Let's just recall some graphical techniques from perturbation theory. To simplify the notation for present consider a self-interacting scalar ϕ^4-theory

$$\mathcal{L} = \frac{(1+b)}{2}\partial_\mu\phi\partial^\mu\phi - \frac{(m^2+a)}{2}\phi^2 - \frac{(\lambda+c)}{4!}\phi^4 .$$

The generating functional $Z[J]$ is simply

$$Z[J] = \int[d\phi]\, e^{i\int d^4x\,[\mathcal{L} + J\phi]} = \langle 0|T e^{i\int d^4x J\phi}|0\rangle$$

Differentiating wrt J yields the n-point or Green functions

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \frac{\delta^n}{i\delta J(x_1)\cdots i\delta J(x_n)} Z[J]\Big|_{J=0} .$$

Perturbation theory is obtained by separating

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_I = \mathcal{L} - \mathcal{L}_0 = \mathcal{L}_I[\phi, \partial\phi]$$

Then

$$Z[J] = e^{i\int d^4z\, \mathcal{L}_I\left[\frac{\delta}{i\delta J(z)}\right]} Z_0[J] \qquad \left(\begin{array}{c}\text{Gell-Mann-Low}\\ \text{Expansion}\end{array}\right)$$

where the free field generating functional $Z_0[J]$ can be explicitly evaluated

$$Z_0[J] = \int \{d\varphi\} e^{i\int d^4x (\mathcal{L}_0 + J\varphi)}$$

$$= e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y) J(y)}$$

where in this case the scalar Feynman propagator is

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \underbrace{\frac{i}{p^2-m^2+i\epsilon}}_{=\tilde{\Delta}_F(p)}$$

Thus the in-field or free field 2-point function

$$\langle in\, 0 | T \phi_{in}(x) \phi_{in}(y) | 0\, in \rangle = \Delta_F(x-y)$$

$$\Rightarrow \quad \langle in\, 0 | T \tilde{\phi}_{in}(p) \phi_{in}(0) | 0\, in \rangle = \frac{i}{p^2-m^2+i\epsilon}$$

Applying the Gell-Mann - Low expansion we obtain the Green functions as a sum over Feynman graphs

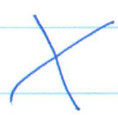
$$\langle 0| T\,\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\, e^{-i\sum_{i=1}^{n} p_i x_i} \sum_{\Gamma\in G^{(n)}}' \alpha(\Gamma)(2\pi)^4\delta_\Gamma \,\times$$

$$\times \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p,k)$$

where $G^{(n)}$ = set of all Feynman diagrams with n-external lines, I_Γ is the Feynman integrand for diagram Γ having n-external lines with momentum p_i and $m(\Gamma)$ loops where


$= \dfrac{i}{p^2-m^2+i\epsilon}$


$= -i(\lambda+c)$

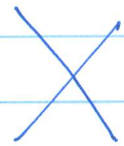

$= -i(a-bp^2)$

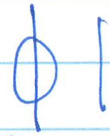
and energy-momentum conservation occurs at each vertex and $\alpha(\Gamma)$ is the symmetry # for Γ.

Often it is useful to consider connected Greens functions. These are made by summing over connected Feynman diagrams only. A connected diagram is one which has all external lines attached to one graph; it is not the product of graphs (i.e. not union of disjoint graphs) each x_i is topologically connected to every other x_j.

ex.

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle^c$$

$=$  $+$  $+ \cdots$

graphs such as  or  are not connected, they are products (union) of connected graphs. The Feynman rules are exactly the same as for the (disconnected) time ordered functions; the only difference is that we are summing over connected diagrams only

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle^c$$

$$= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\, e^{-i\sum_{i=1}^{n} p_i x_i}\, \overbrace{(2\pi)^4\delta^4(p_1+p_2+\cdots+p_n)}^{\text{only 1-Dirac delta function since connected}}\times$$

$$\times \sum_{\Gamma\in G^{(n)}_{conn}}' \alpha(\Gamma)\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p,k)$$

$G^{(n)}_{conn} \subset G^{(n)}$ is the set of connected Feynman diagrams with n-external lines.

Analytically this graphical definition corresponds to the recursive definition for connected Green functions

1) $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle^c$ (i.e. $\langle 0|\phi|0\rangle = 0$)

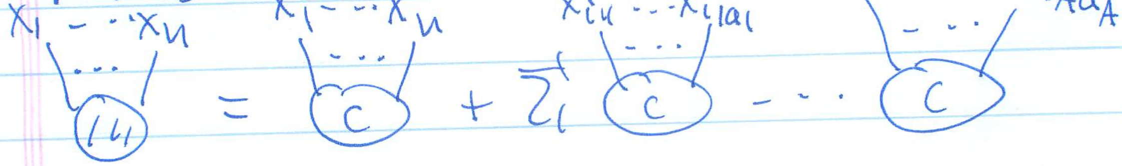
2) $\langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle = \langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle^c$

$$+ \sum'_{\substack{(1,2,3,4)\\ \to (i_{11},i_{12})(i_{21},i_{22})\\ i_{ij} > i_{ij+1}\\ i_{i1} > i_{i+1\,1}}} \langle 0|T\phi(x_{i_{11}})\phi(x_{i_{12}})|0\rangle^c \times \langle 0|T\phi(x_{i_{21}})\phi(x_{i_{22}})|0\rangle^c$$

