Physics 663: Problem Set #4 Due 12 November 2008

1) Show that

$$\delta(\eta^A - \eta'^A) = \eta^A - \eta'^A$$

$$\delta(\bar{\eta}^A - \bar{\eta}'^A) = \bar{\eta}^A - \bar{\eta}'^A ,$$

(See page -373- of Lecture Notes.)

2) Fill in the details for the $O(2) = U(1)$ sigma model with fermions

"wrong" sign mass^2 $\Leftrightarrow$ SSB

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma' \partial^\mu \sigma' + \partial_\mu \pi \partial^\mu \pi) + \frac{m^2}{2}(\sigma'^2 + \pi^2)$$

$$- \frac{\lambda}{8}(\sigma'^2 + \pi^2)^2$$

$$+ \frac{1}{2}\bar{\Psi} i \overleftrightarrow{\not{\partial}} \Psi - g \bar{\Psi}(\sigma' + i\gamma_5 \pi)\Psi .$$

The infinitesimal $U(1)$ transformations are

$$[Q, \pi] = -i\sigma' \equiv i\delta\pi$$

$$[Q, \sigma'] = +i\pi \equiv i\delta\sigma'$$

for the scalars and the (chiral) $U(1)$ transformations for Ψ & $\bar{\Psi}$ are

2)

$$U(\omega)\psi U^{-1}(\omega) = e^{+\frac{i\omega\gamma_5}{2}}\psi$$

$$U(\omega)\bar{\psi} U^{-1}(\omega) = \bar{\psi} e^{+\frac{i\omega\gamma_5}{2}}$$

$$\left(\text{i.e. } \left[U\psi^{\dagger}U^{-1} = \psi^{\dagger} e^{-\frac{i\omega\gamma_5}{2}}\right]\gamma^0\right).$$

So infinitesimally

$$[Q,\psi] = -\tfrac{1}{2}\gamma_5\psi \equiv i\delta\psi$$

$$[Q,\bar{\psi}] = -\tfrac{1}{2}\bar{\psi}\gamma_5 \equiv i\delta\bar{\psi}.$$

$$\Big(\text{Note: } [Q,\psi_L] = \tfrac{1}{2}(1-\gamma_5)[Q,\psi] = \tfrac{1}{2}(1-\gamma_5)(-\tfrac{1}{2}\gamma_5)\psi$$

$$= +\tfrac{1}{2}\left(\tfrac{1}{2}(1-\gamma_5)\psi\right) = +\tfrac{1}{2}\psi_L$$

$$[Q,\psi_R] = \tfrac{1}{2}(1+\gamma_5)(-\tfrac{1}{2}\gamma_5\psi) = -\tfrac{1}{2}\left((1+\gamma_5)\tfrac{1}{2}\psi\right)$$

$$= -\tfrac{1}{2}\psi_R.\Big)$$

The Chiral symmetry forbids a fermion mass term

$$2[Q,\bar{\psi}\psi] = -(\bar{\psi}\gamma_5)\psi - \bar{\psi}(\gamma_5\psi) = -2\bar{\psi}\gamma_5\psi$$

But allows Yukawa terms:

2) $\delta\left(\bar{\psi}(\sigma' + i\gamma_5 \pi)\psi\right) = 0$, hence $\delta\mathcal{L} = 0$.

Since the potential has a minimum at

$\langle 0|\sigma'|0\rangle \equiv v = \sqrt{\frac{2m^2}{\lambda}}$ (due to the "wrong" sign mass^2 term) we must quantize about $\sigma'(x) = \sigma(x) + v$ with $\langle 0|\sigma(x)|0\rangle = 0$. The U(1) symmetry is <u>spontaneously broken</u>.

