Physics 663: Problem Set #3 Due 29 October 2008

Read Ryder Chapter 6 (Chapter 5).

1) Consider functional integrals over complex functions φ & φ^*

Let $\varphi = \frac{1}{\sqrt{2}}(\sigma + i\pi)$; $\varphi^* = \frac{1}{\sqrt{2}}(\sigma - i\pi)$

with σ, π real functions.

Then since

$$\int \left[\frac{d\sigma}{\sqrt{2\pi}}\right]\left[\frac{d\pi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int dx\,(\sigma^2+\pi^2)} = 1,$$

we have that

$$\int \left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right] e^{-\int dx\,\varphi^*\varphi} = 1$$

with $\left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right] = \left[\frac{d\sigma}{\sqrt{2\pi}}\right]\left[\frac{d\pi}{\sqrt{2\pi}}\right]$.

a) Show that

$$\int \left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right] e^{-\int dx\,[\varphi^*\varphi - iJ^*\varphi - i\varphi^* J]}$$

$$= e^{-\int dx\, J^* J},$$

1a) with

$$J = \frac{1}{\sqrt{2}}\left[J_1 + i J_2\right]$$

$$J^* = \frac{1}{\sqrt{2}}\left[J_1 - i J_2\right] \quad , \quad J_{1,2} \text{ Real.}$$

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b) Show that our Real functional Fourier transform implies

$$Z[J,J^*] = \int \left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right] e^{i\int dx\,(\varphi^* J + J^* \varphi)} \tilde{Z}[\varphi,\varphi^*]$$

and

$$\tilde{Z}[\varphi,\varphi^*] = \int \left[\frac{dJ}{\sqrt{2\pi i}}\right]\left[\frac{dJ^*}{\sqrt{2\pi i}}\right] e^{-i\int dx\,(\varphi^* J + J^* \varphi)} Z[J,J^*].$$

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c) What is

$$\delta[\varphi - \varphi'] \; ?$$

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d) Consider a change of variables for the complex function

$$\sigma'(x) = \int K(x,y)\,\sigma(y)\,dy$$

$$\pi'(x) = \int K(x,y)\,\pi(y)\,dy$$

1.d) with the same symmetric kernel $K(x,y)$.

Show that

$$\left[\frac{d(K\varphi)^*}{\sqrt{2\pi i}}\right]\left[\frac{d(K\varphi)}{\sqrt{2\pi i}}\right] = (\det K^2)\left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right].$$

Hence, show that

$$\int \left[\frac{d\varphi}{\sqrt{2\pi i}}\right]\left[\frac{d\varphi^*}{\sqrt{2\pi i}}\right] e^{-\int dx\,dy\, \varphi^*(x) K(x,y) \varphi(y)} \times e^{i\int dx (J^*\varphi + \varphi^* J)}$$

$$= \left[\frac{1}{\det(K)}\right] e^{-\int dx\,dy\, J^*(x) K^{-1}(x,y) J(y)}.$$