\not{\partial}\bar{\theta}}\,[\omega(x) + \theta^{\alpha}\zeta_{\alpha}(x) + \theta^{2}\sigma]$$
$$\bar{\Lambda}(x,\theta,\bar{\theta}) = e^{+i\theta\not{\partial}\bar{\theta}}\,[\omega^{\dagger}(x) + \bar{\theta}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}(x) + \bar{\theta}^{2}\sigma^{\dagger}] \tag{3.4.12}$$

and expanding the exponentials we find

$$\begin{aligned}(\bar{\Lambda} - \Lambda) = {} & (\omega^{\dagger} - \omega) - \theta^{\alpha}\zeta_{\alpha} + \bar{\theta}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}} - \theta^{2}\sigma + \bar{\theta}^{2}\sigma^{\dagger} \\ & + \theta\sigma^{\mu}\bar{\theta}\, i\partial_{\mu}(\omega + \omega^{\dagger}) - \frac{i}{2}\bar{\theta}^{2}\,\theta\not{\partial}\bar{\zeta} \\ & + \frac{i}{2}\theta^{2}\,\zeta\overleftarrow{\not{\partial}}\bar{\theta} \qquad - \frac{1}{4}\theta^{2}\bar{\theta}^{2}\partial^{2}(\omega^{\dagger} - \omega)\,.\end{aligned} \tag{3.4.13}$$

Hence recalling our expansion for V we find that the gauge transformations of the component fields are

$$C' - C = i(\omega^{\dagger} - \omega)$$
$$\chi'_{\alpha} - \chi_{\alpha} = -i\zeta_{\alpha} \quad ; \quad \bar{\chi}'^{\dot{\alpha}} - \bar{\chi}^{\dot{\alpha}} = i\bar{\zeta}^{\dot{\alpha}}$$
$$M' - M = -2i\sigma \quad ; \quad M'^{\dagger} - M^{\dagger} = 2i\sigma^{\dagger}$$
$$V'_{\mu} - V_{\mu} = -\partial_{\mu}(\omega + \omega^{\dagger})$$
$$\lambda'_{\alpha} - \lambda_{\alpha} = (\not{\partial}\bar{\zeta})_{\alpha} \quad ; \quad \bar{\lambda}'^{\dot{\alpha}} - \bar{\lambda}^{\dot{\alpha}} = +(\zeta\overleftarrow{\not{\partial}})^{\dot{\alpha}}$$
$$D' - D = -i\partial^{2}(\omega^{\dagger} - \omega)$$

Thus we can always choose a Λ and $\Lambda^{\dagger}$ such that C, χ, $\bar{\chi}$, M, $M^{\dagger}$ are <u>gauged away</u>! More specifically choose the field dependent gauge transformation

$$\omega^{\dagger} - \omega = + iC \quad \rightarrow \quad C' = 0$$

$$\zeta_{\alpha} = -i\,\chi_{\alpha} \quad \rightarrow \quad \chi'_{\alpha} = 0$$

$$\zeta^{\dot{\alpha}} = +i\,\bar{\chi}^{\dot{\alpha}} \quad \rightarrow \quad \bar{\chi}'^{\dot{\alpha}} = 0$$

$$\sigma = -\frac{i}{2} M \quad \rightarrow \quad M' = 0 \tag{3.4.15}$$

$$\sigma^{\dagger} = \frac{i}{2} M^{\dagger} \quad \rightarrow \quad M'^{\dagger} = 0 \;!!$$

Thus we still have the parameters $(\omega + \omega^{\dagger})$ at our disposal - that is only V_{μ} still transforms under a restricted gauge transformation $\Lambda = \omega = \Lambda^{\dagger}$. Thus in this Wess-Zumino gauge

$$V = \theta\sigma^{\mu}\bar{\theta}V_{\mu} + \frac{1}{2}\theta^{2}\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \frac{1}{2}\bar{\theta}^{2}\theta^{\alpha}\lambda_{\alpha} + \frac{1}{4}\theta^{2}\bar{\theta}^{2}D \tag{3.4.16}$$

where $V'_{\mu} = V_{\mu} - 2\partial_{\mu}\omega$

$$\lambda'_{\alpha} = \lambda_{\alpha},\quad \bar{\lambda}'^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}} \tag{3.4.17}$$

$$D' = D$$

(where now we define

$$\lambda + i\partial\!\!\!/\bar{\chi} \rightarrow \lambda$$

$$\bar{\lambda} - i\chi\overleftarrow{\partial\!\!\!/} \rightarrow \bar{\lambda}$$

$$D + \partial^{2}C \rightarrow D\;) \tag{3.4.18}$$

under the restricted gauge transformations $\Lambda = \Lambda^{\dagger} = \omega = \omega^{\dagger}$.

