CH. 1. THE STANDARD MODEL

1.0. Introduction

Gauge theories have proven to be a successful framework for relativisitic quantum mechanical models of elementary particle interactions. Recently a standard model of these interactions has emerged which consistently describes all known facts of elementary particle physics. This model is based on the electroweak gauge group SU(2) x U(1) of Glashow-Weinberg-Salam (GWS) and the SU(3) color gauge group of quantum chromodynamics (QCD). The fundamental particles which interact according to these gauge symmetries belong to the three lowest spin representations of the Lorentz group.

These are classified as follows:

**Matter fields: (spin 1/2 fermions)

*Leptons: non-strongly interacting fermions.

$$e^-,\mu^-,\tau^-$$

$$\nu_e,\nu_\mu,\nu_\tau$$

*Quarks: the strongly interacting constituents of hadrons: q_m^a

a = 1,2,3 or R,G,B: the three types of SU(3)-color involved in QCD interactions.

m = u,d,c,s,t,b...?: the different flavors involved in SU(2) × U(1) electroweak interactions

**Gauge bosons: (spin 1)

*Photon γ: mediates electromagnetic interactions.

*Intermediate Vector Bosons (IVB's): $W^\pm,Z^\circ$: mediate charged and neutral current weak interactions.

*Gluons G_b^a: mediate the strong interactions. (There are 8 gluons, since they transform as the adjoint representation of the color SU(3) gauge group, $\sum_{a=1}^{3} G_a^a = 0$.)

**Higgs bosons: (spin 0) ϕ^+, ϕ^o: these are responsible for spontaneously breaking the electro-weak SU(2) x U(1) symmetry. Three bosons are "eaten" to give the $W^{\pm}$, Z^o mass, while $\eta = \mathrm{Re}\phi^o$ survives and has mass $\simeq$ 100 GeV.

1.1. The Glashow-Weinberg-Salam (GWS)-Model of Electroweak Interactions

1.1.1. Particle Spectrum and Symmetric Lagrangian

The model is based on the gauge group SU(2) x U(1). SU(2) is isomorphic to the group of 2 x 2 unitary matrices with determinant = 1. It has $2^2 - 1 = 3$ generators T^i, $i = 1,2,3$, satisfying

$$[T^i, T^j] = i\varepsilon^{ijk} T^k \tag{1.1.1}$$

where ε^{ijk} is the Levi-Civita tensor ($\varepsilon^{123} = +1$). T^i are called the weak isospin generators. Since this is a 3 parameter group, there are 3 gauge bosons: A^i_μ, $i = 1,2,3$.

There are 2 particularly important representations of T^i:

1) The fundamental rep. given by Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.1.2}$$

$$T^i_{ab} = \frac{1}{2}\sigma^i_{ab} \qquad a,b = 1,2\ ; \quad i = 1,2,3$$

$$[T^i, T^j] = i\varepsilon^{ijk}\, T^k$$

2) Regular, real or adjoint rep. given by the structure constants of the group

$$T^i_{jk} = -i\varepsilon_{ijk} \qquad ijk = 1,2,3$$

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix} \qquad T^2 = \begin{bmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{1.1.3}$$

$$T^3 = \begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[T^i, T^j] = i\varepsilon^{ijk}\, T^k .$$

U(1) is the one parameter, abelian group of phase transformations. Its generator is called the hypercharge: $\mathcal{Y}$, the single associated gauge boson is B_μ.

Each of these subgroups has its own coupling constant; g and g' for SU(2) and U(1), respectively.

The gauge bosons transform <u>inhomogeneously</u> under the group. Consider the SU(2) subgroup first, if $U(\omega) = e^{+igT^i\omega^i}$ and $U^{-1}(\omega) = e^{-ig\underline{T}\cdot\underline{\omega}}$ where T^i is any matrix rep. of $\mathcal{T}^i$ and $\underline{T}\cdot\underline{\omega} = T^i\omega^i$

$$\underline{T}\cdot\underline{A}'_\mu = U(\omega)\ \underline{T}\cdot\underline{A}_\mu\ U^{-1}(\omega) - \frac{i}{g}\,(\partial_\mu\ U(\omega))\ U^{-1}(\omega)\ . \tag{1.1.4}$$

Infinitesimally this yields

$$U(\omega) \simeq 1 + ig\ \underline{T}\cdot\underline{\omega}\ ; \qquad U^{-1} \simeq 1 - ig\ \underline{T}\cdot\underline{\omega}\ . \tag{1.1.5}$$

Hence $\underline{A}_\mu$ transforms as

$$\begin{aligned}
\underline{T}\cdot\underline{A}'_\mu &= U[\underline{T}\cdot\underline{A}_\mu - \frac{i}{g}\,U^{-1}(\partial_\mu U)]U^{-1} \qquad\qquad (1.1.6)\\
&= (1 + ig\underline{\omega}\cdot\underline{T})[\underline{A}_\mu\cdot\underline{T} + \frac{i}{g}(-ig\ \underline{T}\cdot\partial_\mu\underline{\omega})](1-ig\underline{\omega}\cdot\underline{T})\\
&= \underline{T}\cdot\underline{A}_\mu + ig\omega^j A^k_\mu(T_jT_k - T_kT_j) + \underline{T}\cdot\partial_\mu\underline{\omega}\\
&= \underline{T}\cdot\underline{A}_\mu + ig\ \omega j A^k_\mu(i\varepsilon^{jki}T_i) + \underline{T}\cdot\partial_\mu\underline{\omega}\\
&= \underline{T}\cdot A_\mu + \underline{T}\cdot(\partial_\mu\underline{\omega} - g\underline{\omega}\times\underline{A}_\mu)
\end{aligned}$$

where we have taken $\underline{\omega}$ to be infinitismal. Thus we see that the transformation of A_μ is independent of $\underline{T}$, that is for infinitesimal $\underline{\omega}$

$$\underline{A}'_\mu = \underline{A}_\mu + (\partial_\mu \underline{\omega} + g \underline{A} \times \underline{\omega}) . \qquad (1.1.7)$$

For the U(1) subgroup, $U(\theta) = e^{+ig'\theta y}$ and $U^{-1}(\theta) = e^{-ig'\theta y}$, where y is an eigenvalue of Y (i.e., just a number): the hypercharge of the representation under consideration. Therefore

$$y B'_\mu = U y B_\mu U^{-1} - \frac{i}{g'} [\partial_\mu U] U^{-1}$$

$$= y B_\mu + y(\partial_\mu \theta) \qquad (1.1.8)$$

whether θ is finite or infinitismal.

Each gauge boson transforms only under its associated subgroup; being invariant under the remaining subgroups. That is $\underline{A}_\mu$ transforms under SU(2) only, under U(1) transformations $\underline{A}'_\mu = \underline{A}_\mu$ it is invariant. Likewise B_μ transforms under U(1) only, under SU(2) transformations $B'_\mu = B_\mu$ it is invariant.

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The gauge invariant kinetic energy term for the gauge bosons (Yang-Mills fields) is built from the antisymmetric, covariant field strength tensor for each group:

1) U(1) :

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\mu B_\nu$$

$$B'_{\mu\nu} = \partial_\nu B'_\nu - \partial_\nu B'_\mu$$

$$= B_{\mu\nu} + \partial_\mu(\partial_\nu \theta) - \partial_\nu(\partial_\mu \theta) = B_{\mu\nu} . \qquad (1.1.9)$$

Accordingly, $-\frac{1}{4} B_{\mu\nu}B^{\mu\nu} = -\frac{1}{4} B'_{\mu\nu}B'^{\mu\nu}$ is a Lorentz and U(1) invariant: [$-\frac{1}{4}$ is a conventional normalization factor].

2) SU(2):

Let us try the above construction on SU(2). We start by considering

$$F^i_{\mu\nu} \overset{?}{=} \partial_\mu A^i_\nu - \partial_\nu A^i_\mu .$$

From
$$\partial_\nu \underline{A}'_\nu - \partial_\nu \underline{A}'_\mu = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu \tag{1.1.10}$$
$$+ g(\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu) \times \underline{\omega}$$
$$+ g(\underline{A}_\nu \times \partial_\nu \underline{\omega} - \underline{A}_\mu \times \partial_\nu \underline{\omega})$$

we get
$$\underline{F}'_{\mu\nu} \cdot \underline{F}'^{\mu\nu} = \underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} + \cancelto{0}{2g\underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} \times \underline{\omega}}$$
$$+ 2g\underline{F}_{\mu\nu} \cdot (\underline{A}^\nu \times \partial^\mu \underline{\omega} - \underline{A}^\mu \times \partial^\nu \underline{\omega})$$
$$+ g^2 (\underline{F}_{\mu\nu} \times \underline{\omega}) \cdot (\underline{F}^{\mu\nu} \times \underline{\omega}) \tag{1.1.11}$$
$$+ 2g^2 (\underline{F}_{\mu\nu} \times \underline{\omega}) \cdot (\underline{A}^\nu \times \partial^\mu \underline{\omega} - \underline{A}^\mu \times \partial^\nu \underline{\omega})$$
$$+ g^2 (\underline{A}^\nu \times \partial^\mu \underline{\omega} - \underline{A}^\mu \times \partial^\nu \underline{\omega})^2$$
$$\neq \underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} \ !!$$

that is, $F^i_{\mu\nu}$ as defined above is not a suitable building block for an invariant kinetic energy term. We have to replace the simple derivative ∂_μ with the covariant derivative D_μ. This is defined as

$$D_\mu \equiv \partial_\mu - ig\underline{T} \cdot \underline{A}_\mu \tag{1.1.12}$$

Then we define $F^i_{\mu\nu}$ as, where again we use any rep. T^i of T^i,

$$\underline{T} \cdot \underline{F}_{\mu\nu} \equiv D_\mu (\underline{T} \cdot \underline{A}_\nu) - D_\nu (\underline{T} \cdot \underline{A}_\mu)$$
$$= \underline{T} \cdot (\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu) - ig(\underline{T} \cdot \underline{A}_\mu)(\underline{T} \cdot \underline{A}_\nu)$$
$$+ ig(\underline{T} \cdot \underline{A}_\nu)(\underline{T} \cdot \underline{A}_\mu)$$
$$= \underline{T} \cdot (\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu) - igA^j_\mu A^k_\nu (T^j T^k - T^k T^j)$$
$$= \underline{T} \cdot (\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu) + g\varepsilon^{jki} A^j_\mu A^k_\nu T^i \tag{1.1.13}$$
$$= \underline{T} \cdot [\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + g\underline{A}_\mu \times \underline{A}_\nu]$$

Thus we get:

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + g\underline{A}_\mu \times \underline{A}\nu \tag{1.1.14}$$

or
$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon^{ijk} A^j_\mu A^k_\nu$$

Let's check that $\underline{F}_{\mu\nu}$ does indeed tranform homogeneously:

First we evaluate

$$\begin{aligned} g\underline{A}'_\mu \times A'_\nu = g\underline{A}_\mu \times \underline{A}_\nu + g\partial_\mu \underline{\omega} \times \underline{A}_\nu + g\underline{A}_\mu \times \partial_\nu \underline{\omega} \\ + g^2 \underline{A}_\mu \times (\underline{A}_\nu \times \underline{\omega}) + g^2 (\underline{A}_\mu \times \underline{\omega}) \times \underline{A}_\nu \end{aligned} \tag{1.1.15}$$

Now use the Jacobi identity (cyclic perm. of triple product)

$$[T_i,[T_j,T_k]] + [T_j,[T_k,T_i]] + [T_k,[T_i,T_j]] = 0 \tag{1.1.16}$$

and
$$[T_i,T_j] = i\varepsilon_{ijk} T^k$$

to obtain the well known identity

$$\varepsilon_{ij\ell}\varepsilon_{km\ell} + \varepsilon_{jk\ell}\varepsilon_{im\ell} + \varepsilon_{ki\ell}\varepsilon_{jm\ell} = 0 \quad ; \tag{1.1.17}$$

this leads to

$$\begin{aligned} g\underline{A}'_\mu \times \underline{A}' = g\underline{A}_\mu \times \underline{A}_\nu + g(\partial_\mu \underline{\omega} \times A_\nu + \underline{A}_\mu \times \partial_\nu \omega \\ + g^2 (\underline{A}_\mu \times \underline{A}_\nu) \times \underline{\omega} \end{aligned} \tag{1.1.18}$$

Recalling eq. (1.1.10) we secure

$$\underline{F}'_{\mu\nu} = \underline{F}_{\mu\nu} + g\underline{F}_{\mu\nu} \times \underline{\omega} \tag{1.1.19}$$

which is homogeneous as required! Hence we have for infinitesimal transformations

$$\underline{A}'_{\mu} = \underline{A}_{\mu} + g\underline{A}_{\mu} \times \underline{\omega} + \partial_{\mu}\underline{\omega}$$

that

$$-\frac{1}{4}\underline{F}'_{\mu\nu}\cdot\underline{F}'^{\mu\nu} = -\frac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} + 2g\underline{F}_{\mu\nu}\cdot(\underline{F}^{\mu\nu}\times\underline{\omega})(-\frac{1}{4})$$

$$-\frac{1}{4}g^2(\underline{F}_{\mu\nu}\times\underline{\omega})\cdot(\underline{F}^{\mu\nu}\times\underline{\omega}) \tag{1.1.20}$$

The second term on the r.h.s. vanishes identically, and the third drops for infintesimal $\underline{\omega}$. So

$$-\frac{1}{4}\underline{F}'_{\mu\nu}\cdot\underline{F}'^{\mu\nu} = -\frac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} \tag{1.1.21}$$

Thus, the SU(2) x U(1) invariant pure Yang-Mills part of the GWS Lagrangian is given by

$$L_{YM} = -\frac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1.1.22}$$

Note that due to the non-linearity of the SU(2) field strength tensor $F^{i}_{\mu\nu}$, $-\frac{1}{4}\underline{F}_{\mu\nu}\,\underline{F}^{\mu\nu}$ now includes trilinear and quartic self-interaction terms for the non-abelian gauge field A^{i}_{μ}.]

We next consider how the fermions are incorporated in the model. The GWS model is based on a chiral SU(2) gauge invariance; that is, parity violation is built into the model by assigning the right and left handed fermions to different representations of SU(2). All the right-handed fermions (see Appendix A for Dirac matrix and other conventions)

$$\psi_R \equiv R \equiv \frac{1}{2}(1+\gamma_5)\psi \equiv \gamma_+\psi \tag{1.1.23}$$

are SU(2) singlets

$$\psi_R' \equiv \psi_R \tag{1.1.24}$$

while all left-handed fermions

$$\psi_L \equiv L \equiv \frac{1}{2}(1-\gamma_5)\psi \equiv \gamma_-\psi \tag{1.1.25}$$

are in the fundamental rep. of SU(2) (i.e. doublets, two dimensional representations of SU(2) called 2 of SU(2)). As mentioned above the fundamental representation of T^i is given by the Pauli-matrices: $T^i = \frac{1}{2}\sigma^i$ thus for finite $\underline{\omega}$

$$\psi_L' = U(\omega)\psi_L \quad \text{where} \quad U(\omega) = e^{+ig\underline{T}\cdot\underline{\omega}} \tag{1.1.26}$$

For infinitesimal $\underline{\omega}$:

$$\psi_L' = \psi_L + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma}\,\psi_L \tag{1.1.27}$$

Both ψ_L and ψ_R transform non-trivially under Y

$$\begin{aligned} \psi_L' &= U(\theta)\psi_L = e^{ig' y_L \theta}\psi_L \\ \psi_R' &= U(\theta)\psi_R = e^{ig' y_R \theta}\psi_R \end{aligned} \tag{1.1.28}$$

or for infinitesimal θ

$$\psi'_L = \psi_L + ig'y_L\theta\psi_L$$

$$\psi'_R = \psi_R + ig'y_R\theta\psi_R \ . \tag{1.1.29}$$

The quantum numbers y_L, y_R are chosen so that the quantum numbers of $\psi_{L,R}$ for T^3 and Y are such that the electric charge Q is given by $Q = T^3 + Y$, where the diagonal operator for SU(2) is the third component of isospin T^3. (An alternate convention is: $Q = T^3 + Y/2$, but then $2g' \to g'$ so factors of 1/2 compensate in the Lagrangian.)