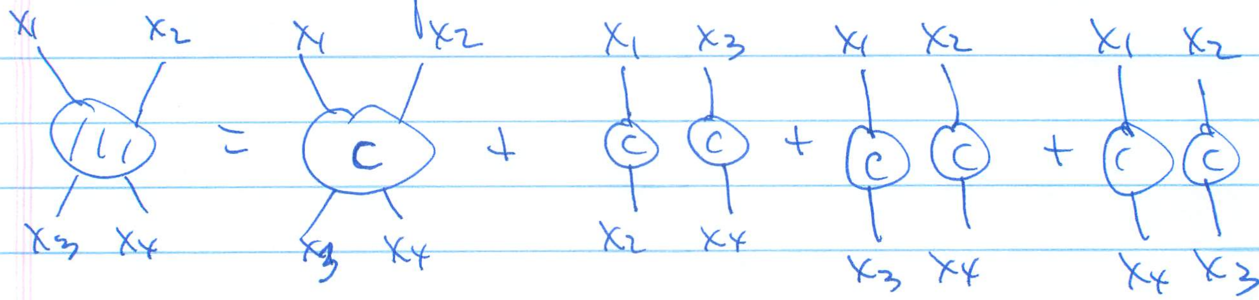
⋮

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \sum'_{\substack{(1,\dots,n)\\ \to (i_{11}\cdots i_{1a_1})\cdots(i_{A1}\cdots i_{Aa_A})\\ (\text{with } i_{ij} > i_{ij+1},\ i_{i1} > i_{i+1\,1})}} \langle 0|T\phi(x_{i_{11}})\cdots\phi(x_{i_{1a_1}})|0\rangle^c \times \cdots \times \langle 0|T\phi(x_{i_{A1}})\cdots\phi(x_{i_{Aa_A}})|0\rangle^c$$

That is



So for example



Notice each $\langle 0|T\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)|0\rangle^c$ has only one $(2\pi)^4\delta^4(p_1+\cdots+p_n)$ delta function since each  cannot be written as the product (union)  with a subset of $\{x_i\}$ on different graphs in the product.

Now we can also introduce a generating functional for the connected functions call it $Z^c[J]$

$$Z^c[J] \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!}\int d^4x_1\cdots d^4x_n\, J(x_1)\cdots J(x_n)\,\times$$

$$\times\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle^c$$

Again cryptically we write this as

$$Z^c[J] = \langle 0| T\, e^{i\int dx\, J(x)\phi(x)} |0\rangle^c$$

Notice with the above definition we have

$$Z[J] = e^{Z^c[J]}$$

$$= \sum_{A=0}^{\infty} \frac{1}{A!} \overbrace{Z^c[J] \cdots Z^c[J]}^{A \text{ times}}$$

$$= \sum_{A=0}^{\infty} \frac{1}{A!} \left(\sum_{a_1=0}^{\infty} \frac{i^{a_1}}{a_1!} \int dx_{11} \cdots dx_{1a_1} J(x_{11}) \cdots J(x_{1a_1}) \langle 0| T \phi(x_{11}) \cdots \phi(x_{1a_1}) |0\rangle^c \right) \cdots$$

$$\cdots \left(\sum_{a_A=0}^{\infty} \frac{i^{a_A}}{a_A!} \int dx_{A1} \cdots dx_{Aa_A} J(x_{A1}) \cdots J(x_{Aa_A}) \langle 0| T \phi(x_{A1}) \cdots \phi(x_{Aa_A}) |0\rangle^c \right)$$

$$= \sum_{A=0}^{\infty} \sum_{\substack{a_1=0 \\ \vdots \\ a_A=0}}^{\infty} \frac{i^{a_1+ \cdots +a_A}}{A!\, a_1! \cdots a_A!} \int dx_{11} \cdots dx_{1a_1} \cdots dx_{Aa_A}$$

$$J(x_{11}) \cdots J(x_{Aa_A}) \langle 0| T \phi(x_{11}) \cdots \phi(x_{1a_1}) |0\rangle^c \cdots$$

$$\cdots \langle 0| T \phi(x_{A1}) \cdots \phi(x_{Aa_A}) |0\rangle^c$$

Now the X_{ij} are just dummy integration variables

So we can ~~Re-write~~ group ~~these~~ the sum ~~for~~ fixed $a_1+a_2+\cdots+a_A=n$

or

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n \, J(x_1) \cdots J(x_n) \times$$

$$\times \sum_{\substack{(1\cdots n) \\ \to (i_{11}\cdots i_{1a_1})\cdots(i_{A1}\cdots i_{Aa_A}) \\ i_{ij} > i_{ij+1},\ i_{i1} > i_{i+1\,1}}} \langle 0|T\phi(x_{i_{11}})\cdots\phi(x_{i_{1a_1}})|0\rangle^c \cdots \langle 0|T\phi(x_{i_{A1}})\cdots\phi(x_{i_{Aa_A}})|0\rangle^c$$

Since for fixed n there are

$\dfrac{n!}{a_1!\cdots a_A!}$ ways to pick $(x_{i_{11}}\cdots)\cdots(x_{i_{A1}}\cdots x_{i_{Aa_A}})$ out of $(x_1\cdots x_n)$ $i_{ij} > i_{ij+1}$

when $a_1+a_2+\cdots+a_A=n$.

and there are $A!$ ways to break $x_1\cdots x_n$ into the A classes $(x_{11}\cdots x_{1a_1})\cdots(x_{A1}\cdots x_{Aa_A})$

i.e. $A!$ ways to shuffle the A-classes around.