The W-Z gauge is not a supercovariant gauge, however, that is no longer do SUSY transformations commute with gauge transformations. However, if we allow (operator) field dependent gauge transformations we can show that the algebra will again close and gauge transformations will commute with our covariant susy transformations (See B. Dewit and D. Z. Freedman PRD 12 2286). Hence in the W-Z gauge $e^{\pm gV}$ simplifies

$$e^{\pm gV} = 1 \pm g\ \theta\sigma^{\mu}\bar{\theta}V_{\mu} \pm \frac{g}{2}\theta^2\bar{\theta}\bar{\lambda} \pm \frac{g}{2}\bar{\theta}^2\theta\lambda$$

$$\pm \frac{g}{4}\theta^2\bar{\theta}^2 D + \frac{g^2}{4}\theta^2\bar{\theta}^2\ V_{\mu}V^{\mu} \tag{3.4.19}$$

(where $\theta\sigma^{\mu}\bar{\theta}\ \theta\sigma^{\nu}\bar{\theta} = +\frac{1}{4}\theta^2\bar{\theta}^2_{Tr}(\sigma^{\mu}\ \bar{\sigma}^{\nu})$

$= \frac{1}{2}\theta^2\bar{\theta}^2\ g^{\mu\nu}$ was used) .

Since in the non-Abelian case

$$V' = V + ig(\bar{\Lambda} - \Lambda) + \ldots$$

we are always able to transform to the W-Z gauge for which $C = \chi = \bar{\chi} = M = M^{\dagger} = 0$ and $\Lambda = \Lambda^{\dagger} = \omega = \omega^{\dagger}$.

So let's expand our action in terms of components in the W-Z gauge; it will still be invariant under the (restricted) gauge transformations

and

$$\begin{aligned} V'_{\mu} &= V_{\mu} - 2\partial_{\mu}\omega \\ \lambda'_{\alpha} &= \lambda_{\alpha} \\ D' &= D \\ \bar{\lambda}'^{\dot{\alpha}} &= \bar{\lambda}^{\dot{\alpha}} \\ \phi'_{\pm} &= e^{\pm ig\omega}\ \phi_{\pm} \\ \bar{\phi}'_{\pm} &= e^{\mp ig\omega}\ \bar{\phi}_{\pm} \ . \end{aligned} \tag{3.4.20}$$

First let's transform to the WZ gauge and expand

$$\begin{aligned} \Gamma_{ym} &= -2Z \int dV\ V\ D\bar{D}\bar{D}D \\ &= -2Z \int d^4x[16D^2 + 16i\lambda\overleftrightarrow{\not{\partial}}\bar{\lambda} - 8\ F_{\mu\nu}F^{\mu\nu}] \end{aligned} \tag{3.4.21}$$

where

$$F_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu} \ ,$$

$$\Gamma_K =$$

$$\int dV[Z_+\bar{\phi}_+ e^{gV}\,\phi_+ + Z_-\bar{\phi}_- e^{-gV}\phi_-]$$

$$= \int dV[Z_+\bar{\phi}_+\phi_+(1+g\ \theta\not{V}\bar{\theta} + \frac{g}{2}\,\theta^2\bar{\theta}\bar{\lambda} + \frac{g}{2}\bar{\theta}^2\theta\lambda \quad .$$

$$+ \frac{g}{4}\,\theta^2\bar{\theta}^2(D + gV^2)) \tag{3.4.22}$$

$$+ Z_-\bar{\phi}_-\phi_-(1-g\ \theta\not{V}\bar{\theta} - \frac{g}{2}\,\theta^2\bar{\theta}\bar{\lambda} - \frac{g}{2}\,\bar{\theta}^2\theta\lambda$$

$$- \frac{g}{4}\,\theta^2\bar{\theta}^2\ (D - gV^2))]$$

$$= \int dV[Z_+(e^{+i\theta\not{\partial}\bar{\theta}}\ (\bar{A}_+ + \bar{\theta}\bar{\psi}_+ + \bar{\theta}^2\bar{F}_+))\ (e^{-i\theta\not{\partial}\bar{\theta}}\ (A_+ + \theta\psi_+ + \theta^2F_+))\ x$$

$$(1 + g\theta\not{V}\bar{\theta} + \frac{g}{2}\theta^2\bar{\theta}\bar{\lambda} + \frac{g}{2}\,\bar{\theta}^2\theta\lambda + \frac{g}{4}\,\theta^2\bar{\theta}^2(D + gV^2))$$

$$+ (+ \to -,\ g \to -g)]$$

$$= \int d^4x\ \ \{16Z_+[(\partial_\mu - \frac{ig}{2}\,V_\mu)A_+^\dagger][(\partial_\mu + \frac{ig}{2}\,V_\mu)A_+]$$