In the GWS model, the fermions are arranged in 3 families, or generations. Each of these consists of a SU(2) doublet of left-handed quarks, a SU(2) doublet of left-handed leptons, 2 right-handed quark SU(2) singlets and one right-handed lepton SU(2) singlet. More explicitly, these are as follows:

1) The Electron Family:

$$\begin{bmatrix} \nu_e \\ e \end{bmatrix}_L , \begin{bmatrix} u \\ d \end{bmatrix}_L , e_R , u_R , d_R \ .$$

2) The Muon Family:

$$\begin{bmatrix} \nu_\mu \\ \mu \end{bmatrix}_L , \begin{bmatrix} c \\ s \end{bmatrix}_L , \mu_R , c_R , s_R \ .$$

3) The Tau Family:

$$\begin{bmatrix} \nu_\tau \\ \tau \end{bmatrix}_L , \begin{bmatrix} t \\ b \end{bmatrix}_L , \tau_R , t_R , b_R \ .$$

Since $T^3 \leftrightarrow \frac{1}{2}\sigma^3$ the isospin quantumnumbers of the doublets are

$$+\frac{1}{2} \text{ for the upper field}$$

$$-\frac{1}{2} \text{ for the lower field} \ .$$

We can then list the T^3,y,Q quantum numbers for the fields:

	T^3	y	$Q = T^3 + y$
$(\nu_e, \nu_\mu, \nu_\tau)_L$	+ 1/2	- 1/2	0
$(e, \mu, \tau)_L$	- 1/2	- 1/2	-1
$(u, c, t)_L$	+ 1/2	+ 1/6	+ 2/3
$(d, s, b)_L$	- 1/2	+ 1/6	- 1/3
$(e, \mu, \tau)_R$	0	-1	-1
$(u, c, t)_R$	0	+ 2/3	+ 2/3
$(d, s, b)_R$	0	- 1/3	- 1/3

(Note we have suppressed the SU(3) color indices of the quarks: i.e., each u,d,c,s,t,b stands for 3 fields, one for each color R,G,B.) In general let us call each lepton doublet ℓ_{mL}, and each quark doublet q_{mL}, where $m = \begin{matrix} e,\mu,\tau \\ 1,2,3 \end{matrix}$ for the 3 families, and each right-handed field by e_{mR}, u_{mR}, d_{mR} where

$$
\begin{aligned}
e_{eR} &= e_R \;, & u_{eR} &= u_R \;, & d_{eR} &= d_R \\
e_{\mu R} &= \mu_R \;, & u_{\mu R} &= c_R \;, & d_{\mu R} &= s_R \\
e_{\tau R} &= \tau_R \;, & u_{\tau R} &= t_R \;, & d_{\tau R} &= b_R
\end{aligned}
$$

(Often we will suppress the generation index m also.)

The fermion kinetic energy terms in the SU(2) × U(1) invariant Lagrangian are also obtained by replacing ∂_μ by the appropriate covariant derivative D_μ defined analogously to eq.(1.1.12).

First recall the variations of the fields under our SU(2) x U(1) transformations: (ω,θ infintesimal)

$$\ell_L' = \ell_L + \frac{i}{2} g\underline{\omega}\cdot\underline{\sigma}\ell_L - \frac{ig'\theta}{2}\ell_L$$
$$q_L' = q_L + \frac{i}{2} g\underline{\omega}\cdot\underline{\sigma}\, q_L + \frac{ig'\theta}{6} q_L$$
$$e_R' = e_R - ig'\theta e_R$$
$$u_R' = u_R + \frac{2i}{3} g'\theta\, u_R$$
$$d_R' = d_R - \frac{ig'\theta}{3} d_R \qquad (1.1.30)$$

[These equations apply for each family.]

Consequently, the covariant derivatives are $D_\mu \equiv \partial_\mu - ig\,\underline{T}\cdot\underline{A}_\mu - ig'yB_\mu$,

$$D_\mu\ell_L \equiv (\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu + \frac{ig'}{2}B_\mu)\ell_L$$
$$D_\mu q_L \equiv (\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{6}B_\mu)q_L$$
$$D_\mu e_R = (\partial_\mu + ig'B_\mu)e_R$$
$$D_\mu u_R = (\partial_\mu - \frac{2i}{3}g'B_\mu)u_R$$
$$D_\mu d_R = (\partial_\mu + \frac{i}{3}g'B_\mu)d_R \qquad (1.1.31)$$

The SU(2) x U(1) gauge transformation of these covariant derivatives is, for example, given by

$$(D_\mu\ell_L)' = D_\mu'\ell_L'$$

$$= (\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}' + \frac{ig'}{2}B_\mu')(\ell_L + \frac{ig}{2}\underline{\omega}\cdot\underline{\sigma}\,\ell_L - \frac{i\theta g'}{2}\ell_L) \qquad (1.1.32)$$

$$= [1 + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{2}\theta][\partial_\mu \ell_L + \frac{ig'}{2}\,B_\mu \ell_L]$$

$$- (\frac{ig}{2})^2\;\underline{\sigma}\cdot\underline{A}_\mu\;\underline{\omega}\cdot\underline{\sigma}\;\ell_L - \frac{ig}{2}\,(-\frac{ig'}{2})\theta\;\underline{\sigma}\cdot\underline{A}_\mu\ell_L - \frac{ig^2}{2}\;\underline{\sigma}\cdot\underline{A}_\mu \times \underline{\omega}\;\ell_L\;.$$

Using

$$\underline{\sigma}\cdot\underline{A}\;\underline{\sigma}\cdot\underline{\omega} \;=\; A^j\omega^k\sigma_j\sigma_k \tag{1.1.33}$$

$$= A^j\omega^k[i\epsilon_{jki}\sigma_i + \delta_{jk}]$$

$$= i\;\underline{\sigma}\cdot\underline{A}\times\underline{\omega} + \underline{A}\cdot\underline{\omega}\;,$$

we obtain

$$(D_\mu\ell_L)' \;=\; [1 + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{2}\,\theta][\partial_\mu\ell_L - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu\ell_L + \frac{ig'}{2}\,B_\mu\ell_L] \tag{1.1.34}$$

or equivalently

$$(D_\mu\ell_L)' \;=\; [1 + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{2}\,\theta](D_\mu\ell_L) \tag{1.1.35}$$

That is if

$$\ell_L' \;=\; U(\omega,\theta)\ell_L \tag{1.1.36}$$

with $U(\omega,\theta) = U(\omega)U(\theta)$ then

$$(D_\mu\ell_L)' \;=\; U(\omega,\theta)(D_\mu\ell_L) \quad \text{also !} \tag{1.1.37}$$

Similarly for the other terms. Thus we can indeed turn the globally invariant fermion kinetic terms

$$\bar{\ell}_L i\not{\partial}\ell_L + \bar{q}_L i\not{\partial}q_L + \bar{e}_R i\not{\partial}e_R + \bar{u}_R i\not{\partial}u_R + \bar{d}_R i\not{\partial}d_R \tag{1.1.38}$$

into locally gauge invariant ones by replacing the ordinary derivatives ∂_μ with the appropriate D_μ covariant derivatives.

Before doing this recall that $\psi_L \equiv \gamma_-\psi$, so

$$\overline{(\psi_L)} \equiv \psi_L^+ \gamma^o = \psi^+ \frac{1}{2} (1-\gamma_5)\gamma^o = \bar{\psi} \frac{1}{2} (1+\gamma_5)$$

$$(= (\bar{\psi})_R)$$

$$\overline{(\psi_L)}\, \gamma^\mu \psi_L = \bar{\psi}\, \gamma_+ \gamma^\mu \gamma_-\, \psi = \bar{\psi}\, \gamma^\mu \gamma_- \gamma_-\, \psi \tag{1.1.39}$$

$$= \bar{\psi}\, \gamma^\mu \gamma_-\, \psi ,$$

$$\overline{(\psi_R)}\, \gamma^\mu \psi_L = \bar{\psi}\, \gamma^\mu \gamma_+ \gamma_-\, \psi = 0!$$

So the kinetic terms are of the form

$$\overline{(\psi_L)} i \not\partial\, \psi_L\, , \quad \overline{(\psi_R)} i \not\partial\, \psi_R\, . \tag{1.1.40}$$

Now if (the notation $\bar{\psi}_L \equiv \overline{(\psi_L)}$ etc. unless stated otherwise)

$$\psi_L' = U\psi_L \tag{1.1.41}$$

then

$$\psi_L'^{+} = \psi_L^+ U^+ = \psi_L^+ U^{-1} \tag{1.1.42}$$

and

$$\bar{\psi}_L = \bar{\psi}_L U^{-1} \quad . \tag{1.1.43}$$

Similarly $\bar{\psi}_R' = \bar{\psi}_R U^{-1}$. Thus we have for the covariant deriavtive kinetic energy terms

$$(\bar{\psi}_L i \not{D} \psi_L)' = \bar{\psi}_L' i \not{D}' \psi_L' = \bar{\psi}_L U^{-1} U i \not{D} \psi_L$$

$$= \bar{\psi}_L i \not{D} \psi_L \, , \tag{1.1.44}$$

i.e., it is gauge invariant. Thus the SU(2) × U(1) gauge invariant fermion kinetic energy terms may be written as follows:

$$L_F = \bar{\ell}_L i\not{D}\ell_L + \bar{q}_L i\not{D} q_L + \bar{e}_R i\not{D} e_R + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R \ . \quad (1.1.45)$$

(recall that we are implicitly summing over each family: $e,\mu,\tau,\ldots$).

At this point we note that not only are the gauge bosons massless but also the fermions are massless:

$$\begin{aligned} \bar{\psi}\psi &= \bar{\psi}(\gamma_+ + \gamma_-)\psi = \bar{\psi}\gamma_+\psi + \bar{\psi}\gamma_-\psi \\ &= \psi^+\gamma^o\gamma_+\gamma_+\psi + \psi^+\gamma^o\gamma_-\gamma_-\psi \\ &= \psi^+\gamma_-\gamma^o\psi_R + \psi^+\gamma_+\gamma^o\psi_L \\ &= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \end{aligned} \quad (1.1.46)$$

But for SU(2) x U(1) all LH fermions are doublets and RH fermions are singlets, thus we cannot make a SU(2) x U(1) invariant mass term! We must introduce Higgs scalar bosons to spontaneously break the local SU(2) x U(1) to $U(1)_{em}$ so that only the photon remains massless and the fermions gain mass. Now to accomplish this dual task we introduce a complex scalar Higgs field ϕ in the doublet of SU(2) (4 hermitian (real) fields)

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^o \end{bmatrix} \qquad \begin{array}{lll} \phi^+ & \text{has} & Q = +1 \\ \phi^o & \text{has} & Q = 0 \end{array} \quad (1.1.47)$$

since $Q = T_3 + y$,

$$\phi \text{ has } y = +\frac{1}{2}$$

So for ω,θ infinitesimal

$$\begin{aligned} \phi' &= \phi + \frac{ig}{2}\underline{\omega}\cdot\underline{\sigma}\phi + \frac{ig'}{2}\theta\phi \\ D_\mu\phi &= [\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{2}B_\mu]\phi \end{aligned} \quad (1.1.48)$$

and as usual $\phi' = U(\omega,\theta)\phi$ for finite (ω,θ) .

We can make invariant Yukawa interaction terms of the form

$$\bar{q}_L \phi \, d_R \quad , \quad \bar{\ell}_L \phi e_R \tag{1.1.49}$$

Now

$$\bar{q}_L \phi \quad \text{is SU(2) invariant}$$

but since q_L has $y = +\frac{1}{6}$, $\bar{q}_L$ has $y = -\frac{1}{6}$

$$\text{and} \quad \phi \text{ has } y = +\frac{1}{2}$$

$$\bar{q}_L \phi \quad \text{has } y = +\frac{1}{2} - \frac{1}{6} = +\frac{1}{3}$$

Since d_R has $y = -\frac{1}{3}$

$$\bar{q}_L \, \phi \, d_R \text{ is also U(1) invariant.}$$

similarly for $\bar{\ell}_L \phi \; e_R$. Of course we can sum over the families and have inter-family couplings

$$\begin{aligned} & \Gamma^d_{mn} \, \bar{q}_{mL} \; \phi \; d_{nR} + \text{h.c.} \\ = \; & \Gamma^d_{mn} \, \bar{q}_{mL} \, \phi \, d_{nR} + \Gamma^{d*}_{mn} \, d^{\dagger}_{nR} \, \phi^{+} \gamma^{o} q_{mL} \\ = \; & \Gamma^d_{mn} \, \bar{q}_{mL} \, \phi \, d_{nR} + \Gamma^{d*}_{mn} \, \bar{d}_{nR} \, \phi^{+} \, q_{mL} \end{aligned} \tag{1.1.50}$$

and

$$\Gamma^e_{mn} \, \bar{\ell}_{mL} \, \phi \, e_{nR} + \text{h.c.} \tag{1.1.51}$$

$$= \Gamma^e_{mn} \, \bar{\ell}_{mL} \, \phi \, e_{nR} + \Gamma^{e*}_{mn} \, \bar{e}_{nR} \, \phi^{+} \, \ell_{mL}$$

However, this looks at first as if there is no u mass since $\bar{q}_L \phi \; u_R$ has $y = -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = 1$ ($Q = +1 \neq 0$ also!) But we can also couple to the hermitean conjugate of ϕ; $\phi^{\dagger}$

$$\phi' = \phi + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma}\phi + \frac{ig'}{2}\,\theta\phi$$

$$\phi^{\dagger\prime} = \phi^{\dagger} - \frac{ig}{2}\phi^{\dagger}\underline{\omega}\cdot\underline{\sigma}^{\dagger} - \frac{ig'}{2}\,\theta\phi^{\dagger} \tag{1.1.52}$$

Now $\sigma^{\dagger} = \sigma$

So $\phi^{\dagger\prime} = \phi^{\dagger}U^{-1}$ if $\phi' = U\phi$!

That is if ϕ transforms as $(2, +\frac{1}{2})$ under $(SU(2), U(1))$

then $\phi^{\dagger}$ transforms as $(2^{*}, -\frac{1}{2})$ under $(SU(2), U(1))$.