Substituting $\sigma' = \sigma + v$ into the Lagrangian we find

$$\mathcal{L} = \tfrac{1}{2}\left[\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi\right] - \tfrac{1}{2}(2m^2)\sigma^2$$

$$- \frac{\lambda}{8}\left[(\sigma^2 + \pi^2)(\sigma^2 + \pi^2 + 4v\sigma)\right]$$

$$+ \frac{i}{2}\bar{\psi}\overleftrightarrow{\not\partial}\psi - (gv)\bar{\psi}\psi - g\bar{\psi}(\sigma + \gamma_5 \pi)\psi .$$

The fermions now get a mass $M = gv$ as a result of the spontaneous symmetry breaking! $\mathcal{L}$ is still

2.) invariant under the (chiral) $U(1)$ with $\sigma' = \sigma + v$. π is a massless Goldstone boson while σ is the massive $\mu^2 = 2m^2 = \lambda v^2$ Higgs field. The ground state (vacuum) is <u>not</u> invariant however

$$\langle 0| \begin{pmatrix} \pi \\ \sigma' \end{pmatrix} |0\rangle = \langle 0| U^\dagger U \begin{pmatrix} \pi \\ \sigma' \end{pmatrix} U^\dagger U |0\rangle$$

$$= \langle 0'| \begin{bmatrix} c\pi + s\sigma' \\ c\sigma' - s\pi \end{bmatrix} |0'\rangle$$

If $|0'\rangle = |0\rangle$ then

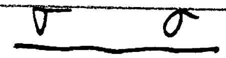
$$\langle 0| \begin{pmatrix} \pi \\ \sigma' \end{pmatrix} |0\rangle = \langle 0| \begin{pmatrix} \pi c + \sigma' s \\ \sigma' c - \pi s \end{pmatrix} |0\rangle .$$

Since $\langle 0|\pi|0\rangle = 0 = \langle 0|\sigma|0\rangle$ we find

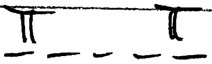
$$\langle 0| \begin{pmatrix} 0 \\ v \end{pmatrix} |0\rangle = \langle 0| \begin{pmatrix} vs \\ vc \end{pmatrix} |0\rangle ,$$

a contradiction! Thus $|0\rangle \neq U|0\rangle$!

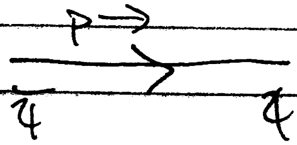
2) Find the Feynman Rules:

1) 

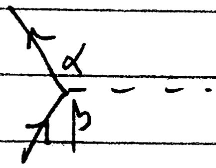
$$\frac{i}{p^2 - 2m^2}$$



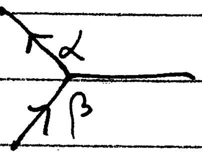
$$\frac{i}{p^2}$$



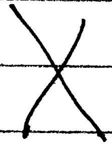
$$\frac{i}{\not{p} - M}$$

2) 

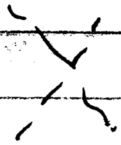
$$-ig\,\gamma_{5\alpha\beta}$$



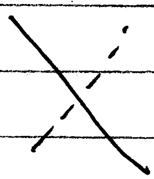
$$-ig\,\delta_{\alpha\beta}$$

3) 

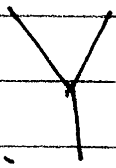
$$-i3\lambda$$



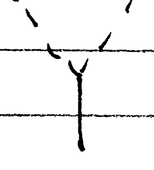
$$-i3\lambda$$



$$-i\lambda$$

4) 

$$-i3\lambda v$$



$$-i\lambda v$$

3.) Consider coupling the Higgs sector of the Standard Model to quarks and leptons

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_F + \mathcal{L}_{Yuk} \quad \text{where}$$

a) $$\mathcal{L}_\phi = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi)$$

$$V(\phi^\dagger \phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

with $$\phi = \begin{bmatrix} \phi_+ \\ \phi_0' \end{bmatrix} \qquad \phi_+, \phi_0' \text{ complex scalars.}$$

b) $$\mathcal{L}_F = \bar{l}_{mL}\, i\not\partial\, l_{mL} + \bar{q}_{mL}\, i\not\partial\, q_{mL}$$

$$+ \bar{e}_{mR}\, i\not\partial\, e_{mR} + \bar{u}_{mR}\, i\not\partial\, u_{mR}$$

$$+ \bar{d}_{mR}\, i\not\partial\, d_{mR}$$

Recall: $$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi$$

$$\psi_R = \tfrac{1}{2}(1+\gamma_5)\psi$$

and

3.) The lepton doublet is

$$l_{eL} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$ left handed $SU(2)_L$ lepton doublet for each generation $(e, \mu, \tau) = m$ (or $m = 1,2,3$)

i.e. $l_{\mu L} = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$, $l_{\tau L} = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$.

