Appendix

$$\mathcal{L} = Z\,\partial_\mu \bar{c}\,\partial^\mu c - m^2 \bar{c} c$$

$$\Rightarrow \quad (Z\partial^2 + m^2) c = 0$$
$$(Z\partial^2 + m^2)\bar{c} = 0$$

$c, \bar{c} \in$ Grassmann alg.

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{c}} = -Z\dot{\bar{c}}$$

$$\bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{c}}} = +Z\dot{c}$$

So CAR

$$\delta(x^0-y^0)\{\pi(x), c(y)\} = -i\delta^4(x-y)$$

$$\delta(x^0-y^0)\{\bar{\pi}(x), \bar{c}(y)\} = -i\delta^4(x-y)$$

$\Rightarrow$

$$\delta(x^0-y^0)\{\dot{\bar{c}}(x), c(y)\} = +\frac{i}{Z}\delta^4(x-y)$$

$$\delta(x^0-y^0)\{\dot{c}(x), \bar{c}(y)\} = -\frac{i}{Z}\delta^4(x-y)$$

Now $\left(\delta(x^0-y^0)\{\dot{\bar{c}}(x), c(y)\} = +\frac{i}{Z}\delta^4(x-y)\right)^\dagger$

$$\delta(x^0-y^0)\{\dot{\bar{c}}^\dagger(x), c^\dagger(y)\} = -\frac{i}{Z}\delta^4(x-y)$$

$$\delta(x^0-y^0)\{\dot{\bar{c}}^\dagger(x), c^\dagger(y)\} = -\frac{i}{Z}\delta^4(x-y)$$

So if $c^\dagger = \bar{c}$; $\bar{c}^\dagger = c$ we retrieve other CAR

We could also choose $\begin{array}{l} C^\dagger = C \\ \bar{C}^\dagger = -\bar{C} \end{array}$
and we find CAR's do not mix, just get back what we started with!

Choose conventional assignment first

$$C^\dagger = \bar{C} \quad ; \quad \bar{C}^\dagger = C .$$

$$C(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{+ikx} \right]$$

$$\bar{C}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[b(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{+ikx} \right]$$

and $\omega_k = \sqrt{\vec{k}^2 + m^2/2}$

$$a(\vec{k}) = i\int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} C(x) = i\int d^3x\, e^{ikx} \left[\dot{C}(x) - i\omega_k C(x) \right]$$

$$b^\dagger(\vec{k}) = i\int d^3x\, C(x) \overleftrightarrow{\partial_0} e^{-ikx} = i\int d^3x\, e^{-ikx} \left[-i\omega_k C(x) - \dot{C}(x) \right]$$

$$a^\dagger(\vec{k}) = i\int d^3x\, \bar{C}(x) \overleftrightarrow{\partial_0} e^{-ikx} = i\int d^3x\, e^{-ikx} \left[-i\omega_k \bar{C}(x) - \dot{\bar{C}}(x) \right]$$

$$b(\vec{k}) = i\int d^3x\, e^{+ikx} \overleftrightarrow{\partial_0} \bar{C}(x) = i\int d^3x\, e^{+ikx} \left[\dot{\bar{C}}(x) - i\omega_k \bar{C}(x) \right]$$

So

$$\{a(\vec{k}), a^\dagger(\vec{l})\} = \int d^3x\, d^3y\, e^{ikx}\, e^{-ily} \times \{\mathring{C}(x) - i\omega_k C(x), \mathring{\bar{C}}(y) + i\omega_l \bar{C}(y)\}\Big|_{x^0=y^0}$$

$$= \int d^3x\, d^3y\, e^{ikx-ily}\Big[i\omega_l \tfrac{-i}{Z}\delta^3(\vec{x}-\vec{y}) - i\omega_k \tfrac{+i}{Z}\delta^3(\vec{x}-\vec{y})\Big]$$

$$= \int d^3x\, d^3y\, e^{i(k-l)x}\left(\frac{\omega_l}{Z} + \frac{\omega_k}{Z}\right)\delta^3(\vec{x}-\vec{y})\Big|_{x^0=y^0}$$

$$\boxed{\{a(\vec{k}), a^\dagger(\vec{l})\} = +(2\pi)^3\, \frac{2\omega_k}{Z}\, \delta^3(\vec{k}-\vec{l})}$$

$$\{b(\vec{k}), b^\dagger(\vec{l})\} = \int d^3x\, d^3y\, e^{ikx}\, e^{-ily} \times \{\mathring{\bar{C}}(x) - i\omega_k \bar{C}(x), i\omega_l C(y) + \mathring{C}(y)\}\Big|_{x^0=y^0}$$

$$= \int d^3x\, d^3y\, e^{ikx-ily}\Big[i\omega_l \tfrac{+i}{Z}\delta^3(x-y) - i\omega_k \tfrac{-i}{Z}\delta^3(x-y)\Big]\Big|_{x^0=y^0}$$

$$= \int d^3x\, d^3y\, e^{i(k-l)x} - \left[\frac{\omega_l}{Z} + \frac{\omega_k}{Z}\right]\delta^3(x-y)$$

$$\boxed{\begin{aligned} &= -(2\pi)^3\, \frac{2\omega_k}{Z}\, \delta^3(\vec{k}-\vec{l}) \\ &= \{b(\vec{k}), b^\dagger(\vec{l})\} \end{aligned}}$$

all other anti-commut. vanish.

So $a^\dagger(\vec{k}),\ b^\dagger(\vec{k})$ create

$a(\vec{k}),\ b(\vec{k})$ annihilate

$$|(m,n)\rangle = a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_m)\, b^\dagger(\vec{l}_1)\cdots b^\dagger(\vec{l}_n)\,|0\rangle$$

$$\langle(1,0)|(1,0)\rangle = \langle 0|a(\vec{k})\,a^\dagger(\vec{k}')|0\rangle$$

$$= +(2\pi)^3\,\frac{2\omega_k}{Z}\,\delta^3(\vec{k}-\vec{k}')$$

$$\langle(0,1)|(0,1)\rangle = \langle 0|\,b(\vec{l})\,b^\dagger(\vec{l}')|0\rangle$$

$$= -(2\pi)^3\,\frac{2\omega_k}{Z}\,\delta^3(\vec{l}-\vec{l}')$$

So <u>negative norm</u> for one of the types of particles whatever the choice of Z.

So check # operators $\quad [AB,C] = A\{B,C\} - \{A,C\}B$

$$[a^\dagger(\vec{k})a(\vec{k}),\ a^\dagger(\vec{l})] = +(2\pi)^3\frac{2\omega_k}{Z}\,\delta^3(\vec{k}-\vec{l})\,a^\dagger(\vec{l})$$

$$[b^\dagger(\vec{k})b(\vec{k}),\ b^\dagger(\vec{l})] = -(2\pi)^3\frac{2\omega_k}{Z}\,\delta^3(\vec{k}-\vec{l})\,b^\dagger(\vec{l})$$

So we can define positive energy by using indefinite metric $\eta = (-1)^b$ $b = \#$ of b particles

So

$$H = \int \frac{d^3k}{(2\pi)^3 \frac{2\omega_k}{2}} \omega_k \left[a^\dagger(\vec{k}) a(\vec{k}) + (-1)^b b^\dagger(\vec{k}) b(\vec{k}) \right]$$

etc

S-matrix is pseudo unitary

$$S^{-1} = \eta S^\dagger \eta$$