But for SU(2) 2 and 2^{*} are equivalent:

$$\phi^{\dagger\prime} = \phi^{\dagger} - \frac{ig}{2}\,\phi^{\dagger}\,\underline{\omega}\cdot\underline{\sigma} \quad \text{if } \phi \text{ is a 2 of SU(2)}$$

So
$$\phi_a^{*\prime} = \phi_a^{*} - \frac{ig}{2}\,(\underline{\omega}\cdot\underline{\sigma})_{ba}\,\phi_b^{*}$$

$$= \phi_a^{*} - \frac{ig}{2}\,(\omega\cdot\sigma^{T})_{ab}\,\phi_b^{*} \tag{1.1.53}$$

but
$$-\sigma^{iT} = (i\sigma^{2})^{\dagger}\,\sigma^{i}(i\sigma^{2}) \tag{1.1.54}$$

hence
$$\phi^{*\prime} = \phi^{*} + \frac{ig}{2}\,(i\sigma^{2})^{\dagger}\,\underline{\omega}\cdot\underline{\sigma}(i\sigma^{2})\phi^{*} \tag{1.1.55}$$

this implies
$$(i\sigma^{2}\phi^{*})' = (i\sigma^{2}\,\phi^{*}) + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma}(i\sigma^{2}\phi^{*})\ . \tag{1.1.56}$$

Defining
$$\tilde{\phi} \equiv i\sigma^{2}\phi^{*} \tag{1.1.57}$$

then
$$\tilde{\phi}' = \tilde{\phi} + \frac{ig}{2}\,\underline{\omega}\cdot\underline{\sigma}\,\tilde{\phi}$$

that is
$$\tilde{\phi} \text{ is a 2 of SU(2), like } \phi!$$

$$\tilde{\phi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi^{-} \\ \phi^{0\dagger} \end{bmatrix} = \begin{bmatrix} \phi^{0\dagger} \\ -\phi^{-} \end{bmatrix} \quad \text{a} \quad (2, -\tfrac{1}{2}) \tag{1.1.58}$$

We can write the Yukawa coupling as

$$\bar{q}_L \tilde{\phi} u_R$$

where

$\bar{q}_L \tilde{\phi}$: SU(2) singlet, $y = -\frac{1}{6} - \frac{1}{2} = -\frac{2}{3}$

u_R: SU(2) singlet, $y = +\frac{2}{3}$

So

$\Gamma^u_{mn} \bar{q}_{mL} \tilde{\phi} u_{nR} + \text{h.c.}$ is an SU(2)xU(1) invariant

Thus we have the Yukawa interaction terms:

$$\begin{aligned} L_{yuk} &= \Gamma^e_{mn} \bar{\ell}_{mL} \phi e_{nR} \\ &+ \Gamma^d_{mn} \bar{q}_{mL} \phi d_{nR} \\ &+ \Gamma^u_{mn} \bar{q}_{mL} \tilde{\phi} U_{nR} \quad + \text{h.c.} \end{aligned} \tag{1.1.59}$$

As we will see not all the $\Gamma^{e,d,u}_{mn}$ are observable due to our similar T_3, y's for each generation and similarly for the RH fields i.e. we can redefine some of the couplings away. In addition to the Yukawa interaction we have the gauge couplings and the Higgs self-interaction

$$\begin{aligned} L_\phi &= (D_\mu \phi)^\dagger D^\mu \phi - V(\phi^\dagger \phi) \\ V(\phi^\dagger\phi) &= \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2 \end{aligned} \tag{1.1.60}$$

where

$$D_\mu \phi = (\partial_\mu - \frac{ig}{2} \underline{\sigma} \cdot \underline{A}_\mu - \frac{ig'}{2} B_\mu)\phi \tag{1.1.61}$$

So we finally have the SU(2) x U(1) gauge invariant GWS Lagrangian: gathering terms:

$$L = L_{ym} + L_F + L_\phi + L_{yuk}$$

1) $L_{ym} = -\frac{1}{4} \underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$

where

$$\begin{aligned} \underline{F}_{\mu\nu} &= \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + g\underline{A}_\mu \times \underline{A}_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned} \tag{1.1.62}$$

2) $$L_F = \bar{\ell}_{mL}\, i\not{D}\ell_{mL} + \bar{q}_{mL}\, i\not{D}q_{mL} + \bar{e}_{mR}\, i\not{D}e_{mR} \qquad (1.1.63)$$

$$+ \bar{u}_{mR}\, i\not{D}u_{mR} + \bar{d}_{mR}\, i\not{D}\, d_{mR}$$

where

$$D_\mu \ell_L = (\partial_\mu - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu + \frac{ig'}{2}\,B_\mu)\ell_L$$

$$D_\mu q_L = (\partial_\mu - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{6}\,B_\mu)q_L$$

$$D_\mu e_R = (\partial_\mu + ig'B_\mu)e_R \qquad (1.1.64)$$

$$D_\mu u_R = (\partial_\mu - \frac{2i}{3}\, g'B_\mu)u_R$$

$$D_\mu d_R + (\partial_\mu + \frac{i}{3}\, g'B_\mu)d_R$$

3) $$L_\phi = (D_\mu\phi)^\dagger\, (D^\mu\phi) - V(\phi^\dagger\phi) \qquad (1.1.65)$$

where

$$V(\phi^\dagger\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \qquad (1.1.66)$$

$$D_\mu\phi = (\partial_\mu - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{2}\,B_\mu)\phi$$

4) $$L_{yuk} = \Gamma^e_{mn}\,\bar{\ell}_{mL}\,\phi\, e_{nR} + \Gamma^d_{mn}\,\bar{q}_{mL}\,\phi\, d_{nR} \qquad (1.1.67)$$

$$+ \Gamma^u_{mn}\,\bar{q}_{mL}\,\tilde{\phi}\, u_{nR} + \text{H.C.}$$

where

$$\tilde{\phi} = i\sigma^2\phi^* = \begin{bmatrix}\phi^{o\dagger}\\ -\phi^-\end{bmatrix} \qquad (1.1.68)$$

Let's recall the SU(2)xU(1) gauge transformations which leave this Lagrangian invariant:

1) $$\underline{T}\cdot\underline{A}'_\mu = U(\omega,\theta)\,\underline{T}\cdot\underline{A}_\mu\, U^{-1}(\omega,\theta) - \frac{i}{g}\,(\partial_\mu\, U(\omega,\theta))U^{-1}(\omega,\theta)$$

$$(1.1.69)$$

1') $$\underline{A}'_\mu = \underline{A}_\mu + [\partial_\mu\underline{\omega} + g\,\underline{A}_\mu \times \underline{\omega}]$$

where

$$U(\omega,\theta) = e^{+ig\underline{\omega}\cdot T} \qquad (1.1.70)$$

since $\underline{A}_\mu$ is U(1) invariant.

$$2)\ \ell'_L = U_\ell(\omega,\theta)\ell_L \qquad (1.1.71)$$

$$2')\ \ell'_L = \ell_L + \frac{i}{2}\, g\, \underline{\omega}\cdot\underline{\sigma}\ \ell_L - \frac{ig'}{2}\,\theta\ell_L$$

where

$$U_\ell(\omega,\theta) = e^{+\frac{i}{2} g\underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{2}\theta} \qquad (1.1.72)$$

<u>Note</u>: $\bar{\ell}'_L = \bar{\ell}_L U_\ell^{-1}(\omega,\theta)$

$$(1.1.73)$$

$$\bar{\ell}'_L = \bar{\ell}_L - \frac{i}{2}\, g\ \bar{\ell}_L\underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{2}\ \ \theta\bar{\ell}_L$$

$$3)\ q'_L = U_q(\omega,\theta)q_L \qquad (1.1.74)$$

$$3')\ q'_L = q_L + \frac{i}{2}g\ \underline{\omega}\cdot\underline{\sigma}\ q_L + \frac{ig'}{6}\,\theta q_L$$

where

$$U_q(\omega,\theta) = e^{+\frac{i}{2} q\ \underline{\omega}\cdot\underline{\sigma} - \frac{ig'}{6}\theta} \qquad (1.1.75)$$

$$4)\ e'_R = e^{-ig'\theta}\ e_R$$

$$(1.1.76)$$

$$4')\ e'_R = (1 - ig'\theta)e_R$$

$$5)\ u'_R = e^{+\frac{2i}{3}g'\theta}\ u_R$$

$$(1.1.77)$$

$$5')\ u'_R = (1 + \frac{2i}{3}\ g'\theta)u_R$$

$$6)\ d'_R = e^{-\frac{ig'}{3}\theta}\ dR$$

$$(1.1.78)$$

$$6')\ d'_R = (1 - \frac{ig'}{3}\ \theta)d_R$$

$$7)\ \ \phi' = U_\phi(\omega,\theta)\phi$$

$$7')\ \ \phi' = [1 + \frac{ig}{2}\ \underline{\omega}\cdot\underline{\sigma} + \frac{ig'}{2}\ \theta]\phi$$

$$(1.1.79)$$

where

$$U_\phi(\omega,\theta) = e^{+\frac{ig}{2}\ \underline{\omega}\cdot\underline{\sigma} + \frac{ig'}{2}\theta} \qquad (1.1.80)$$

1.1.2. SPONTANEOUS SYMMETRY BREAKING, PARTICLE MASSES, COUPLING CONSTANTS, MIXING ANGLE, ETC.

We now desire to spontaneously break the SU(2) x U(1) symmetry down to $U(1)_{EM}$ Consider the vacuum expectation of $\langle\phi\rangle \neq 0$

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\v\end{bmatrix} \qquad v = \text{real.} \tag{1.1.81}$$

Then

$$\sigma^i\begin{bmatrix}0\\v\end{bmatrix} \neq 0$$ so the vac. is not invariant under all SU(2) transformations

and $$y\begin{bmatrix}0\\v\end{bmatrix} \neq 0$$ since ϕ has $y = +1/2$ so U(1) is also broken.

But

$$Q\begin{bmatrix}0\\v\end{bmatrix} = (T_3 + y)\begin{bmatrix}0\\v\end{bmatrix} = \left(\frac{1}{2}\sigma^3 + \frac{1}{2}1\right)\begin{bmatrix}0\\v\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0\\v\end{bmatrix} = 0.$$

The Q, that is EM symmetry, is not broken. i.e. ϕ_o has zero charge.
However, since

$$\langle 0|\phi|0\rangle = \langle 0|U^{-1}U\phi U^{-1}U|0\rangle \tag{1.1.82}$$

$$= \langle 0'|\phi'|0'\rangle$$

if $|0\rangle$ is invariant $\quad U|0\rangle = |0\rangle$,

So $$\langle 0|\phi|0\rangle = \langle 0|\phi'|0\rangle \tag{1.1.83}$$

thus $$\frac{1}{\sqrt{2}}\begin{bmatrix}0\\v\end{bmatrix} = U_\phi(\omega,\theta)\,\frac{1}{\sqrt{2}}\begin{bmatrix}0\\v\end{bmatrix} \tag{1.1.84}$$

Contradiction! Hence $U|0\rangle \neq |0\rangle$ that is

$|0\rangle$ is <u>not</u> invariant

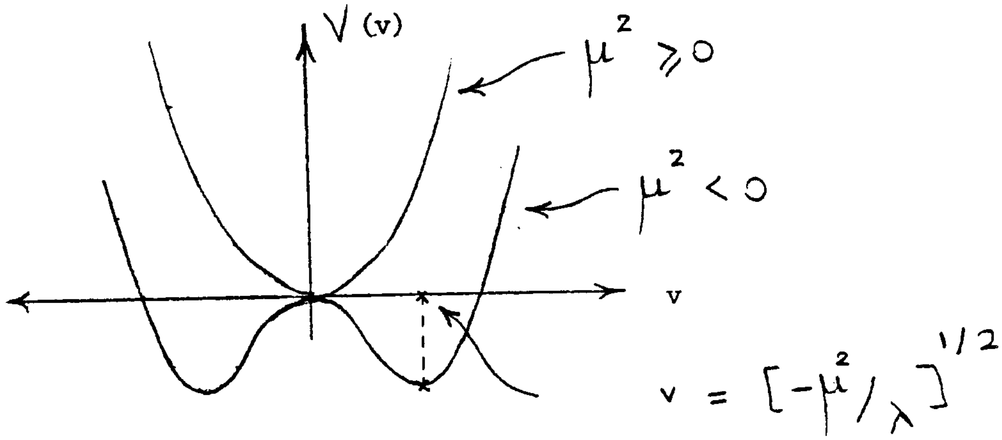
Now we ask to look at the minimum of the potential V for $\phi = \frac{1}{\sqrt{2}}\binom{0}{v}$

$$V(v) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{1.1.85}$$

$$\phi^\dagger\phi = \boxed{\phi^{+\dagger} \quad \phi_o^\dagger}\begin{bmatrix}\phi^+ \\ \phi^o\end{bmatrix} = \phi^-\phi^+ + \phi_o^\dagger\phi_o \tag{1.1.86}$$

but
$$\langle\phi^+\rangle = 0; \ \langle\phi^o\rangle = v/\sqrt{2} \quad \rightarrow \phi^+\phi = \frac{1}{2}v^2 \tag{1.1.87}$$

hence
$$\begin{aligned} V(v) &= \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^4 \\ &= \frac{1}{2}v^2(\mu^2 + \frac{1}{2}\lambda v^2) \end{aligned} \tag{1.1.88}$$



Now $V' = v(\mu^2 + \lambda v^2)$ if $\mu^2 > 0$ the minimum of V is at $v = 0$; this is the symmetric solution. If $\mu^2 < 0$ V has extrema at $v = 0$

$$v = \sqrt{-\mu^2/\lambda} \ .$$

$V'' = \mu^2 + 3\lambda v^2$ at $v = 0$, $V'' = \mu^2 < 0 \rightarrow v = 0$ maximum

while at $v = \sqrt{-\mu^2/\lambda} \rightarrow V'' = \mu^2 - 3\lambda\mu^2/\lambda$

$$= -2\mu^2 > 0 \text{, a minimum}$$

So we choose the spontaneously broken mode minimum

$$v = \sqrt{-\mu^2/\lambda} \quad \mu^2 < 0 \tag{1.1.89}$$

We can eliminate the Goldstone bosons by the Higgs-Kibble transformation:

defining 4 new fields $\underline{\xi}$, η by

$$\phi = e^{\frac{-i\underline{\xi}\cdot\underline{\sigma}}{2v}} \begin{bmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{bmatrix} \tag{1.1.90}$$

$\underline{\xi}$ are Goldstone bosons and η is a Higgs meson

We can now exploit the gauge invariance of the theory to transform away the $\underline{\xi}$'s.

Make the SU(2) gauge transformation with

$$\vec{\omega} = \frac{\vec{\xi}}{2gv} \tag{1.1.91}$$

then

$$\phi' = U_\phi(\vec{\omega} = \frac{\vec{\xi}}{2gv}, \theta = 0)\phi \tag{1.1.92}$$

so that

$$\phi' = \begin{bmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{bmatrix} \text{ only.} \tag{1.1.93}$$

We must re-write the Lagrangian in terms of ϕ'; A'_μ, ℓ', q', e', u', d'.

The form of the Lag. is the same so we just drop all primes and replace ϕ with $\frac{v+\eta}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$ everywhere:

So 1) L_{ym} remains unchanged.