So $\cancel{\sum_{n=0}^{\infty} \to \sum_{A=0}^{\infty} \frac{1}{A!}}$

$$\sum_{n=0}^{\infty} \sum_{\substack{(1\cdots n) \\ \to (i_{1\cdots})\cdots(i_{A1}\cdots)}} \frac{1}{n!} \longleftrightarrow \sum_{A=0}^{\infty} \frac{1}{A!} \sum_{a_1 \cdots a_A = 0}^{\infty} \frac{1}{a_1!\cdots a_A!}$$

thus we have $= Z[J]$ ✓.

4) Graphical Rules for $Z\{J\}$, $Z^c[J]$

1) for each n $\quad \int dx_1 \cdots dx_n \, J(x_1) \cdots J(x_n) = \times$ in x-space

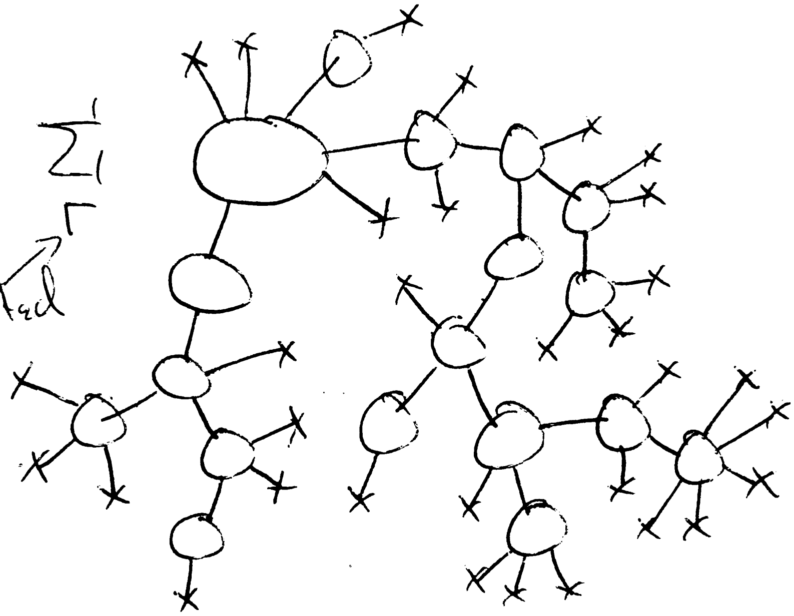
$\times = \int \frac{dp_1}{(2\pi)^4} \cdots \frac{dp_n}{(2\pi)^4} \tilde{J}(-p_1) \cdots \tilde{J}(-p_n)$ in p-space

2) same as $G^{(n)}(x_1, \ldots, x_n)$ Rules

4b

$$Z^c[J] = \sum_{\Gamma}{}'$$

connected

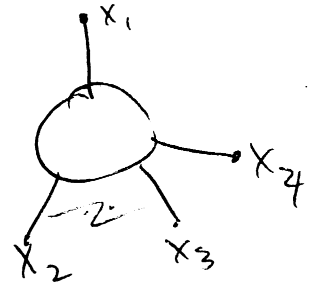


where each "blob"

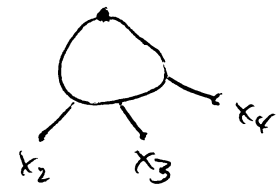
etc. is same

One particle irreducible diagram

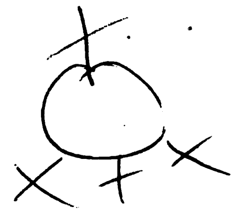
One particle irreducible diagrams are those connected diagrams which remain connected *after* the removal of one line. For example

1)  is not 1PI

Since Remove any external line • x_1

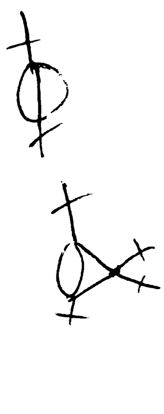


and x_1 is separated from x_2, x_3, x_4 So first 1PI diagrams have the external lines removed symbolically

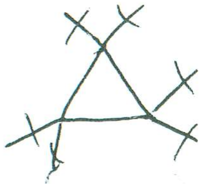


a line is drawn through them.

2)  not 1PI since Remove line and

 becomes disconnected

3)  is 1PI

 is 1PI etc.

-insert -306'-

The Feynman Rules for 1PI graphs are

1) each external line is amputated $\times = 1$

2) Same Rules for the Rest of diagram

3) for the $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle^{\text{prop, 1PI}}$

$$\equiv -\left[\langle 0|T\phi(x_1)\phi(x_2)|0\rangle^c\right]^{-1}$$

So

$$\int dx_2 \langle 0|T\phi(x_1)\phi(x_2)|0\rangle^{P} \langle 0|T\phi(x_2)\phi(x_3)|0\rangle^c$$

$$\equiv -\delta^4(x_1-x_3)$$

in momentum space this becomes

Again the Feynman rules are the same - we are only summing over 1-PI graphs

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle^{1PI}$$
$$= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4} e^{-ip_i x_i} (2\pi)^4\delta^4(p_1+\cdots+p_n)\times$$
$$\times \sum_{\Gamma\in G^{(n)}_{1PI}} \alpha(\Gamma)\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4} I_\Gamma(p,k)$$