Likewise

$$\underbrace{q_{mL}}_{m=1} = \begin{pmatrix} u \\ d \end{pmatrix}_L$$ left handed $SU(2)_L$ quark doublet for each generation $m = 1,2,3$.

$$\underbrace{q_{mL}}_{m=2} = \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \underbrace{q_{mL}}_{m=3} = \begin{pmatrix} t \\ b \end{pmatrix}_L$$

Finally the Right handed fields are all $SU(2)_L$ <u>singlets</u>

$$\left.\begin{array}{l} (e_R, u_R, d_R) \\ (\mu_R, c_R, s_R) \\ (\tau_R, t_R, b_R) \end{array}\right\} = \left\{ (e_{mR}, u_{mR}, d_{mR}) \right.$$

3.) The $U_Y(1)$ weak hypercharge quantum numbers are defined so that the electric charge Q is $Q = T_3 + Y$

	T_3	Y	$Q = T_3 + Y$
ν_{eL}	$+\frac{1}{2}$	$-\frac{1}{2}$	0
e_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
u_L	$+\frac{1}{2}$	$+1/6$	$+2/3$
d_L	$-\frac{1}{2}$	$+1/6$	$-1/3$
e_R	0	-1	-1
u_R	0	$+2/3$	$+2/3$
d_R	0	$-1/3$	$-1/3$

Likewise for ν_μ, μ, c, s & ν_τ, τ, t, b.

Note: Fermion mass terms are not allowed by $SU(2)_L \times U(1)_Y$ symmetry

$$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$$

doublets singlets

3.) $$\mathcal{L}_{Yuk} = \Gamma^e_{mn}\,\bar{l}_{mL}\,\phi\, e_{nR} + \Gamma^d_{mn}\,\bar{q}_{mL}\,\phi\, d_R$$
$$+\Gamma^u_{mn}\,\bar{q}_{mL}\,\tilde{\phi}\, u_{nR} + h.c.$$

with
$$\tilde{\phi} \equiv i\sigma^2\phi^* = \begin{bmatrix} \phi_0'^{+} \\ -\phi_- \end{bmatrix}.$$

The $SU(2)_L \times U(1)_Y$ transformations are

$$\phi + \delta\phi \equiv U\phi U^{-1} = e^{\frac{ig}{2}\vec{\omega}\cdot\vec{\tau} + \frac{ig'}{2}\theta}\phi$$
$$= \left[1 + \frac{ig}{2}\vec{\omega}\cdot\vec{\tau} + \frac{ig'}{2}\theta\right]\phi$$

$$l_L + \delta l_L \equiv U l_L U^{-1} = l_L + \frac{ig}{2}\vec{\omega}\cdot\vec{\tau}\, l_L - \frac{ig'}{2}\theta\, l_L$$

$$q_L + \delta q_L \equiv U q_L U^{-1} = q_L + \frac{ig}{2}\vec{\omega}\cdot\vec{\tau}\, q_L + \frac{ig'}{6}\theta\, q_L$$

$$e_R + \delta e_R \equiv U e_R U^{-1} = e_R - ig'\theta\, e_R$$

$$u_R + \delta u_R \equiv U u_R U^{-1} = u_R + \frac{2i}{3} g'\theta\, u_R$$

$$d_R + \delta d_R \equiv U d_R U^{-1} = d_R - \frac{ig'}{3}\theta\, d_R$$

(i.e. $-i[Q,\phi] \equiv \delta\phi$) for each m ex. l_{mL}, q_{mL}, etc.

3) What are the Feynman Rules when you spontaneously break the $SU(2)_L \times U(1)_Y$ symmetry down to $U(1)_{em}$, the $U(1)$ of electromagnetism,

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{em}$$

by letting $\phi_0' = \phi_0 + \frac{v}{\sqrt{2}}$ with $\langle 0|\phi_0|0\rangle = 0$ in order to minimize the potential?