2) L_F remains unchanged

These do not involve ϕ

3) $$L_\phi = \left[\left(\partial_\mu - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{2}\,B_\mu\right)\begin{bmatrix}0\\ \dfrac{v+\eta}{\sqrt{2}}\end{bmatrix}\right]^\dagger$$

$$\times\left(\partial_\mu - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{2}\,B_\mu\right)\begin{bmatrix}0\\ \dfrac{v+\eta}{\sqrt{2}}\end{bmatrix}$$

$$-\left[\frac{\mu^2}{2}(v+\eta)^2 + \frac{\lambda}{4}(v+\eta)^4\right] \qquad (1.1.94)$$

$$= \frac{1}{2}\left[\begin{bmatrix}0 & \partial_\mu\eta\end{bmatrix} + \frac{ig}{2}\begin{bmatrix}0 & v+\eta\end{bmatrix}\underline{\sigma}\cdot\underline{A}_\mu + \frac{ig'}{2}\begin{bmatrix}0 & v+\eta\end{bmatrix}B_\mu\right]$$

$$\times\left[\begin{bmatrix}0\\ \partial^\mu\eta\end{bmatrix} - \frac{ig}{2}\,\underline{\sigma}\cdot\underline{A}^\mu\begin{bmatrix}0\\ v+\eta\end{bmatrix} - \frac{ig}{2}\,B^\mu\begin{bmatrix}0\\ v+\eta\end{bmatrix}\right] - V(v+\eta)$$

$$L_\phi = \frac{1}{2}\,\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{8}(v+\eta)^2\begin{bmatrix}0 & 1\end{bmatrix}\left[g\,\underline{\sigma}\cdot\underline{A}^\mu + g'B^\mu \quad g\underline{\sigma}\cdot\underline{A}_\mu + g'B_\mu\right]\begin{pmatrix}0\\1\end{pmatrix}$$

$$-V(\eta+v)$$

Now

$$g\,\underline{\sigma}\cdot\underline{A}^{\mu} + g'B^{\mu} = \begin{bmatrix} gA_3^{\mu} + g'B^{\mu} & g(A_1^{\mu} - iA_2^{\mu}) \\ g(A_1^{\mu} + iA_2^{\mu}) & -gA_3^{\mu} + g'B^{\mu} \end{bmatrix}, \tag{1.1.95}$$

it is convenient to define

$$\begin{aligned} W_{\mu}^{\pm} &\equiv \frac{1}{\sqrt{2}}\,(A_{\mu}^{1} \mp iA_{\mu}^{2}) \\ Z_{\mu} &\equiv \frac{gA_{\mu}^{3} - g'B_{\mu}}{\sqrt{g^2 + g'^2}} \\ A_{\mu} &\equiv \frac{gB_{\mu} + g'A_{\mu}^{3}}{\sqrt{g^2 + g'^2}} \end{aligned} \tag{1.1.96}$$

So

$$\begin{aligned} &\boxed{0\ 1}\,[g\,\underline{\sigma}\cdot\underline{A}^{\mu} + g'B^{\mu}]^2 \binom{0}{1} \\ &= 2g^2 W_{\mu}^{+}W^{-\mu} + (g^2 + g'^2)Z_{\mu}Z^{\mu} \end{aligned} \tag{1.1.97}$$

Thus

$$\begin{aligned} L_{\phi} = \frac{1}{2}\,\partial_{\mu}\eta\partial^{\mu}\eta - [\frac{\mu^2}{2}\,(v + \eta)^2 + \frac{\lambda}{4}\,(v + \eta)^4] \\ + \frac{1}{8}\,(v + \eta)^2[2g^2 W_{\mu}^{+}W^{-\mu} + (g^2 + g'^2)Z_{\mu}Z^{\mu}] \end{aligned} \tag{1.1.98}$$

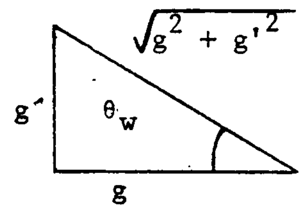
Since $V' = 0$ the terms linear in η vanish, we find

$$L_\phi = \frac{1}{2}\,\partial_\mu\eta\partial^\mu\eta - \frac{\mu^2}{2}\,\eta^2 - \frac{\lambda}{4}[\eta^4 + 4v\eta^3 + 6v^2\eta^2]$$
$$+ \frac{1}{8}(v^2 + 2\eta v + \eta^2)[2g^2 W^+_\mu W^{-\mu} + (g^2 + g'^2)Z_\mu Z^\mu] \tag{1.1.99}$$

Now we define $M_w \equiv \frac{gv}{2}$, the mass of the $W^\pm$, then

$$M_z^2 = \frac{(g^2 + g'^2)}{g^2}\,M_w^2 \tag{1.1.100}$$

Now let $\frac{g'}{g} = \tan\theta_w$



with θ_w = weak or Weinberg angle.

So

$$M_z = \frac{M_w}{\cos\theta_w} \quad . \tag{1.1.101}$$

Also

$$e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}} = g\sin\theta_w \tag{1.1.102}$$

and

$$Z_\mu = \cos\theta_w\,A^3_\mu - \sin\theta_w\,B_\mu$$
$$A_\mu = \sin\theta_w A^3_\mu + \cos\theta_w B_\mu \quad . \tag{1.1.103}$$

There is no $A_\mu A^\mu$ term so the photon is massless as it should be since $U(1)_{EM}$ is unbroken.

The $[\text{mass}]^2$ of η is

$$+\mu^2 + 3\lambda v^2 = m_\eta^2 = +\mu^2 - 3\mu^2 = -2\mu^2$$

$$m_\eta^2 = -2\mu^2 > 0 \quad . \tag{1.1.104}$$

The remaining terms are higgs meson self-interaction terms and higgs vector interactions. So we finally secure

$$\begin{aligned} L_\phi = {} & \frac{1}{2}\,\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}\,m_\eta^2\eta^2 - \frac{\lambda}{4}(\eta^4 + 4v\eta^3) \\ & + gM_w\,\eta W_\mu^+W^{-\mu} + \frac{\frac{1}{2}\,gM_w}{\cos^2\theta_w}\,\eta Z_\mu Z^\mu \\ & + \frac{1}{4}\,g^2\,\eta^2\,W_\mu^+W^{-\mu} + \frac{1}{8}\,\frac{g^2}{\cos^2\theta_w}\,Z_\mu Z^\mu\eta^2 \\ & + M_w^2\,W_\mu^+\,W^{-\mu} + \frac{1}{2}\,M_z^2\,Z_\mu Z^\mu \end{aligned} \tag{1.1.105}$$

We can now re-write L_{YM} and L_F in terms of A_μ, Z_μ, $W_\mu^\pm$

Recall

$$\binom{Z}{A} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\binom{A^3}{B}$$

which implies

$$\binom{A^3}{B} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}\binom{Z}{A}$$

that is

$$\begin{aligned} A_\mu^3 &= \cos\theta_w Z_\mu + \sin\theta_w A_\mu \\ B_\mu &= -\sin\theta_w Z_\mu + \cos\,\theta_w A_\mu \end{aligned} \tag{1.1.106}$$

So

$$
\frac{-ig}{2}\,\underline{\sigma}\cdot\underline{A}_\mu + \frac{ig}{2}\,B_\mu
$$

$$
= -\frac{i}{2}\begin{bmatrix} gA^3_\mu - g^{\prime}B_\mu & g(A^{\prime}_\mu - iA^2_\mu) \\ g(A^{\prime}_\mu + iA_\mu{}^2) & -gA^3_\mu \quad g^{\prime}B_\mu \end{bmatrix}
$$

$$
= -\frac{i}{2}\begin{bmatrix} Z\sqrt{g^2+g^{\prime 2}} & +g\sqrt{2}\,W^+ \\ +g\sqrt{2}\,W^- & \frac{g^{\prime 2}-g^2}{\sqrt{g^2+g^{\prime 2}}}\,Z - \frac{2gg^{\prime}}{\sqrt{g^2+g^{\prime 2}}}\,A \end{bmatrix}_\mu . \tag{1.1.107}
$$

This yields

$$
\bar{\ell}_{mL}\, i\not{\partial}\, \ell_{mL} = \begin{bmatrix} \bar{\nu}_{mL} & \bar{e}_{mL} \end{bmatrix} i\not{\partial} \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix}
$$

$$
+ \begin{bmatrix} \bar{\nu}_{mL} & \bar{e}_{mL} \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{g^2+g^{\prime 2}}\,\not{Z} & +\sqrt{2}\,g\,\not{W}^+ \\ +\sqrt{2}g\,\not{W}^- & \frac{g^{\prime 2}-g^2}{\sqrt{g^2+g^{\prime 2}}}\,\not{Z} - \frac{2gg^{\prime}}{\sqrt{g^2+g^{\prime 2}}}\,\not{A} \end{bmatrix} \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix} \tag{1.1.108}
$$

and

$$\bar{e}_R \, i\not{D} e_R = \bar{e}_R \, i\not{\partial} e_R + \bar{e}_R(-g')[-\sin\theta_w \not{Z} + \cos\theta_w \not{A}]e_R \tag{1.1.109}$$

$$= \bar{e}_R i \, \not{\partial} e_R + \bar{e}_R \, [\frac{g'^2}{\sqrt{g^2 + g'^2}} \not{Z} - \frac{gg'}{\sqrt{g^2 + g'^2}} \not{A}]e_R$$

We can combine these two to give

$$\bar{\ell}_{mL} i\not{D} \ell_{mL} + \bar{e}_R \, i\not{D} e_R$$

$$= \bar{\nu}_L \, i\not{\partial} \nu_L + \bar{e} i\not{\partial} e \tag{1.1.110}$$

$$+ \frac{\sqrt{g^2 + g'^2}}{2} \, Z_\mu \, [\bar{\nu}_L \gamma^\mu \nu_L + \frac{g'^2 - g^2}{g'^2 + g^2} \bar{e}_L \, \gamma^\mu \, e_L + \frac{g'^2}{g^2 + g'^2} \, \bar{e}_R \gamma^\mu \, e_R]$$

$$- \frac{gg'}{\sqrt{g^2 + g'^2}} \, A_\mu \, [\bar{e} \, \gamma^\mu \, e]$$

$$+ \frac{g}{\sqrt{2}} [\bar{\nu}_L \not{W}^+ e_L + \bar{e}_L \not{W}^- \nu_L]$$

Next we consider

$$\bar{q}_L i\not{\partial} q_L + \bar{u}_R i\not{\partial} u_R + \bar{d}_R\ i\not{\partial} d_R$$

$$= \quad \bar{u} i\not{\partial} u + \bar{d} i\not{\partial} d$$

$$+ \frac{\sqrt{g^2 + g'^2}}{2} \quad Z_\mu [\bar{u}_L \gamma^\mu u_L + \frac{g'^2 - g^2}{g'^2 + g^2} \bar{d}_L \gamma^\mu d_L \tag{1.1.111}$$

$$- \frac{4}{3} \frac{g'2}{g^2 + g'2} \ \bar{u}\ \gamma^\mu u - \frac{4}{3} \frac{g'^2}{g^2 + g'^2}\ d_L \gamma^\mu d_L$$

$$+ \frac{2}{3} \frac{g'^2}{g^2 + g'^2}\ d_R \gamma^\mu d_R]$$

$$+ \frac{gg'}{\sqrt{g^2 + g'^2}} \quad A_\mu [-\bar{d}_L \gamma^\mu d_L + \frac{2}{3}\ \bar{u}\ \gamma^\mu u$$

$$+ \frac{2}{3}\ \bar{d}_L \gamma^\mu d_L - \frac{1}{3}\ \bar{d}_R \gamma^\mu d_R]$$

$$+ \frac{g}{\sqrt{2}} [\bar{u}_L\ \not{W}^+ d_L + \bar{d}_L\ \not{W}^- u_L]$$

Putting this altogether L_F can be written as

$$L_F = \bar{\nu}_L\, i\not\partial \nu_L + \bar{e} i\not\partial e + \bar{u} i\not\partial u + \bar{d} i\not\partial d$$

$$+ eA_\mu J^\mu_{em} + \frac{g}{2\sqrt{2}} (J^\mu_w W^-_\mu + J^{\mu\dagger}_w W^\dagger_\mu) \qquad (1.1.112)$$

$$+ \sqrt{\frac{g^2 + g'^2}{2}}\; J^\mu_z Z_\mu$$

where the electromagnetic current J^μ_{em} is given by

$$J^\mu_{em} = [+ \frac{2}{3} \bar{u}\, \gamma^\mu u - \frac{1}{3} \bar{d}\, \gamma^\mu d - \bar{e}\, \gamma^\mu e] \qquad (1.1.113)$$

The charged weak current J^μ_w is defined by

$$J^\mu_w = 2(\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L\, \gamma^\mu\, u_L)$$

$$= (\bar{e}\, \gamma^\mu (1-\gamma_5)\nu + \bar{d}\, \gamma^\mu (1-\gamma_5) u) \qquad (1.1.114)$$

The weak neutral current is given by

$$J^\mu_z = \bar{\nu}_L \gamma^\mu \nu_L - \frac{g^2 - g'^2}{g^2 + g'^2} \bar{e}_L \gamma_\mu e_L$$

$$+ \frac{g'^2}{g^2 + g'^2} \bar{e}_R\, \gamma^\mu\, e_R$$

$$- \frac{4}{3}\; \frac{g'^2}{g^2 + g'^2} \bar{\mu}\, \gamma^\mu \mu + \bar{u}_L\, \gamma^\mu u_L$$

$$- \frac{g^2 - 1/3\, g'^2}{g^2 + g'^2}\; \bar{d}_L \gamma^\mu d_L \qquad (1.1.115)$$

$$+ \frac{2}{3} \frac{g'^2}{g^2 + g'^2}\; \bar{d}_R \gamma^\mu\, d_R$$

$$= \bar{\nu}_L \gamma^\mu \nu_L + 2\sin^2\theta_w \, \bar{e}\gamma^\mu e - \bar{e}_L \gamma^\mu e_L$$

$$- \frac{4}{3} \sin^2\theta_w \, \bar{u}\gamma^\mu u + \bar{u}_L \gamma^\mu u_L$$

$$+ \frac{2}{3} \sin^2\theta_w \, \bar{d}\gamma^\mu d - \bar{d}_L \gamma^\mu d_L$$

So we have the 3 types of currents:

1) $$J^\mu_w = (\bar{e}_m \gamma^\mu (1-\gamma_5)\nu_m + \bar{d}_m \gamma^\mu (1-\gamma_5) u_m) \qquad (1.1.116)$$

2) $$J^\mu_{em} = q_M \bar{\psi}_M \gamma^\mu \psi_M \qquad (1.1.117)$$

$$= + \frac{2}{3} \bar{u}_m \gamma^\mu u_m - \frac{1}{3} \bar{d}_m \gamma^\mu d_m - \bar{e}_m \gamma^\mu e_m$$

3) $$J^\mu_z = \bar{\psi}_M \gamma^\mu T^3_M (1-\gamma_5)\psi_M$$

$$-2q_M \sin^2\theta_w \bar{\psi}_M \gamma^\mu \psi_M \qquad (1.1.118)$$

$$= \bar{\nu}_L \gamma^\mu \nu_L - \bar{e}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu u_L - \bar{d}_L \gamma^\mu d_L$$

$$+ 2 \sin^2\theta_w (\bar{e} \gamma^\mu e - \frac{2}{3} \bar{u}\gamma^\mu u + \frac{1}{3} \bar{d} \gamma^\mu d)$$

where T^3_M is value of T^3 for ψ_M (i.e., 0, $\pm \frac{1}{2}$) and q_M is the charge and M sums over all fermions.