Again $G^{(n)}_{1PI} \subseteq G^{(n)}_{conn} \subseteq G^{(n)}$ is the set of all 1PI Fey. dia. with n-amputated external lines each Γ is made with the usual rules (except only put 1 for amputated lines $= 1$)

$$\frac{i}{p^2-m^2+i\epsilon} = \xrightarrow{\ p\ }$$

$$= -i(\lambda+c)$$

$$= -i(a-bp^2) \quad \text{etc.}$$

and $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle^{1PI} = -\left[\langle 0|T\phi(x_1)\phi(x_2)|0\rangle^c\right]^{-1}$

$$\int dx_2 \int \frac{dp}{(2\pi)^4} e^{-ip(x_1-x_2)} \langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^P$$

$$\int \frac{dq}{(2\pi)^4} e^{-iq(x_2-x_3)} \langle 0|T\tilde{\phi}(q)\phi(0)|0\rangle^c$$

$$= -\int \frac{dp}{(2\pi)^4} e^{-ip(x_1-x_3)}$$

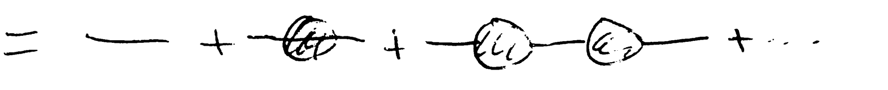
$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_1-x_3)} \Big(\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^P \langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^c$$

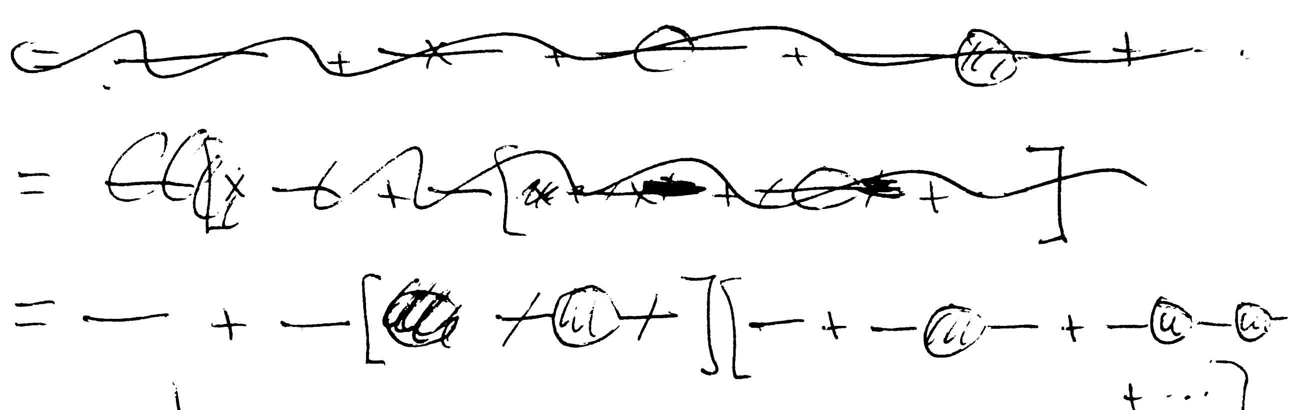
$$= -1 \Big)$$

So

$$\boxed{\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^P = \frac{-1}{\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^c}}$$

Now we have

$\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle =$ 



Sum geometric series - Schwinger-Dyson expansion where the blob is 1PI and is called the proper self energy $-i\Pi(p) \equiv$

So $=$ $+$ $+ \cdots$

$= \;—\; + \;—\;[\quad]\langle 0|T\tilde{\phi}(p)\,\phi(0)|0\rangle$

$\Rightarrow$

$$\langle 0|T\tilde{\phi}(p)\,\phi(0)|0\rangle\left[1 - \quad\right] = \;—$$

So

$$\langle 0|T\tilde{\phi}(p)\,\phi(0)|0\rangle\left[1 - \frac{i}{p^2-m^2}(-i\Pi(p))\right] = \frac{i}{p^2-m^2}$$

$$\boxed{\Rightarrow \quad \langle 0|T\tilde{\phi}(p)\,\phi(0)|0\rangle = \frac{i}{p^2-m^2-\Pi(p)}}$$

So $\langle 0|T\tilde{\phi}(p)\,\phi(0)|0\rangle^{P} = i\left(p^2-m^2-\Pi(p)\right)$

$$-i\Pi(p) = \quad + \quad + \cdots$$
$$= -i(a-bp^2) - i\Pi'(p)$$

So

$$\boxed{\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle^P = i\left[(1+b)p^2-(m^2+a)-\Pi'(p)\right]}$$

5) Analytically we can most easily define the 1PI functions $\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle^P$ through a Legendre transform of $Z[J]$.

Towards this end introduce the generating functional for 1PI functions — called the effective action:

$$\Gamma[\varphi] \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\int dx_1\cdots dx_n\,\varphi(x_1)\cdots\varphi(x_n)\times$$

$$\times\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle^{1PI}$$

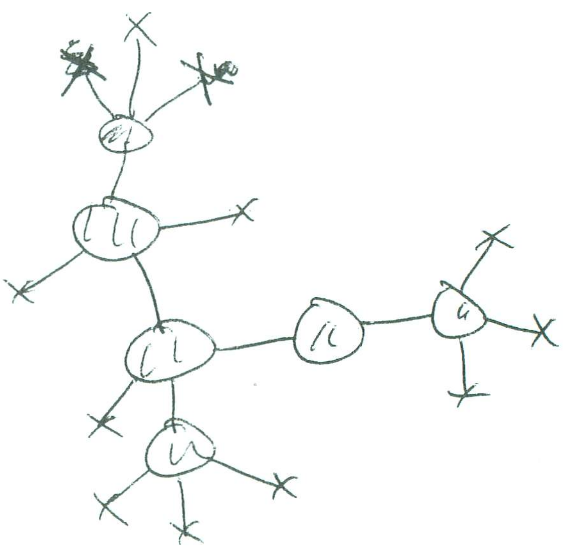
(the word effective has many different uses in field theory, so make sure you know the context)