$$+ 16Z_+\ \frac{i}{4}\,[\psi_+(\overrightarrow{\not{\partial}} - \frac{ig}{2}\,\not{V})\bar{\psi}_+ - \psi_+(\overleftarrow{\not{\partial}} + \frac{ig}{2}\,\not{V})\bar{\psi}_+]$$

$$+ 16Z_+\ F_+^\dagger F_+$$

$$+ 4gZ_+(A_+^\dagger\psi_+\lambda - A_+\bar{\psi}_+\bar{\lambda} + A_+A_+^\dagger D)$$

$$+ (+ \to -,\ g \to -g)\} \quad .$$

So we see that this looks just like an ordinary gauge theory with gauge covariant derivatives

$$D_\mu A_\pm \equiv (\partial_\mu \pm \frac{ig}{2}\,V_\mu)A_\pm$$

$$\left(D_\mu A_\pm^\dagger \equiv (\partial_\mu \mp \frac{ig}{2}\,V_\mu)A_\pm^\dagger = [D_\mu A_\pm]^\dagger\right)$$

$$D_\mu\psi_\pm \equiv (\partial_\mu \pm \frac{ig}{2}\,V_\mu)\psi_\pm \tag{3.4.23}$$

(and similarly $\quad D_\mu\bar{\psi}_\pm \equiv (\partial_\mu \mp \frac{ig}{2}\,V_\mu)\bar{\psi}_\pm)$.

Since $\lambda, \bar{\lambda}$, V_μ are in the adjoint representation they transform as singlets for U(1). Thus the SQED action in the W-Z gauge is given by

$$\Gamma = \Gamma_{ym} + \Gamma_K + \Gamma_\phi \tag{3.4.24}$$

1) $$\Gamma_{ym} = -64Z \int d^4x[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2} \lambda\overleftrightarrow{\partial}\bar{\lambda} + \frac{1}{2} D^2] \tag{3.4.25}$$

with $\quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$

2) $$\begin{aligned}\Gamma_K = 16Z_+ \int d^4x[&(D_\mu A_+)^\dagger (D^\mu A_+) \\ &+ \frac{i}{4} \psi_+\overleftrightarrow{\not{D}} \bar{\psi}_+ + F_+^\dagger F_+ \\ &+ \frac{g}{4} (A_+^\dagger\psi_+\lambda - A_+\bar{\psi}_+\bar{\lambda} + A_+A_+^\dagger D)] \\ + 16Z_- \int d^4x[&(D_\mu A_-)^\dagger (D^\mu A_-) \\ &+ \frac{i}{4} \psi_-\overleftrightarrow{\not{D}} \bar{\psi}_- + F_-^\dagger F_- \\ &- \frac{g}{4} (A_-^\dagger\psi_-\lambda - A_-\bar{\psi}_-\bar{\lambda} + A_-A_-^\dagger D)]\end{aligned} \tag{3.4.26}$$

where

$$\begin{aligned} D_\mu A_\pm &\equiv (\partial_\mu \pm \frac{ig}{2} V_\mu)A_\pm \\ D_\mu \psi_\pm &\equiv (\partial_\mu \pm \frac{ig}{2} V_\mu)\psi_\pm \end{aligned} \tag{3.3.27}$$

3) And finally the usual looking mass terms

$$\begin{aligned}\Gamma_\phi = &-16m \int d^4x[A_+F_- + A_-F_+ - \frac{1}{2} \psi_+\psi_-] \\ &-16m \int d^4x[A_+^\dagger F_-^\dagger + A_-^\dagger F_+^\dagger - \frac{1}{2} \bar{\psi}_+\bar{\psi}_-]\end{aligned} \tag{3.4.28}$$

So once again we see how the superfield action can be reduced to a more familiar ordinary field action. SQED looks like ordinary QED with a photon A_μ, massive Dirac electron

$$\psi = \begin{pmatrix} \psi_+ \\ \overline{\psi}_- \end{pmatrix} ,$$

a neutral massless Majorana fermion λ, and massive charged scalars $A_\pm$. The Yukawa and quartic couplings are required to have the same coupling constant g by SUSY as well as the electron and scalar masses being degenerate. Let's note that if we desire to give the vector superfield a supersymmetric mass we must add the term $\frac{1}{2} M^2 \int dVV^2$ to the action. Since it is not gauge invariant we cannot transform to the W-Z gauge in order to expand the action. The mass term however is easy to work out and it yields

$$\int dVV^2 = 8\int d^4x \; [CD+\chi\lambda-\overline{\chi}\overline{\lambda}+MM^+-V_\mu V^\mu] \; . \qquad (3.4.29)$$

Since C now couples with D and χ with λ these (C,χ) become dynamic degrees of freedom--so a massive vector multiplet describes a massive vector V_μ, two spin 1/2 fermions, λ and χ, and a scalar C, all with the same mass.

(The general expression for $\int dVVD\overline{D}\overline{D}DV$ is left as an exercise.)