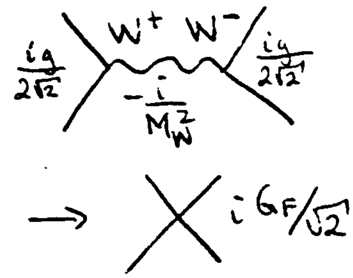
The masses of M_z and M_w can be estimated by the low energy effective theory. Recall $W^{\pm}$ has a propagator like

$$\frac{g_{\mu\nu}}{p^2 - M_w^2}$$

and Z^0 like

$$\frac{g_{\mu\nu}}{p^2 - M_z^2} \tag{1.1.119}$$

at low energy. For a charged current interaction we find



$$L_c^{eff} = \frac{g^2}{8} (J_w^{\mu} J_{w\mu}^{\dagger} + \text{h.c.}) \frac{-1}{M_w^2}$$

$$= + \frac{g^2}{8M_w^2} (J_w^{\mu} J_{w\mu}^{+} + \text{h.c.})$$

$$\equiv + \frac{G_F}{\sqrt{2}} (J_w^{\mu} J_{w\mu}^{+} + \text{h.c.}) \tag{1.1.120}$$

that is

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_w^2} = \frac{1}{2v^2} \quad . \tag{1.1.121}$$

Now recall $e = g\sin\theta_w$; $e^2 = 4\pi\alpha$

So

$$m_w^2 = \frac{\sqrt{2}}{G_F} \frac{e^2}{8\sin^2\theta_w} = \frac{(37\text{ GeV})^2}{\sin^2\theta_w} \tag{1.1.122}$$

$$M_w = \frac{37\text{ GeV}}{\sin\theta_w} \tag{1.1.123}$$

$$M_z = \frac{M_w}{\cos\theta_w} = \frac{75\text{ GeV}}{\sin 2\theta_w} \qquad \text{to lowest order.}$$

4) Finally we must evaluate the Yukawa Lagrangian to determine the fermion masses.

$$L_{yuk} = \Gamma_{mn}^{e} \bar{\ell}_{mL} \begin{bmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{bmatrix} e_{nR} + \Gamma_{mn}^{d} \bar{q}_{mL} \begin{bmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{bmatrix} d_{nR}$$

$$+ \Gamma_{mn}^{u} \bar{q}_{mL} \begin{bmatrix} \frac{v+\eta}{\sqrt{2}} \\ 0 \end{bmatrix} u_{nR} \quad + \text{h.c.} \tag{1.1.124}$$

$$= \Gamma^e_{mn} \left[\frac{v + \eta}{\sqrt{2}}\right] \bar{e}_{mL}\, e_{nR} + \Gamma^d_{mn} \left[\frac{v + \eta}{\sqrt{2}}\right] \bar{d}_{mL}\, d_{nR}$$

$$+ \Gamma^u_{mn} \left[\frac{v + \eta}{\sqrt{2}}\right] \bar{u}_{mL}\, u_{nR} + \text{h.c.}$$

$$= -\bar{e}_L M^e e_R + \bar{e}_L h^e e_R \eta$$

$$- \bar{u}_L M^u u_R + \bar{u}_L h^u u_R \eta$$

$$- \bar{d}_L M^d d_R + \bar{d}_L h^d d_R \eta \qquad + \text{h.c.}$$

where

$$M^e_{mn} \equiv \frac{-v}{\sqrt{2}} \Gamma^e_{mn} \text{ ; similarly for } M^u \text{ and } M^d$$

and

$$h^e_{mn} \equiv \frac{1}{\sqrt{2}} \Gamma^e_{mn} = \frac{-1}{v} M^e_{mn} \tag{1.1.125}$$

$$= \frac{-g}{2M_W} M^e_{mn} \text{ , similarly for } h^u \text{ and } h^d.$$

The mass matrix is in general not diagonal; the weak interaction fields we have been using are not the "physical" fields i.e. mass matrix eigenfields. Generally the Γ_{mn} are complex and non-hermitian. But we can still diagonalize the mass matrix by a left-right transformation for $M^{e,u,d}$; We have 3 x 3 matrices call them M with real, positive eigenvalues m_1, m_2, m_3 that is

$$M_{diag} = \begin{bmatrix} m_1 & & 0 \\ & m_2 & \\ 0 & & m_3 \end{bmatrix} = A_L^\dagger M A_R \tag{1.1.126}$$

(i.e. $(m_1, m_2, m_3) = (m_e, m_\mu, m_\tau)$ or (m_u, m_c, m_L) or (m_d, m_s, m_b)).
where A_L and A_R are 3 x 3 unitary matrices ($A_L = A_R$ if M is hermitian). Further we can almost determine A_L and A_R uniquely by noting that $MM^\dagger$ and $M^\dagger M$ are hermitian:

$$(MM^\dagger)^\dagger = MM^\dagger \; ; \; (M^\dagger M)^\dagger = M^\dagger M \quad . \tag{1.1.127}$$

and that

$$\begin{aligned} A_L^\dagger MM^\dagger A_L &= A_L^\dagger M \; A_R A_R^\dagger M^\dagger A_L \\ &= M_{diag} M_{diag}^\dagger = \begin{bmatrix} m_1^2 & & 0 \\ & m_2^2 & \\ 0 & & m_3^2 \end{bmatrix} \end{aligned} \tag{1.1.128}$$

$$\begin{aligned} A_R^\dagger M^\dagger M \; A_R &= A_R^\dagger M^\dagger \; A_L A_L^\dagger M \; A_R = M_{diag}^\dagger \; M_{diag} \\ &= \begin{bmatrix} m_1^2 & & 0 \\ & m_2^2 & \\ 0 & & m_3^2 \end{bmatrix} \end{aligned} \tag{1.1.129}$$

We can determine A_L and A_R by diagonalizing $MM^\dagger$, $M^\dagger M$ using simple linear algebra. This fixes A_L and A_R only up to 3 arb. phases for R and 3 for L that is if we let

$$A_L' = A_L K_L \quad ; \quad A_R' = A_R K_R$$

$$K_{\substack{L\\R}} = \begin{bmatrix} e^{i\phi_{1\substack{L\\R}}} & & 0 \\ & e^{i\phi_{2\substack{L\\R}}} & \\ 0 & & e^{i\phi_{3\substack{L\\R}}} \end{bmatrix} \tag{1.1.130}$$

;

Then

$$\begin{aligned} & A_L'^\dagger \; MM^\dagger \; A_L' \\ &= K_L^\dagger A_L^\dagger \; MM^\dagger \; A_L K_L = K_L^\dagger M_{diag}^2 \; K_L \\ &= M_{diag}^2 . \end{aligned} \tag{1.1.131}$$

and also

$$A'^{\dagger}_{R}\ M^{\dagger} M\ A'_{R} = M^{2}_{diag} \qquad (1.1.132)$$

So $A'_{\substack{R\\L}}$ obey the same equations as $A_{\substack{R\\L}}$. However, we still can use

$A^{+}_{L}\ M\ A_{R} = M_{diag.}$ to determine the phase differences

$\phi_{iL} - \phi_{iR}$ since

$$A'^{+}_{L}\ MA'_{R} = K^{+}_{L}\ M_{diag}\ K_{R} \qquad (1.1.133)$$

$$= \begin{bmatrix} m_1 e^{-i(\phi_{1L} - \phi_{1R})} & & 0 \\ & m_2 e^{-i(\phi_{2L} - \phi_{2R})} & \\ 0 & & m_3 e^{-i(\phi_{3L} - \phi_{3R})} \end{bmatrix} .$$

Each term must be <u>real</u> and <u>positive</u>

$$\phi_{iL} = \phi_{iR} \quad ! \qquad (1.1.134)$$

$$\equiv \phi_i$$

So $K_L = K_R = K$ only are arbitrary.

(Alternatively we can redefine phases for the fields

$$u'_L = K_L^\dagger u_L$$

$$u'_R = K_R^\dagger u_R$$

L_{yuk} maintains the same form only if $K_L = K_R$.)

So back to mass eigenstates. Defining new fields (concentrating on u_L, u_R for the moment)

$$u_L^W = A_L^u u_L \qquad (1.1.135)$$

$$u_R^w = A_R^u u_R$$

where I've now put a superscript W(eak) on all the fields in the SU(2) x U(1) interaction basis - i.e. give the fields we've been working with in the Lagrangian a superscript W. The mass-eigenfields now have no superscript.

Similarly for

$$d_L^w = A_L^d d_L \qquad e_L^w = A_L^e e_L \qquad (1.1.136)$$

$$d_R^w = A_R^d d_R \qquad e_R^w = A_R^e e_R \quad .$$

The mass and Yukawa interaction terms now become diagonalized

$$
\begin{aligned}
L_{yuk} &= -\bar{u}_L^w M^u u_R^w - \bar{u}_L^w h^u u_R^w \eta + \ldots \\
&= -\bar{u}_L A_L^{u\dagger} M^u A_R^u u_R - \frac{g}{2M_w} \bar{u}_L A_L^{u\dagger} M^u A_R^u u_R \eta + \ldots \\
&= -\bar{\upsilon}_L \begin{bmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{bmatrix} \upsilon_R - \frac{g}{2M_w} \eta \bar{\upsilon}_L \begin{bmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{bmatrix} \upsilon_R \\
&\quad + \ldots
\end{aligned}
\qquad (1.1.137)
$$

$$= -\left[1 + \frac{g}{2M_w}\eta\right]\left\{ \begin{bmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{bmatrix} \begin{bmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{bmatrix} \begin{bmatrix} u_R \\ c_R \\ t_R \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \bar{u}_R & \bar{c}_R & \bar{t}_R \end{bmatrix} \begin{bmatrix} m_u & & \\ & m_c & \\ & & m_t \end{bmatrix} \begin{bmatrix} u_L \\ c_L \\ t_R \end{bmatrix} \right\} + \ldots$$

Finally yielding

$$\begin{aligned} L_{yuk} = -\left[1 + \frac{g}{2M_w}\eta\right][m_u\bar{u}u + m_c\bar{c}c + m_t\bar{t}t \\ + m_d\bar{d}d + m_s\bar{s}s + m_b\bar{b}b \\ + m_e\bar{e}e + m_\mu\bar{\mu}\mu + m_\tau\bar{\tau}\tau \end{aligned} \tag{1.1.138}$$

So L_{yuk} only measures the eigenvalues - i.e. the masses. We would like to determine the rest of $\Gamma^{e,d,u}_{mn}$, i.e. A_L, A_R. To do this we express the gauge couplings in L_F in terms of the mass eigen

Since T^3 and y (hence Q) are the same for each generation we find J^μ_{EM}, J^μ_z have the same form [no change]. Thus the GWS with left-handed-fermion fields in doublets has <u>no</u> flavor changing neutral currents [no strangeness changing currents:GIM]. However, the J^μ_w becomes

$$\begin{aligned} J^\mu_w &= \bar{e}^w\,\gamma^\mu(1-\gamma_5)\nu^w + \bar{d}^w\,\gamma^\mu(1-\gamma_5)u^w \\ &= \bar{e}\,\gamma^\mu(1-\gamma_5)A_L^{\dagger e}\,\nu^w + \bar{d}\gamma^\mu(1-\gamma_5)A_L^{d\dagger}A_L^u u \\ &\equiv \bar{e}\,\gamma^\mu(1-\gamma_5)\nu + \bar{d}\,\gamma^\mu(1-\gamma_5)\,A_c u \end{aligned} \tag{1.1.139}$$

where

$$A_{c\,mn} \equiv (A_L^{d\dagger}\, A_L^u)_{mn}$$

and (1.1.140)

$$\nu_L \equiv A_L^{e\dagger}\, \nu_L^w$$

since neutrinos are massless and we just associate ν_{mL} with e_{mL}. A_c is the generalized Cabibbo matrix - it fixes the flavor structure of J_w^μ. So through gauge interactions we can determine some of the $\Gamma_{mn}^{e,d,u}$. Note that as before

$$\ell_{mL} = \begin{bmatrix} \nu_m \\ e_m \end{bmatrix}_L$$

and (1.1.141)

$$q_{mL} = \begin{bmatrix} A_{cmn}\, u_n \\ \\ d_m \end{bmatrix}_L$$

transform as SU[2] doublets, while u_{mR}, d_{mR}, e_{mR} are SU(2) singlets. A_R cannot be measured in the GWS model,since it never appears in the currents.We can only determine some of the A_L; that is A_c which appears in J_w^μ.

A_c are all the Higgs couplings the GWS model can measure!

A_c is a 3 x 3 unitary matrix since the A_L are and

$$A_c = A_L^{d\dagger}\, A_L^u \qquad (1.1.142)$$

so it has 3^2 real parameters. But recall there are 3 arb. phases for each A_L. It appears we can choose 6 of the A_c parameters at will, they are not observable and can be chosen for convenience. Actually, 5 of the phases only can be chosen to put A_c in a convenient form. The other phase does not occur in the Lagrangian and so is irrelevant.

So A_c depends only on 9 - 5 = 4 real parameters. 3 of the parameters represent rotations of the families into each other. The $4^{\underline{th}}$ parameter, an observable phase angle, measures the amount of CP violation in the original Yukawa couplings. Kobayashi and Maskawa first expressed A_c this way and so A_c is called the KM matrix.

A_{KM} in this 3 family (6 quark) GWS model can be written as

$$A_{KM} = \begin{bmatrix} C_1 & -S_1C_2 & -S_1S_2 \\ +S_1C_3 & C_1C_2C_3 + S_2S_3e^{-i\delta} & C_1S_2C_3 - C_2S_3e^{-i\delta} \\ S_1S_3 & C_1C_2S_3 - S_2C_3e^{-i\delta} & +\ C_1S_2S_3 + C_2C_3e^{-i\delta} \end{bmatrix} \quad (1.1.143)$$

θ_1, θ_2, θ_3 are the 3 rotation angles, and δ is the CP violating phase.

Phenomenology of the KM version of the GWS model:

1) A_{KM} not well determined

a) $|c_1| = 0.9737 \pm 0.0025$

$|s_3| = 0.28^{+0.21}_{-0.28}$

from β-decay and semi-leptonic hyperon decay.

b) $K_L - K_S$ mass difference $\rightarrow$

$0.1 < |S_2| < 0.7$

c) ε parameter for CP violation in Kaons $\rightarrow$ $s_2s_3\sin\delta = 0[10^{-3}]$.

2) Neutral current interactions:

ν - hadron, ν-e, e-hadron

$\rightarrow \sin^2\theta_W = 0.229 \pm 0.009$ [exp.]

$\pm\ 0.005$ [theory].

3) Higgs mass $m_\eta = \sqrt{-2\mu^2} = \sqrt{2\lambda}\quad v$

$v = 246$ GeV from G_F

$m_\eta = \sqrt{2\lambda}\quad 250$ GeV

various arguments: $m_\eta < 200$ GeV .

4) Charmed hadron decay, non-leptonic hyperon and kaon decays, $\Delta I = \frac{1}{2}$, CP violation, $K_L - K_S$ mass difference: all seem to be compatible with the model.