Again more cryptically this becomes

$$\Gamma[\varphi] = \langle 0 | T e^{+\int d^4x\, \varphi(x)\phi(x)} |0\rangle^{1PI}$$

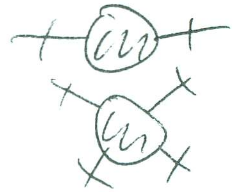
Now let's recall

$$Z^c[J] = \sum_{\Gamma \in G_{conn}}$$



$$\left(G_{conn} = \bigcup_{n=0}^{\infty} G^n_{conn} \right)$$

where each blob



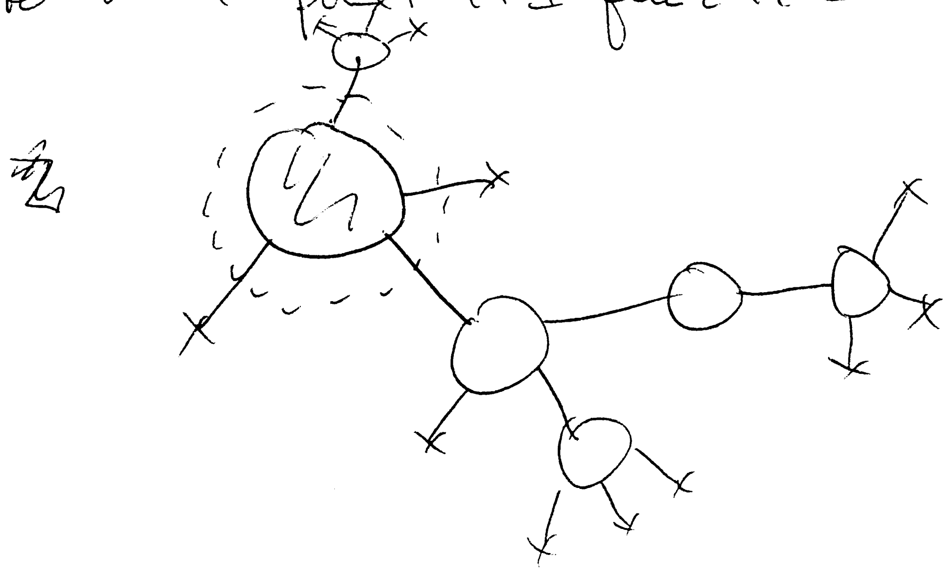
is 1PI

and each $-\!\!\times \iff \int d^4x\, J(x)$

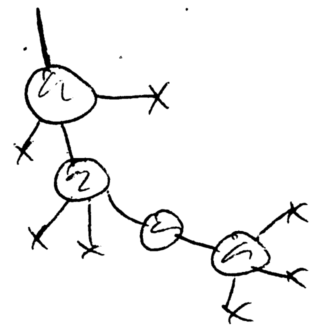
Notice each Γ contains contributions to many different vertex functions.

i.e.

we could ~~ase~~ view this as a contribution to a 4-point 1PI function



~~of course there is~~ where we have identified the $\sum_{\gamma \in G_{conn}}$



$\equiv \varphi(x)$

etc.

as the source φ of $\Gamma[\varphi]$

So we have identified $\varphi(x) = \dfrac{\delta Z^c[J]}{i\,\delta J(x)}$

But this graph occurs again as a contrib. for each 4 point 1PI function, not just the designated (as well as in n point 1PI functions) one, as well as for each 2 point 1PI function

So we cannot simply set $\Gamma[\varphi] = Z^c[J]$
with $\varphi = \frac{\delta Z^c}{i\delta J}$

we have further combinatorics to worry about. We must count the # of times each ~~(4-point 1PI)~~ m-point 1PI function & 2 point 1PI function can occur so that $\Gamma[\varphi]$ can be given in terms of $Z^c[J]$ times these combinatoric factors.

That is ~~turning things around~~

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, J(x_1) \cdots J(x_n)$$

$$\sum_{\Gamma \in G^{(n)}_{conn}} C_n(\Gamma) \langle 0 | T\,\phi(x_1) \cdots \phi(x_n) | 0 \rangle^{conn}_{\Gamma}$$

where C_n is the # of times the same diagram $\Gamma \in G^n_{conn}$ contributes to $\Gamma[\varphi]$

Now let's show the combinatoric factor $C_n(\Gamma)$ depends only on n. Consider each vertex blob
the # of these count the # of times such "vertex" appears in Γ as a "vertex", and each time we have a ~~dotted~~ line we can have

$$— \; + \; —①— \; + \; —①—①— \; + \cdots$$

So the # of lines count the # of times Γ contributes to the 2-point 1PI function since

$$\langle 0 | T\,\phi(x)\phi(y) | 0 \rangle^{1PI} = -\Delta_F^{-1}(x-y)$$

these contribute with a minus sign

So

$$C_n(\Gamma) = [\text{\# of vertex blobs}] - [\text{\# of 2 point contributions}]$$

$$= V - L - n$$

of "vertices" of Γ — # of internal lines of Γ — # of external lines

~~but since~~ but we found # of loops m

$m = L - V + 1$ since we count the blobs as "vertices" V this "effective" diagram has no loops $m = 0 = L - V + 1$

$$\Rightarrow \quad C_n(\Gamma) = C_n = 1 - n \quad !!$$

Thus

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots dx_n (1-n) J(x_1) \ldots J(x_n) \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle^{con}$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots dx_n J(x_1) \ldots J(x_n) \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle^{con}$$