5) CERN UA1 Preliminary measurement of $w^\pm$ mass

$$M_w = 81 \pm 5 \text{ GeV!}$$

1.2 QUANTUM CHROMODYNAMICS

1.2.1. Particle Representations and the Symmetric SU(3) Color Lagrangian

The theory of strong interactions is based on the gauge group SU(3) of color. SU(3) is the group of 3 x 3 unitary matrices with determinant 1. There are $3^2 - 1 = 8$ generators for this group and they obey the commutation relations

$$[T^i, T^j] \equiv if_{ijk}T^k \tag{1.2.1}$$

where

T^i $i = 1\ldots,8$ are the generators of the group and f_{ijk} is the completely anti-symmetric structure constants for the group. These are given by: (recall for SU(2) $f_{ijk} = \varepsilon_{ijk}$ $i,j,k = 1,2,3$)

$$\begin{aligned} f_{123} &= +1 \\ f_{147} = f_{246} = f_{257} = f_{345} &= +\frac{1}{2} \\ f_{156} = f_{367} &= -\frac{1}{2} \\ f_{458} &= \frac{1}{2}\sqrt{3} \\ f_{678} &= \frac{1}{2}\sqrt{3} \end{aligned} \tag{1.2.2}$$

all others [not related by permutations] = 0. As in the SU(2) case we have 2 representations of the T^i that are very important.

1) The fundamental representation of SU(3) called 3 (triplet of SU(3)) is given by the 8 Gell-Mann λ^i matrices which are the analogs of the three Pauli matrices for SU(2). These 3 x 3 matrices are

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda^4 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{1.2.3}$$

$$\lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Then we represent $T^i \to T^i \equiv \frac{1}{2}\lambda^i$ and $[T^i, T^j] = if_{ijk} T^k$. Also we note that $Tr[\lambda^i \lambda^j] = 2\delta^{ij}$ and the anti-commutator is

$$\{\lambda^i, \lambda^j\} = \frac{4}{3}\delta^{ij} + 2d_{ijk}\lambda^k \tag{1.2.4}$$

where d_{ijk} is completely symmetric in its indices and its non-zero elements are given by

$$\begin{aligned} d_{118} &= d_{228} = d_{338} = -d_{888} = \sqrt{1/3} \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = \\ &= d_{355} = -d_{366} = -d_{377} = +\frac{1}{2} \\ d_{448} &= d_{558} = d_{668} = d_{778} = \frac{-1}{2\sqrt{3}} \end{aligned} \tag{1.2.5}$$

Every 3 x 3 unitary matrix $U(\phi_1, \ldots, \phi_8)$ of determinant 1 has $3^2 - 1 = 8$ independent matrix elements and is given by

$$U = e^{+ig_s \underline{\phi} \cdot \underline{T}} = e^{+\frac{ig_s}{2}\phi_i \lambda^i} \tag{1.2.6}$$

2) The second important representation is the adjoint representation given by the 8 x 8 matrices made from the structure constants themselves:

$$T^i_{jk} \equiv -i\, f_{ijk} \tag{1.2.7}$$

i.e.

$$T^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

etc.

Then $[T^i, T^j] = if_{ijk}T^k$ as a consequence of the Jacobi identity

$$[T^i, [T^j, T^k]] + [T^j, [T^k, T^i]] + [T^k, [T^i, T^j]] \equiv 0 \qquad \text{yielding}$$

$$[f_{jk\ell}f_{i\ell m} + f_{ki\ell}f_{j\ell m} + f_{ij\ell}\, f_{k\ell m}]T^m = 0$$

So

$$f_{jk\ell}\, f_{i\ell m} - f_{ik\ell}f_{j\ell m} = f_{ij\ell}\, f_{\ell km} \tag{1.2.8}$$

Since f_{ijk} is completely anti-symmetric thus

$$(-i)f_{ik\ell}\,(-i)f_{j\ell m} - (-i)f_{jk\ell}(-i)f_{i\ell m} = if_{ij\ell}(-i)f_{\ell km}$$

$$\rightarrow [T^i, T^j] = if_{ijk}T^k \ .$$

Since SU(3) is an 8 parameter group we have 8 vector bosons $G_\mu{}^i, i = 1,\dots,8$ These transform inhomogeneously under a gauge transformation; for any representation T^i of SU(3)

$$\text{let } U(\underline{\phi}) = e^{+ig_s\underline{\phi}\cdot\underline{T}} \quad ; \quad U^\dagger = U^{-1} = e^{-ig_s\underline{\phi}\cdot\underline{T}} \tag{1.2.9}$$

and the transformation is defined analogously to (1.1.4)

$$\underline{T}\cdot\underline{G}'_\mu = U(\underline{\phi})\underline{T}\cdot\underline{G}_\mu U^{-1}(\underline{\phi}) - \frac{i}{g_s}(\partial_\mu U(\underline{\phi}))U^{-1}(\underline{\phi}) \ . \tag{1.2.10}$$

Infinitesimally $\quad U \simeq 1 + ig_s\underline{\phi}\cdot\underline{T} \ ; \ U^{-1} \simeq 1 - ig_s\underline{\phi}\cdot\underline{T} \qquad (1.2.11)$

and we find that

$$\underline{T}\cdot\underline{G}'_\mu = \underline{T}\cdot\underline{G}_\mu + \underline{T}\cdot\partial_\mu\underline{\phi} + ig_s[\underline{\phi}\cdot\underline{T}, \underline{T}\cdot G_\mu] \tag{1.2.12}$$

$$= \underline{T}\cdot\underline{G}_\mu + T^k[\partial_\mu \phi^k + g_s f_{kji} G^j_\mu \phi^i]$$

Thus

$$G^{i'}_\mu = G^i_\mu + \partial_\mu \phi_i + g_s f_{ijk} G^j_\mu \phi^k \tag{1.2.13}$$

or defining $(\underline{A} \times \underline{B})_i \equiv f_{ijk} A_j B_k$

$$\underline{G}'_\mu = \underline{G}_\mu + \partial_\mu\underline{\phi} + g_s\underline{G}_\mu \times \underline{\phi} \tag{1.2.14}$$

As in the SU(2) case we can form the anti-symmetric covariant field strength tensor by using the covariant derivative:

$$D_\mu \equiv \partial_\mu - ig_s\underline{T}\cdot\underline{G}_\mu \tag{1.2.15}$$

$$\underline{T}\cdot\underline{F}_{\mu\nu} \equiv D_\mu(\underline{T}\cdot\underline{G}_\nu) - D_\nu(\underline{T}\cdot\underline{G}_\mu) \tag{1.2.16}$$

which yields

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{G}_\nu - \partial_\nu \underline{G}_\mu + g_s \underline{G}_\mu \times \underline{G}_\nu \qquad (1.2.17)$$

or

$$F^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu + g_s f_{ijk} G^j_\mu G^k_\nu$$

Once again $\underline{F}_{\mu\nu}$ transforms homogeneously

using $\quad U^{-1}U = 1 \rightarrow \partial_\mu U^{-1}U = -U^{-1}\partial_\mu U$,

$$\underline{T}\cdot\underline{F}'_{\mu\nu} = U(\underline{\phi})\ \underline{T}\cdot\underline{F}_{\mu\nu} U^{-1}(\underline{\phi}) \quad . \qquad (1.2.18)$$

Infinitesimally we have

$$\underline{T}\cdot\underline{F}'_{\mu\nu} = \underline{T}\cdot\underline{F}_{\mu\nu} + ig_s[\underline{\phi}\cdot\underline{T}, \underline{T}\cdot\underline{F}_{\mu\nu}]$$

$$= T^i(F^i_{\mu\nu} + g_s f_{ijk} F^j_{\mu\nu}\ \phi^k]$$

$$\underline{F}'_{\mu\nu} = \underline{F}_{\mu\nu} + g_s \underline{F}_{\mu\nu} \times \underline{\phi} \qquad (1.2.19)$$

analogously to the SU(2) case equation (1.1.19).

Thus the SU(3) invariant pure Yang-Mills part of the QCD Lagrangian is

$$L_{ym} = -\frac{1}{4}\underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} \qquad (1.2.20)$$

The leptonic matter fields are invariant under the SU(3) color transformations as are the Higgs fields. For each flavor the quarks transform as a fundamental triplet of SU(3): let $q^a_f \varepsilon \{u^a, d^a, c^a, s^a, t^a, b^a\}$, where f indicates the flavor u, d, c, s, t, or b; a = 1,2,3 = color = R,G,B . The fundamental representation is given by $T^i \equiv \frac{1}{2}\lambda^i$ and

$$q' = U(\underline{\phi})q = e^{\frac{ig_s}{2}\underline{\phi}\cdot\underline{\lambda}}\ q \qquad (1.2.21)$$

where q and q' are column vectors with 3 rows. Thus infinitestimally

$$q'^a = q^a + \frac{ig_s}{2}(\underline{\phi}\cdot\underline{\lambda})^{ab}\ q^b \qquad (1.2.22)$$

Thus the covariant derivative is defined by

$$D_\mu q = [\partial_\mu - \frac{ig_s}{2}\underline{\lambda}\cdot\underline{G}_\mu]q \qquad (1.2.23)$$

and as usual

$$(D_\mu q)' = [\partial_\mu - \frac{ig_s}{2} \underline{\lambda}\cdot\underline{G}'_\mu]q'$$
$$= [\partial_\mu - \frac{ig_s}{2} U\underline{\lambda}\cdot\underline{G}_\mu U^{-1} - (\partial_\mu U)U^{-1}]U\, q$$

which gives $$(D_\mu q)' = UD_\mu q \tag{1.2.24}$$

The invariant kinetic energy terms then become

$$\bar{q}\ i\not{D}\ q \tag{1.2.25}$$

since $$\bar{q}' = \bar{q}U^\dagger = \bar{q}U^{-1}$$

$$\rightarrow$$

we have $$(\bar{q}\ i\not{D}q)' = (\bar{q}\ i\not{D}q) \tag{1.2.26}$$

Note that we must sum over flavors and colors so

$$\bar{q}\ i\not{D}\, q = \bar{q}^a_f\ i\not{D}^{ab}\ q^b_f \tag{1.2.27}$$

We see that SU(3) invariant mass terms are also allowed (ignoring the fact that these break SU(2) x U(1) but assume they arise from the Higgs mechanism discussed previously). Thus the current quark mass term is $-\sum_f m_f\, \bar{q}^a_f q^a_f$. The mass "matrix" is taken diagonal since the kinetic energy terms are unchanged under the A_L and A_R transformations. Thus the QCD Lagrangian is

$$L^{QCD} = -\frac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} + \bar{q}\ i\not{D}\, q - m\bar{q}q \tag{1.2.28}$$

1.2.2. Asymptotic Freedom and Confinement

The most important property of QCD is its asymptotic freedom. That is the momentum dependent coupling constant decreases to zero as the momentum increases due to the anti-screening properties of the gluons. The renormalization group equations provide the formal techniques for discussing the asymptotic properties of a theory.

Generally speaking the normalization of field operators is determined from the asymptotic conditions which state that a field $\phi(x) \xrightarrow[x_o \to \pm\infty]{LSZ} \phi_{\substack{out \\ in}}(x)$ where $\phi_{\substack{out \\ in}}$ are the free outgoing and incoming field operators. Thus ϕ is normalized to the free field, that is the propagator is normalized on mass shell with residue one.

When one deals with massless theories, on-shell normalization is usually not possible due to the infrared singular behavior of the Green functions. Thus we must normalize the field operators off mass shell at some normalization mass $\mu^2 < 0$. In addition, the coupling constant should be defined at the off-shell mass μ^2. As will be seen, the observables are independent of this field normalization. This invariance of the observables with respect to the field normalization gives rise to the differential equation known as the renormalization group equation.

In order to be concrete let's consider the self-interacting scalar ϕ^4 model with renormalized (finite) parameters mass m and coupling constant g. The field operators (and observables) are only determined uniquely when we specify normalization conditions on the propagator and 4 point function. That is we must specify the relation of m,g to the mass of the particles and their interaction strength. Usually these conditions are specified on mass shell, but even for this massive theory let's specify them off-mass shell so as to avoid possible IR singularities in the Green functions as $m \to 0$. Further, m is a mass parameter and is not necessarily the physical mass m_p (i.e. the position of the pole in the propagator) thus we can fix its value at some momentum which we take to be zero for convenience,

$$\Delta_F^{-1}(0,m^2,\mu^2,g) = m^2 \qquad (1.2.29)$$

where $(\Delta_F^{-1})\Delta_F$ is the (inverse) propagator and $\mu^2 \leq m^2$ the normalization mass to be discussed below.

The normalization of the field ϕ is specified usually by normalizing the residue of the propagator on shell to be 1

$$-i(k^2-m_p^2)\Delta_F(k^2)\Big|_{k^2=m_p^2} = 1 \qquad (1.2.30)$$

but keeping in mind that this diverges as $m_p \to 0$ we normalize the field off mass shell:

$$-i(k^2-m_p^2)\Delta_F(k^2,m^2,\mu^2,g) = 1 \quad \text{at} \quad k^2=\mu^2 \qquad (1.2.31)$$

Finally the coupling constant must also be fixed. This is done by setting it equal to the vertex function at a specified off-shell momentum:

$$\Gamma(k_1,k_2,k_3,k_4,m^2,\mu^2,g) = -ig \qquad (1.2.32)$$

$$\text{at} \quad k_i^2 = \mu^2 \ ; \quad (k_i+k_j)^2 = \frac{4}{3}\mu^2 \quad i\neq j \ ,$$

where Γ is the 4 point vertex (1PI) function. Thus to summarize there is no unique choice of renormalized parameters m and g and these must be defined in the theory. We choose the conditions

$$1) \quad \Delta_F^{-1}(0,m^2,\mu^2,g) = m^2 \tag{1.2.33}$$

$$2) \quad \Gamma(k_1,k_2,k_3,k_4,m^2,\mu^2,g) = -ig$$

$$\text{at} \qquad k_i^2 = \mu^2 \; ; \qquad (k_i+k_j)^2 = \frac{4}{3}\mu^2 \qquad i\neq j \; . \tag{1.2.34}$$

Further the normalization of the field operator ϕ is fixed by requiring the "off-shell residue" condition

$$3) \quad -i(k^2-m_p^2)\Delta_F(k^2,m^2,\mu^2,g) = 1 \quad \text{at} \quad k^2 = \mu^2 \; , \tag{1.2.35}$$

(where m_p is the physical mass determined from $\Delta_F^{-1}(m_p^2,m^2,\mu^2,g) = 0$; actually we could choose any $[\text{mass}]^2 \neq \mu^2$ here, even 0). These conditions uniquely specify the field operator $\phi = \phi[x,m^2,\mu^2,g]$.

If we change the normalization point $\mu \to \mu'$, we find that new parameters m' and g' can be defined using the normalization conditions (1) and (2) in terms of μ', m', g' such that the new field operator differs only by a new normalization from the old field

$$\phi'(x,m'^2,\mu'^2,g') = Z^{1/2}\,\phi(x,m^2,\mu^2,g) \tag{1.2.36}$$

where the field normalization condition (3), in terms of μ', m' and g', will determine Z.

The renormalization group is defined as the group of transformations

$$\phi \to \phi'(x,m'^2,\mu'^2,g') = Z^{1/2}\,\phi(x,m^2,\mu^2,g) \; . \tag{1.2.37}$$

It is trivial in the sense that the transformation only changes the (arbitrary) normalization of the field operator (the physical parameters like m_p are unchanged). Field operators related by a renormalization $\phi' = Z^{1/2}\,\phi$ are called equivalent. Two sets of parameters m,μ,g and m',μ',g' are equivalent if the corresponding field operators are equivalent. Thus the set of all fields related by $\phi' = Z^{1/2}\,\phi$ and the set of all parameter triples (m,μ,g) are divided into equivalence classes.

Towards determining Z,m',g' it is useful to introduce the effective or running coupling constant which is invariant under renormalization group transformations

$$g(k^2) \equiv g(k^2,m^2,\mu^2,g) \equiv i \prod_{j=1}^{4} \sqrt{-i(k_j^2-m_p^2)\Delta_F(k_j)}\;\Gamma \qquad (1.2.38)$$

where k_i are some constant momenta such as those defined by

$$k_j^2 = k^2 \;; \qquad (k_i+k_j)^2 = \frac{4}{3}\,k^2 , \qquad i\neq j \qquad (1.2.39)$$

Thus for $\phi \to Z^{1/2}\,\phi$ we have $\Delta_F \to Z\Delta_F$ and $\Gamma \to \frac{1}{Z^2}\,\Gamma$ so the running coupling constant $g(k^2) \to g(k^2)$ is a renormalization group invariant. Because of (3) we can re-express (2) as

$$(2) \quad g(\mu^2) = g(k^2,m^2,\mu^2,g)\Big|_{k^2=\mu^2} = g \,. \qquad (1.2.40)$$

We can also introduce the effective residue r, by the dimensionless ratio

$$r(k^2,m^2,\mu^2,g) \equiv -i(k^2-m_p^2)\Delta_F(k^2,m^2,\mu^2,g) \,. \qquad (1.2.41)$$

It is further assumed that the Green's functions obey scale invariance in the sense of engineering dimensional analysis; so if we scale all dimensionful quantities by λ we find

$$G(x_1,\ldots,x_n,\lambda^2 m^2,\lambda^2\mu^2,g) = \lambda^{n-4}G(\lambda x_1,\ldots,\lambda x_n,m^2,\mu^2,g) \,. \qquad (1.2.42)$$

This implies that dimensionless quantities are functions of dimensionless parameters g, k^2/μ^2, m^2/μ^2 only. In particular

$$\begin{aligned} r(k^2,m^2,\mu^2,g) &= r(k^2/\mu^2,m^2/\mu^2,g) \\ g(k^2,m^2,\mu^2,g) &= g(k^2/\mu^2,m^2/\mu^2,g) \,. \end{aligned} \tag{1.2.43}$$

Thus we can write the normalization conditions (2) and (3) as

$$\begin{aligned} r(1,m^2/\mu^2,g) &= 1 \\ g(1,m^2/\mu^2,g) &= g \,. \end{aligned} \tag{1.2.44}$$

We now change $\mu \to \mu'$ and apply the renormalization group transformation

$$\phi'(x,m'^2,\mu'^2,g') = Z^{1/2}\,\phi(x,m^2,\mu^2,g) \,. \tag{1.2.45}$$

The normalization condition (3) for ϕ' now reads

$$-i(k^2-m_p^2)\Delta_F(k^2,m'^2,\mu'^2,g') = 1 \qquad \text{at} \qquad k^2 = \mu'^2 \tag{1.2.46}$$

implying $$-i(k^2-m_p^2)Z\Delta_F(k^2,m^2,\mu^2,g) = 1 \qquad \text{at} \qquad k^2 = \mu'^2 \tag{1.2.47}$$

Thus

$$-i(\mu'^2-m_p^2)\Delta_F(\mu'^2,m^2,\mu^2,g) = Z^{-1} \tag{1.2.48}$$

So

$$Z^{-1} = r(\mu'^2/\mu^2,m^2/\mu^2,g) \,. \tag{1.2.49}$$

Since $g(k^2)$ is a renormalization group invariant

$$g(k^2/\mu'^2,m'^2/\mu'^2,g') = g(k^2/\mu^2,m^2/\mu^2,g) \tag{1.2.50}$$

Setting $k^2 = \mu'^2$ and applying condition (2) for the ϕ' variable $g(1,m'^2/\mu'^2,g') = g'$ we have

$$g' = g(\mu'^2/\mu^2, m^2/\mu^2, g) \tag{1.2.51}$$

Finally since

$$\Delta_F^{-1}(k^2,m'^2,\mu'^2,g') = \frac{1}{Z}\,\Delta_F(k^2,m^2,\mu^2,g) \tag{1.2.52}$$

Normalization condition (1) implies

$$m'^2 = \frac{1}{Z}\,m^2 \quad . \tag{1.2.53}$$

Since the normalization conditions uniquely determine ϕ; then if $\mu \to \mu'$ we find $\phi' = Z^{1/2}\,\phi$ with Z,m',g' uniquely determined above.

If two field operators are equivalent

$$\phi'(x,m'^2,\mu'^2,g') = Z^{1/2}\,\phi(x,m^2,\mu^2,g) \tag{1.2.54}$$

then the Fourier transforms of their time ordered functions are related

$$G(k_1,\ldots,k_n,m'^2,\mu'^2,g') = Z^{n/2}\,G(k_1,\ldots,k_n,m^2,\mu^2,g) \ . \tag{1.2.55}$$

Hence we can differentiate w.r.t. μ' and setting $\mu' = \mu$ we find the standard form of the renormalization group equation

$$\left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta\,\frac{\partial}{\partial g} + \gamma_m\,\frac{\partial}{\partial m^2} + n\gamma\right] G(k_1,\ldots,k_n,m^2,\mu^2,g) = 0 \ , \tag{1.2.56}$$

where

$$\beta = \mu^2 \frac{\partial g'}{\partial \mu'^2}\bigg|_{\mu'=\mu} = \frac{\partial g(\mu'^2/\mu^2, m^2/\mu^2, g)}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\frac{\mu'^2}{\mu^2}=1}$$

which yields

$$\beta = \frac{\partial g(x, m^2/\mu^2, g)}{\partial x}\bigg|_{x=1} \tag{1.2.57}$$

and

$$\gamma = -\frac{1}{2} \frac{\partial \ell n Z}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu} = -\frac{1}{2}\frac{1}{Z} \frac{\partial Z}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu}$$

$$= -\frac{1}{2} r \frac{\partial \frac{1}{r}}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu} = +\frac{1}{2}\frac{1}{r} \frac{\partial r}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu}$$

which yields

$$\gamma = +\frac{1}{2} \frac{\partial r(x, m^2/\mu^2, g)}{\partial x}\bigg|_{x=1} \qquad \text{since } r(1, m^2/\mu^2, g) = 1 \ . \tag{1.2.58}$$

and

$$\gamma_m = \mu^2 \frac{\partial m'^2}{\partial \mu'^2}\bigg|_{\mu'=\mu} = m^2 \frac{\partial \frac{1}{Z}}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu} = -\frac{m^2}{Z^2} \frac{\partial Z}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu}$$

$$= m^2 \frac{\partial r}{\partial \frac{\mu'^2}{\mu^2}}\bigg|_{\mu'=\mu} = m^2 \frac{\partial r(x, m^2/\mu^2, g)}{\partial x}\bigg|_{x=1}$$

which yields

$$\gamma_m = 2m^2\gamma \qquad . \tag{1.2.59}$$

Note that since $g(k^2)$ is a renormalization group invariant

$$g(k^2,m'^2,\mu'^2,g') = g(k^2,m^2,\mu^2,g)$$ which upon differentiation gives

$$[\mu^2 \frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial g} + \gamma_m \frac{\partial}{\partial m^2}]\ g(k^2) = 0\ . \tag{1.2.60}$$

Before studying the solutions of the RGE let's check that physical quantities are RG invariant, that is independent of μ. If $P(m^2,\mu^2,g)$ is a physical quantity then it is independent of μ^2 i.e. $P(m'^2,\mu'^2,g') = P(m^2,\mu^2,g)$ which implies

$$\mu^2 \frac{d}{d\mu'^2} P(m'^2,\mu'^2,g')\Big|_{\mu'=\mu} = 0$$

yielding

$$[\mu^2 \frac{\partial}{\partial\mu^2} + \beta \frac{\partial}{\partial g} + \gamma_m \frac{\partial}{\partial m^2}]\ P(m^2,\mu^2,g) = 0\ . \tag{1.2.61}$$

For example the physical mass, m_p, should be independent of μ. Now

$$\Delta_F^{-1}(k^2,m'^2,\mu'^2,g') = \frac{1}{Z}\ \Delta_F^{-1}(k^2,m^2,\mu^2,g) \tag{1.2.62}$$

so both $\qquad = 0 \quad \text{at} \quad k^2 = m_p^2\ .$

More circuitously we recall the running residue

$$-i(k^2-m_p^2) = \Delta_F^{-1}(k^2,m^2,\mu^2,g)r(k^2,m^2,\mu^2,g) \tag{1.2.63}$$

So defining

$$\mathcal{D} \equiv [\mu^2 \frac{\partial}{\partial\mu^2} + \beta \frac{\partial}{\partial g} + \gamma_m \frac{\partial}{\partial m^2}] \qquad \text{we have} \tag{1.2.64}$$

$$\begin{aligned}[\mu^2 \frac{\partial}{\partial\mu^2} + \beta \frac{\partial}{\partial g} + \gamma_m \frac{\partial}{\partial m^2}]\ im_p^2 \\ = r\mathcal{D}\Delta_F^{-1} + \Delta_F^{-1}\mathcal{D}r\ ,\end{aligned} \tag{1.2.65}$$

from above

$$\mathcal{D}\Delta_F^{-1} = 2\gamma\Delta_F^{-1} \tag{1.2.66}$$

and since r is defined through the two-point function

$$r(k^2,m'^2,\mu'^2,g') = Zr(k^2,m^2,\mu^2,g) . \tag{1.2.67}$$

So

$$\mathcal{D}r = -2\gamma r . \tag{1.2.68}$$

Thus

$$\mathcal{D}m_p^2 = 0 \tag{1.2.69}$$

as required of a physical quantity. Also the S-matrix for any n-particle process is given by

$$S(k_1,\ldots,k_n) \equiv \left|\frac{-i(k_1^2-m_p^2)}{r^{\frac{1}{2}}(k_1^2)} \cdots \frac{-i(k_n^2-m_p^2)}{r^{\frac{1}{2}}(k_1^2)}\right| G(k_1,\ldots,k_n,m^2,\mu^2,g)\Big|_{k_i^2=m_p^2} .$$

$$= r^{1/2}(k_1^2)\Delta_F^{-1}(k_1^2)\ldots r^{1/2}(k_n^2)\Delta_F^{-1}(k_n^2)\; G(k_1,\ldots,k_n,m^2,\mu^2,g)\Big|_{k_i^2=m_p^2} \tag{1.2.70}$$

So

$$\begin{aligned} \mathcal{D}S &= (r^{n/2}\Delta_F^{-n})\mathcal{D}G + (\mathcal{D}r^{n/2}\Delta_F^{-n})G \\ &= -n\gamma(r^{n/2}\Delta_F^{-n}G) + (-n\gamma+2n\gamma)r^{n/2}\Delta_F^{-n}G \\ &= 0 . \end{aligned} \tag{1.2.71}$$

Thus the S-matrix is independent of μ.

We can use the RG to find the behavior of Green function, when we scale all the momenta. In order to simplify the derivation we will consider the theories to be massless (we will look at large momenta compared to mass scales in the theory) the RGE becomes in the massless case

$$[\mu^2 \frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial g} + n\gamma]\ G(k_1,\ldots,k_n,\mu^2,g) \;=\; 0\ . \tag{1.2.72}$$

with $\beta = \beta(g)$; $\gamma = \gamma(g)$ only and the running coupling constant and residue are

$$\begin{aligned} r &= r(k^2/\mu^2,g) \\ \bar{g}(k^2/\mu^2) &\equiv g(k^2/\mu^2,g)\ . \end{aligned} \tag{1.2.73}$$

By dimensional analysis

$$\begin{aligned} G(\lambda k,\mu^2,g) &= \lambda^n G(k,\mu^2/\lambda^2,g) \\ &= \lambda^n Z^{-n/2} G(k,\mu^2,g') \end{aligned} \tag{1.2.74}$$

by the RG where now

$$\begin{aligned} Z^{-1} &= r(\lambda^2,g) \\ g' &= g(\lambda^2,g) = \bar{g}(\lambda^2)\ . \end{aligned} \tag{1.2.75}$$

The RGE can now be used to find r and $\bar{g}$; recall

$$g(k^2/\mu'^2,g') \;=\; g(k^2/\mu^2,g)\ , \tag{1.2.76}$$

differentiating w.r.t. k^2 and setting $k^2 = \mu'^2$ yields

$$\frac{1}{\mu'^2}\left.\frac{\partial g(x,g')}{\partial x}\right|_{x=1} = \frac{1}{\mu^2}\left.\frac{\partial g(x,g)}{\partial x}\right|_{x=\frac{\mu'^2}{\mu^2}} \tag{1.2.77}$$

implying

$$\frac{\mu^2}{\mu'^2}\,\beta(g') \;=\; \left.\frac{\partial g(x,g)}{\partial x}\right|_{x=\frac{\mu'^2}{\mu^2}} \tag{1.2.78}$$

For $\mu'^2/\mu^2 = \lambda^2$ we find a differential equation defining the running coupling constant

$$\lambda^2 \frac{\partial \bar{g}(\lambda^2, g)}{\partial \lambda^2} = \beta(\bar{g}) \quad \text{with the initial condition } \bar{g}(1,g) = g. \tag{1.2.79}$$

Also we have that

$$r(k^2/\mu'^2, g') = Zr(k^2/\mu^2, g) \quad . \tag{1.2.80}$$

Thus

$$(\mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial g}) r = -2\gamma r \tag{1.2.81}$$

letting $k^2/\mu^2 = \lambda^2$ with $\mu^2 \frac{\partial}{\partial \mu^2} = -\lambda^2 \frac{\partial}{\partial \lambda^2}$ this becomes

$$[-\lambda^2 \frac{\partial}{\partial \lambda^2} + \frac{\partial}{\partial g}] r(\lambda^2, g) = -2\gamma r(\lambda^2, g) \tag{1.2.82}$$

with initial condition $r(1,g) = 1$. The solution is

$$r(\lambda^2, g) = e^{+2 \int_g^{\bar{g}(\lambda^2, g)} \frac{\gamma(x)}{\beta(x)} dx} \ . \tag{1.2.83}$$

Let's check this by differentiating

$$[-\lambda^2 \frac{\partial}{\partial \lambda^2} + \beta \frac{\partial}{\partial g}] r = 2 \frac{\gamma(\bar{g})}{\beta(\bar{g})} [(-\lambda^2 \frac{\partial}{\partial \lambda^2} + \beta \frac{\partial}{\partial g})\bar{g}] r - 2 \frac{\gamma(g)}{\beta(g)} \beta(g) r \ . \tag{1.2.84}$$

But $\bar{g}(k^2/\mu'^2, g') = g(k^2/\mu^2, g)$ which gives

$$[-\lambda^2 \frac{\partial}{\partial \lambda^2} + \beta \frac{\partial}{\partial g}] \bar{g} = 0$$ and hence r given by (1.2.83) satisfies (1.2.82).