$$- \sum_{n=1}^{\infty} \frac{i^n}{(n-1)!} \int dx_1 \ldots dx_n J(x_1) \ldots J(x_n) \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle^{con}$$

$$= Z^c[J] - i \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} \int d^4x J(x) \int d^4x_1 \ldots d^4x_{n-1} J(x_1) \ldots J(x_{n-1}) \langle 0 | T \phi(x) \phi(x_1) \cdots \phi(x_{n-1}) | 0 \rangle^{con}$$

let $n-1=m$

$$\Gamma[\varphi] = Z^c[J] - i\int d^4x\, J(x) \sum_{m=0}^{\infty} \frac{i^m}{m!} \times$$

$$\times \int d^4x_1 \dots d^4x_m\, J(x_1)\dots J(x_m) \times$$

$$\times \langle 0| T\phi(x)\,\phi(x_1)\dots\phi(x_m)|0\rangle^{\text{conn}}$$

but we saw that we can identify the source φ of $\Gamma[\varphi]$ as

$$\varphi(x) = \frac{\delta Z^c[J]}{i\delta J(x)} = \frac{\delta}{i\delta J(x)} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n\, J(x_1)\dots J(x_n) \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle$$

$$= \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} \int dx_1 \dots dx_{n-1}\, J(x_1)\dots J(x_{n-1}) \langle 0|T\phi(x)\,\phi(x_1)\dots\phi(x_{n-1})|0\rangle^{c}$$

$$= \sum_{m=0}^{\infty} \frac{i^m}{m!} \int dx_1 \dots dx_m\, J(x_1)\dots J(x_m) \langle 0|T\phi(x)\,\phi(x_1)\dots\phi(x_m)|0\rangle^{\text{con}}$$

So we have

$$\Gamma[\varphi] = Z^c[J] - i\int d^4x\, J(x)\, \varphi(x)$$

$$\cancel{\text{with } \varphi(x) = \frac{\delta Z^c[J]}{i\delta J(x)}}$$

or

$$Z^c[J] = \Gamma[\varphi] + i\int d^4x\, J(x)\, \varphi(x)$$

$$\text{with} \quad \varphi(x) = \frac{\delta Z^c[J]}{i\,\delta J(x)}$$

This Γ and Z^c are related by a LeGendre transform

~~So~~ Notice we can take $\frac{\delta}{\delta\varphi(x)}$

$$\frac{\delta Z^c[J]}{\delta\varphi(x)} = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} + i\,J(x) + i\int d^4y\, \frac{\delta J(y)}{\delta\varphi(x)}\, \varphi(y)$$

but $$\frac{\delta Z^c[J]}{\delta\varphi(x)} = \int d^4y\, \frac{\delta Z^c[J]}{\delta J(y)}\, \frac{\delta J(y)}{\delta\varphi(x)}$$ Chain Rule and $\frac{\delta Z^c[J]}{\delta J(y)} = i\varphi(y)$

$\Rightarrow$

$$\frac{\delta Z^c[J]}{\delta \varphi(x)} = i\int d^4y \, \frac{\delta J(y)}{\delta \varphi(x)} \varphi(y)$$

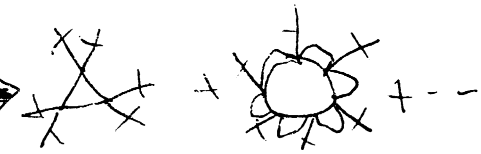
and $\Rightarrow$

$$0 = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} + i J(x)$$

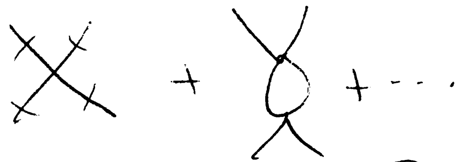
So

$$\boxed{-iJ(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}}$$

We can use the definitions of our action and Legendre transform to more simply find our propagators & Feynman rules, which is useful in more complicated models. Consider $\Gamma[\varphi]$ in the no-loop approximation ($\hbar=0$) this is called tree approximation –

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle^{1PI} \Rightarrow$$ 

for $n>4$ loops only

$$\langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle^{1PI} \Rightarrow$$ 

~~$-i(\lambda+C)$~~ loops

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle^{1PI} =$$

$$= -\Delta_F^{-1}(x_1-x_2) \Rightarrow i\left[(1+b)p^2-(m^2+a)\underbrace{-\Pi(p)}_{\text{loops}}\right]$$

So in the tree approximation –

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle^{1PI}\Big|_{\substack{\text{tree}\\\text{approx.}}} = 0$$

$$\langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle^{1PI}\Big|_{\text{Tree approx}} = \int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}\, e^{-ip_i x_i}$$

$$(2\pi)^4\delta^4(p_1+p_2+\cdots p_4)\left[-i(\lambda+C)\right]$$

and

$$\langle 0| T\,\phi(x_1)\,\phi(x_2)\,|0\rangle^{1PI}\Big|_{\substack{\text{Tree}\\\text{approx}}} = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-ip_i x_i}$$

$$(2\pi)^4\delta^4(p_1+p_2)\; i\left[(1+b)\,p_1^2-(m^2+a)\right]$$

So

$$\Gamma[\varphi]\Big|_{\substack{\text{tree}\\\text{approx}}} \equiv \Gamma_0[\varphi]$$

$$= \frac{1}{2!}\int d^4x_1\,d^4x_2\,\varphi(x_1)\,\varphi(x_2)\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\;\times$$

$$\times\; i\left[-(1+b)\,\partial_{x_1}^2-(m^2+a)\right]e^{-ip_i x_i}\,(2\pi)^4\delta^4(p_1+p_2)$$

$$+\frac{1}{4!}\int d^4x_1\cdots d^4x_4\,\varphi(x_1)\cdots\varphi(x_4)\left[-i(\lambda+c)\right]$$

$$\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}\,e^{-ip_i x_i}\,(2\pi)^4\delta^4(p_1+\cdots+p_4)$$

$$\Gamma_0[\varphi] = \int dx_1\, dx_2\, \frac{i}{2}\varphi(x_1)\varphi(x_2)\left[-(1+b)\partial^2_{x_1} - (m^2+a)\right] \times$$
$$\times \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x_1-x_2)}$$

$$- \frac{i(\lambda+c)}{4!}\int d^4x_1 \cdots d^4x_4\, \varphi(x_1)\cdots\varphi(x_4) \times$$
$$\times \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x_1-x_4)} \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2(x_2-x_4)} \times$$
$$\times \int \frac{d^4p_3}{(2\pi)^4} e^{-ip_3(x_3-x_4)}$$

$$= \int dx_1\, dx_2\, \frac{-i}{2}\varphi(x_1)\varphi(x_2)\left[(1+b)\partial^2_{x_1} + (m^2+a)\right]\delta^4(x_1-x_2)$$

$$- \frac{i(\lambda+c)}{4!}\int d^4x_1 \cdots d^4x_4\, \varphi(x_1)\cdots\varphi(x_4)\,\delta^4(x_1-x_4)$$
$$\delta^4(x_2-x_4)\,\delta^4(x_3-x_4)$$

$$= \int dx_1\, \frac{-i}{2}\varphi(x_1)\left[(1+b)\partial^2_{x_1} + (m^2+a)\right]\varphi(x_1)$$
$$-\frac{i}{4!}(\lambda+c)\int d^4x\, \varphi^4(x)$$

So integrate $\int dx\, \varphi(x)\partial^2\varphi(x) = \cancel{\int dx\, \partial_\lambda(\varphi\partial^\lambda\varphi)} \to 0$

$$- \int dx (\partial_\mu\varphi)(\partial^\mu\varphi)$$

by parts ~~through~~ throwing away surface term since $\varphi(x)$ are ~~[illegible]~~ test functions So

$$\boxed{\begin{aligned} \Gamma_0[\varphi] &= i\int d^4x \left[\frac{(1+b)}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{(m^2+a)}{2}\varphi^2 - \frac{(\lambda+c)}{4!}\varphi^4 \right] \\ \Gamma_0[\varphi] &= i\int d^4x\, \mathcal{L}[\varphi] \end{aligned}}$$

Γ_0 is just i time the action with $\phi \to \varphi$!!

~~Now we can~~

<u>Remarks</u> 1) we can use this to find propagators quickly

given $\mathcal{L}$ we know $\mathcal{L}_{in}$ then

$$\Gamma_{in}[\varphi] = i\int d^4x\, \mathcal{L}_{in}[\varphi] \quad \text{completely!}$$

$$= \int d^4x \, -\frac{i}{2}\varphi(\partial^2+m^2)\varphi$$

So

$$\frac{\delta\Gamma_{in}[\varphi]}{\delta\varphi(x)} = -i(\partial_x^2+m^2)\varphi(x)$$

but $\dfrac{\delta\Gamma_{in}[\varphi]}{\delta\varphi(x)} = -iJ(x)$

and $\varphi(x) = \dfrac{\delta Z^c[J]}{i\,\delta J(x)}$

So

$$-iJ(x) = -i(\partial_x^2+m^2)\frac{\delta Z^c[J]}{i\,\delta J(x)}$$

differentiate wrt $iJ(y)$ & set $J=0$ $\Rightarrow$

$$-i\delta^4(x-y) = (\partial_x^2+m^2)\left.\frac{\delta^2 Z^c[J]}{i\delta J(y)\, i\delta J(x)}\right|_{J=0}$$

$$= (\partial_x^2+m^2)\langle 0|T\,\phi(x)\phi(y)|0\rangle$$

FT $\Rightarrow$

~~Hd~~ $(-p^2+m^2)\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle$

$= -i$

$$\Rightarrow \boxed{\langle 0|T\tilde{\phi}(p)\phi(0)|0\rangle = \frac{i}{p^2-m^2+i\epsilon}}$$ ✓

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2) Symmetries of the action $\int dx\, \mathcal{L}\{\varphi\}$ will ~~become~~ reflected in symmetries of $\Gamma\{\varphi\}$.

Let's consider symmetries in general.

Recall our relation of the S operator to Green functions

$$S = \sum_{k=0}^{\infty} \frac{i^k}{k!}\int dy_1 \cdots dy_k \Big[Z^{-1/2} K_{y_1} \cdots Z^{-1/2} K_{y_n}$$

$$\langle 0|T\phi(y_1)\cdots\phi(y_k)|0\rangle\Big] :\phi_{in}(y_1)\cdots\phi_{in}(y_k):$$

In particular we have the general expression in the case of multiple fields ϕ_i

$$Z^c[J] = \langle 0 | T e^{i\int d^4x\, J^i_{(x)} \phi_i(x)} | 0 \rangle^{\text{connected}}$$

$$\Gamma[\varphi] = \langle 0 | T e^{\int d^4x\, \varphi_i(x) \phi_i(x)} | 0 \rangle^{1PI = \text{proper}}$$