So

$$r(\lambda^2,g) = e^{+2\int_g^{\bar{g}(\lambda^2,g)} \frac{\gamma(x)}{\beta(x)}dx} \tag{1.2.85}$$

and

$$\lambda^2 \frac{\partial\bar{g}(\lambda^2,g)}{\partial\lambda^2} = \beta(\bar{g}) \quad \text{with} \quad \bar{g}(1,g) = g \,. \tag{1.2.86}$$

Hence we find the asymptotic Green function is given by

$$G(\lambda k,\mu^2,g) = \lambda^n e^{+\int_g^{\bar{g}(\lambda^2)} \frac{n\gamma(x)}{\beta(x)}dx} G(k,\mu^2,\bar{g}(\lambda^2)) \,. \tag{1.2.87}$$

So the large momentum ($\lambda k > m$) dynamics is completely controlled by the running coupling constant; whose behavior in turn is determined by the β function.

Recall

$$\beta(g) = \left.\frac{\partial\bar{g}(x,g)}{\partial x}\right|_{x=1} \tag{1.2.88}$$

and in perturbation theory $\bar{g}$ is given as a power series in g and hence so is β. Suppose β begins in order g^3 as it does in gauge theories,

$$\beta(g) = bg^3 \tag{1.2.89}$$

then

$$\lambda^2 \frac{\partial\bar{g}}{\partial\lambda^2} = b\bar{g}^3 \,. \tag{1.2.90}$$

This equation is easier to solve when we introduce the coupling constant

$$\alpha(\lambda^2) \equiv \frac{\bar{g}^2(\lambda^2)}{4\pi} \text{, the fine structure constant.} \tag{1.2.91}$$

Then

$$\lambda^2 \frac{\partial\alpha}{\partial\lambda^2} = 8\pi b\alpha^2 \tag{1.2.92}$$

with $\alpha(1) = \frac{g^2}{4\pi}$ so

$$\alpha(\lambda^2) = \frac{\alpha(1)}{1-8\pi\alpha(1)\, b\, \ell n\lambda^2} \tag{1.2.93}$$

As λ increases $\alpha(\lambda^2)$ increases if $b > 0$ but <u>decreases</u> if $b < 0$, i.e. $\beta < 0$. If $\beta < 0$ the theory is called asymptotically free since $\alpha(\lambda^2) \to 0$ as $\lambda^2 \to \infty$. Note as λ decreases $\alpha(\lambda^2) \to \infty$ as the denominator approaches 0 for the asymptotically free case. This is perhaps an indication of confinement; the interaction strength grows as distance increases; however the perturbation expansion can no longer be valid as $\alpha(\lambda^2)$ grows.

We can apply this type of analysis to gauge theories in general (ignoring the technical difficulties of gauge invariance, scalar fields and masses) to find a similar RGE and running coupling constant $\alpha(\lambda^2)$. For theories including YM fields and fermions a lowest order perturbative calculation of $\beta(g)$ yields

$$\beta = -\frac{g^3}{32\pi^2}\left[\frac{11}{3}\, C_2(G) - \frac{4}{3}\, T_F\right] \tag{1.2.94}$$

where $C_2(G)$ is the quadratic Casimir operator for the adjoint representation of G

$$C_2(G)\,\delta_{ij} = f_{ik\ell}f_{jk\ell}\ . \tag{1.2.95}$$

For example

$$C_2(SU(N)) = N \quad \text{and} \quad C_2(U_1) = 0\ . \tag{1.2.96}$$

T_F is related to the Casimir operator for the fermion representations

$$T_F\delta_{ij} = \frac{1}{2}\,\mathrm{Tr}(T^iT^j) \tag{1.2.97}$$

where we sum over all fermi (left and right) representations. So we see for $T_F < \frac{11}{4}\,C_2(G)$; $\beta < 0$ and the theory is asymptotically free, there are not enough fermions to screen all the non-abelian charges

For QCD $C_2(SU(3)) = 3$ and T_F = number of families = $\frac{1}{2}$ number of flavors (for QCD) = F.

For SU(2) in the GWS model

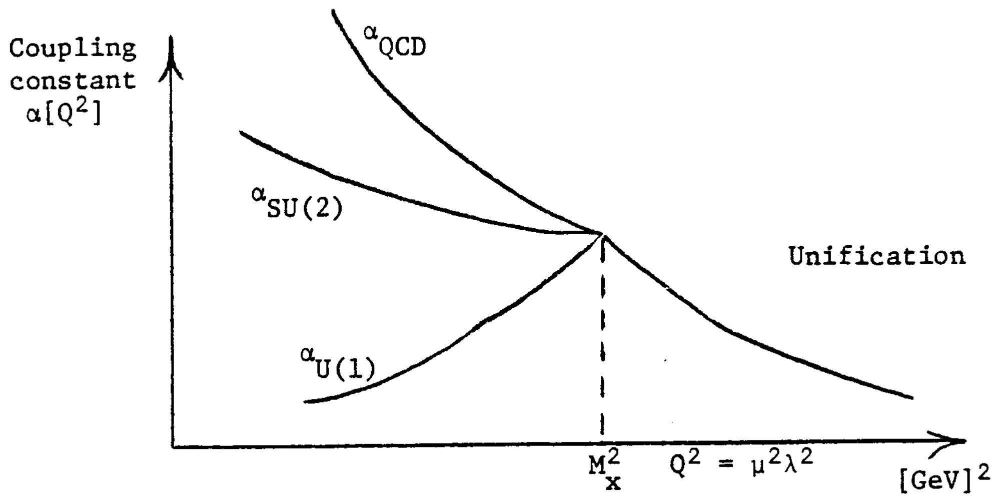
$C_2(SU(2)) = 2$ while again $T_F = F$; and for the U(1) of hypercharge

$$T_F = \frac{1}{2}\,Y_L^2 + \frac{1}{2}\,Y_R^2 = \frac{5}{3}\,F$$

where we are assuming $\lambda > M_w$, M_z or m_q the quark masses so that we can treat the theory as massless (also we have neglected the scalars in the theory necessary for the Higgs mechanism). So we find

$$\begin{aligned} \beta_{QCD} &= -\frac{g_s^3}{32\pi^2}\,[11 - \frac{4}{3}\,F] < 0 \\ \beta_{SU(2)} &= -\frac{g^3}{32\pi^2}\,[\frac{22}{3} - \frac{4}{3}\,F] < 0 \\ \beta_{U(1)} &= +\frac{g'^3}{32\pi^2}\,\frac{20}{9}\,F > 0\,. \end{aligned} \tag{1.2.98}$$

Thus QCD and SU(2) are asymptotically free and $\alpha_s(\lambda^2)$; $\alpha_{SU(2)}(\lambda^2)$ decrease as the momentum λk increases, while $\alpha_{U(1)}(\lambda^2)$ increases. Note that our β differs b a factor of $\frac{1}{2}$ from others, for example, See D.J. Gross in Methods in Field Theory, Les Houches 1975, since our RGE is in terms of $\mu^2\,\frac{\partial}{\partial\mu^2}$ while others use $\mu\,\frac{\partial}{\partial\mu}$ and $\mu\,\frac{\partial}{\partial\mu} = 2\mu^2\,\frac{\partial}{\partial\mu^2}$.



1.3. The Standard Model: Summary

We can summarize our discussion of the standard model of electroweak and strong interactions based on the group SU(3) × SU(2) × U(1) by listing the Lagrangian first in terms of the unshifted fields then in the unitary gauge.

$$L^{SM} = L_{YM} + L_F + L_\phi + L_{Yuk}$$

1) $$L_{YM} = -\frac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} \qquad (1.3.1)$$

where the anti-symmetric covariant field strength tensors are

$$\begin{aligned} \underline{F}_{\mu\nu} &= \partial_\mu\underline{A}_\nu - \partial_\nu\underline{A}_\mu + g\underline{A}_\mu \times \underline{A}_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ \underline{G}_{\mu\nu} &= \partial_\mu\underline{G}_\nu - \partial_\nu\underline{G}_\mu + g_s\underline{G}_\mu \times \underline{G}_\nu \end{aligned} \qquad (1.3.2)$$

2) $$L_F = \bar{\ell}_L^W i\not{D}\ell_L^W + \bar{q}_L^W i\not{D}q_L^W + \bar{e}_R^W i\not{D}e_R^W + \bar{u}_R^W i\not{D}u_R^W + \bar{d}_R^W i\not{D}d_R^W \qquad (1.3.3)$$

where the covariant derivatives are

$$
\begin{aligned}
D_\mu \ell_L^W &= [\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu + \frac{ig'}{2}B_\mu]\ell_L^W \\
D_\mu^{ab} q_L^{Wb} &= [(\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{6}B_\mu)\delta^{ab} - \frac{ig_s}{2}(\underline{\lambda}\cdot\underline{G}_\mu)^{ab}]q_L^{Wb} \\
D_\mu e_R^W &= (\partial_\mu + ig'B_\mu)e_R^W \\
D_\mu^{ab} u_R^{Wb} &= [(\partial_\mu - \frac{2i}{3}g'B_\mu)\delta^{ab} - \frac{ig_s}{2}(\underline{\lambda}\cdot\underline{G}_\mu)^{ab}]u_R^{Wb} \\
D_\mu^{ab} d_R^{Wb} &= [(\partial_\mu + \frac{i}{3}g'B_\mu)\delta^{ab} - \frac{ig_s}{2}(\underline{\lambda}\cdot\underline{G}_\mu)^{ab}]d_R^{Wb}
\end{aligned}
\tag{1.3.4}
$$

where a,b = 1,2,3 = R,G,B and the generation indices have been suppressed.

3)
$$L_\phi = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi) \tag{1.3.5}$$

where the Higgs potential is

$$V(\phi^\dagger\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{1.3.6}$$

and the covariant derivative is

$$D_\mu\phi = (\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\underline{A}_\mu - \frac{ig'}{2}B_\mu)\phi \tag{1.3.7}$$

4)
$$L_{Yuk} = \Gamma^e_{mn}\bar{\ell}^W_{mL}\phi e^W_{nR} + \Gamma^d_{mn}\bar{q}^W_{mL}\phi d^W_{nR} + \Gamma^u_{mn}\bar{q}^W_{mL}\tilde{\phi}\, u^W_{nR} + \text{h.c.} \tag{1.3.8}$$

where

$$\tilde{\phi} = \begin{pmatrix} \phi^{o\dagger} \\ -\phi^- \end{pmatrix} . \tag{1.3.9}$$

In the Higgs unitary gauge we find

1)
$$
\begin{aligned}
L_{YM} = &-\frac{1}{2}K^*_{\mu\nu}K^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} \\
&- ig[\sin\theta_W F^{\mu\nu} - \cos\theta_W H^{\mu\nu}]W^*_\mu W_\nu + \frac{1}{2}g^2[W_\mu W^\mu W^*_\nu W^{*\nu} - (W^\mu W^*_\mu)^2]
\end{aligned}
\tag{1.3.10}
$$

where

$$
\begin{aligned}
K_{\mu\nu} &= d_\mu W^+_\nu - d_\nu W^+_\mu \\
W^\pm_\mu &= \frac{1}{\sqrt{2}}\,[A^1_\mu \mp iA^2_\mu] \\
[W^+_\mu]^* &= W^-_\mu \\
d_\mu &= \partial_\mu - ig[\sin\theta_W A_\mu + \cos\theta_W Z_\mu] \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
H_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu \\
\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} \quad ; \quad \frac{g'}{g} = \tan\theta_W
\end{aligned}
\tag{1.3.11}
$$

2)

$$
\begin{aligned}
L_F = {} & \bar{\nu}_L i\not{\partial}\nu_L + \bar{e}i\not{\partial}e + \bar{u}i\left(\not{\partial} - \frac{ig_s}{2}\,\underline{\lambda}\cdot\underline{\not{G}}\right)u + \bar{d}i\left(\not{\partial} - \frac{ig_s}{2}\,\underline{\lambda}\cdot\underline{\not{G}}\right)d \\
& + eA_\mu J^\mu_{em} + \frac{g}{2\sqrt{2}}\,[J^\mu_W W^-_\mu + J^{\mu+}_W W^+_\mu] + \frac{\sqrt{g^2+g'^2}}{2}\,J^\mu_Z Z_\mu
\end{aligned}
\tag{1.3.12}
$$

where $e = g\sin\theta_W = g'\cos\theta_W$ and

$$
\begin{aligned}
\text{a)}\quad J^\mu_{em} &= q_M \bar{\psi}_M \gamma^\mu \psi_M \\
&= +\frac{2}{3}\,\bar{u}_m\gamma^\mu u_m - \frac{1}{3}\,\bar{d}_m\gamma^\mu d_m - \bar{e}_m\gamma^\mu e_m \\
\text{b)}\quad J^\mu_W &= \bar{e}\,\gamma^\mu(1-\gamma_5)\nu + \bar{d}\,\gamma^\mu(1-\gamma_5)A_{KM}u \\
\text{c)}\quad J^\mu_Z &= \bar{\psi}_M\gamma^\mu T^3_M(1-\gamma_5)\psi_M - 2q_M\bar{\psi}_M\gamma^\mu\psi_M \\
&= \bar{\nu}_L\gamma^\mu\nu_L - \bar{e}_L\gamma^\mu e_L + \bar{u}_L\gamma^\mu u_L - \bar{d}_L\gamma^\mu d_L \\
&\quad + 2\sin^2\theta_W(\bar{e}\gamma^\mu e - \frac{2}{3}\,\bar{u}\gamma^\mu u + \frac{1}{3}\,\bar{d}\gamma^\mu d)\ .
\end{aligned}
\tag{1.3.13}
$$

where A_{KM} is the Kobayashi-Maskawa matrix.

3) $$L_\phi = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}m_\eta^2\eta^2 - \frac{\lambda}{4}(\eta^4+4v\eta^3)$$

$$+ [M_W^2+gM_W\eta + \frac{1}{4}g^2\eta^2]W_\mu^+W^{-\mu} \tag{1.3.14}$$

$$+ \frac{1}{2}[M_Z^2 + \frac{gM_W}{\cos^2\theta_W}\eta + \frac{1}{4}\frac{g^2}{\cos^2\theta_W}\eta^2]Z_\mu Z^\mu$$

where

$$M_W = \frac{gv}{2}\left(\approx \frac{37}{\sin\theta_W}\text{ GeV}\right)$$

$$M_Z = \frac{M_W}{\cos\theta_W}\left(\approx \frac{75}{\sin 2\theta_W}\text{ GeV}\right) \tag{1.3.15}$$

$$m_\eta^2 = -2\mu^2(\lesssim 200\text{ GeV})$$

and

$$\sin^2\theta_W = .229 \pm .009(\pm .005)$$

4) $$L_{Yuk} = -[1+\frac{g}{2M_W}\eta][m_u\bar{u}u + m_c\bar{c}c + m_t\bar{t}t + m_d\bar{d}d + m_s\bar{s}s + m_b\bar{b}b$$

$$+ m_e\bar{e}e + m_\mu\bar{\mu}\mu + m_\tau\bar{\tau}\tau] . \tag{1.3.16}$$