And the Legendre transforms relate the 2 generating functionals

$$\Gamma[\varphi] = Z^c[J] - i\int d^4x\, J^i_{(x)} \varphi_i(x) \Big|_{\varphi_i(x) \equiv \frac{\delta Z^c}{i\delta J^i(x)}}$$

and inversely

$$Z^c[J] = \Gamma[\varphi] + i\int d^4x\, J^i_{(x)} \varphi_i(x) \Big|_{-iJ^i_{(x)} = \frac{\delta \Gamma}{\delta \varphi_i(x)}}$$

(vacuum expectation values of the fields are assumed to be zero here so $\left.\frac{\delta Z^c}{i\delta J^i_{(x)}}\right|_{J=0} = 0$, $\left.\frac{\delta \Gamma}{\delta \varphi_i(x)}\right|_{\varphi=0} = 0$)

Hence applying these to 2-point functions

$$\sum_k{}' \int d^4z \frac{\delta^2 \Gamma}{\delta\varphi_i(x)\,\delta\varphi_k(z)} \frac{\delta^2 Z^c}{i\delta J^k(z)\, i\delta J^j(y)}$$

$$= \int d^4z \sum_k{}' \langle 0|T\,\phi_i(x)\,\phi_k(z)|0\rangle^{1PI} \langle 0|T\,\phi_k(z)\,\phi_j(y)|0\rangle^c$$

$$= \int d^4z \sum_k{}' \Gamma^{(2)}_{ik}(x-z)\, G^{(2)}_{kj}(z-y)$$

$$= \sum_k{}' \int d^4z \frac{-i\delta J^i(x)}{\delta\varphi_k(z)} \frac{\delta\varphi_k(z)}{i\delta J^j(y)}$$

(functional chain rule $\Rightarrow$)

$$= -\frac{\delta J^i(x)}{\delta J^j(y)} = -\delta^i_j\, \delta^4(x-y)$$

Applying the Fourier transform

$$\sum_k{}' \int d^4z \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-z)} \int \frac{d^4q}{(2\pi)^4} e^{-iq(z-y)}\, \tilde{\Gamma}^{(2)}_{ik}(p)\, \tilde{G}^{(2)}_{kj}(q)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \sum_k{}' \tilde{\Gamma}^{(2)}_{ik}(p)\, \tilde{G}^{(2)}_{kj}(p)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left(-\delta^i_j\right)$$

$$\Rightarrow \boxed{\sum_k{}' \tilde{\Gamma}^{(2)}_{ik}(p)\, \tilde{G}^{(2)}_{kj}(p) = -\delta^i_j}$$

Consequently $\boxed{\tilde{G}^{(2)}_{ij}(p) = -\tilde{\Gamma}^{(2)-1}_{ij}(p)}$

i.e.

$$\langle 0|T\tilde{\phi}_i(p)\phi_j(0)|0\rangle = -\tilde{\Gamma}^{(2)-1}_{ij}(p)$$

$$= \left[-\langle 0|T\tilde{\phi}_i(p)\phi_j(0)|0\rangle^{1PI}\right]^{-1}$$

More to the point — suppose the free (Lagrangian) action is given by

$$\Gamma_0[\varphi] = i\int d^4x\, d^4y\, \tfrac{1}{2}\varphi_i(x) K_{ij}(x,y)\varphi_j(y)$$

This is the free field 1-PI function generating functional — So

$$\frac{\delta\Gamma_0}{\delta\varphi_i(x)} = i\int d^4y\, K_{ij}(x,y)\varphi_j(y) = -iJ^i(x)$$

Letting $\varphi_j(y) = \frac{\delta Z^c}{i\delta J^j(y)}$ we have the equation for the propagator

$$\int d^4y\, K_{ij}(x,y)\frac{\delta Z^c}{i\delta J^j(y)} = -J^i(x)$$

$$\Rightarrow \int d^4y\, K_{ij}(x,y)\frac{\delta^2 Z^c}{i\delta J^j(y)\, i\delta J^k(z)} = -\delta^i_k\,\delta^4(x-z)$$

Assuming $K_{ij}(x,y) = -[Z_{ij}\partial_x^2 + m_{ij}^2]\delta^4(x-y)$

$$\Rightarrow \quad -(Z_{ij}\partial_x^2 + m_{ij}^2)\langle 0|T\phi_j(x)\phi_k(z)|0\rangle = -\delta^i_k\delta^4(x-z)$$

Fourier Transforming $\Rightarrow$

$$(Z_{ij}p^2 - m_{ij}^2)\tilde{G}^{(2)}_{jk}(p) = -\delta_{ik}$$

$$\Rightarrow \quad \tilde{G}^{(2)}_{ij}(p) = -(Z_{ij}p^2 - m_{ij}^2)^{-1}$$

Now let's return to the W-Z model to determine the Feynman rules in components & then superspace! Back to p. -295-

$$\Gamma = i\int d^4x\Big[16Z(\partial_\mu A\partial^\mu\bar{A} + \tfrac{i}{4}\psi\overleftrightarrow{\partial}\bar{\psi} + F\bar{F})$$

$$-16m\{2AF + 2\bar{A}\bar{F} - \tfrac{1}{2}\psi\psi - \tfrac{1}{2}\bar{\psi}\bar{\psi}\}$$

$$-12g\{AAF + \bar{A}\bar{A}\bar{F} - \tfrac{1}{2}A\psi\psi - \tfrac{1}{2}\bar{A}\bar{\psi}\bar{\psi}\}\Big]$$

let $Z = \frac{1}{16}$, $m \to \frac{1}{32}m$, $g \to \frac{1}{12}g$

So the bare action is (adding subscript "o" to denote bare quantities)