For deriving the quantum mechanical dynamical equations of motion, it is most convenient to work with the first order formulation. As we already know, since the Lagrangian is invariant under local gauge transformations so will be the equations of motion for the potentials A_μ^a, hence they are not completely determined by the equations. That is we must choose a particular gauge condition which fixes the independent angles $\omega^a(x)$ and hence selects one A_μ^a from its gauge orbit - then the equations of motion determine A_μ^a's time evolution uniquely.

Alternatively stated, when we quantize a theory we do so in terms of the independent dynamical degrees of freedom. That is we identify the independent variables and their canonically conjugate momenta of the system. The Hamiltonian is expressed in terms of these variables alone and describes their time evolution through the Heisenberg equation of motion. We can then represent the Green functions as a path integral over the independent dynamical variables. Since the Lagrangian is gauge invariant not all of the

Euler-Lagrange equations of motion will be independent — they do not determine the arbitrary angles $\omega^a(x)$ — that is what local gauge invariance means. Hence there will be a linear relation among equations of motion (this is the second Noether theorem which we will not dwell upon), this implies that not all the fields A^a_μ are independent. In fact, as we will see, A^a_0 acts as a Lagrange multiplier imposing this functional constraint among the fields. Hence in order to describe the quantum mechanical time evolution of the system correctly we must eliminate the constrained fields in terms of the independent dynamical variables so that we can write the Hamiltonian in terms of them explicitly. The non-independence of the potentials A^a_μ requires a relation between them, this is given by the gauge condition — choosing a gauge. Since the dynamics is independent of the gauge choice we are free to choose it at our convenience. From QED we know that the Coulomb gauge condition allows us to isolate the transverse independent degrees of freedom most easily, however it does so in a non-manifestly Lorentz invariant way. The Lorentz gauge condition although manifestly Lorentz invariant does not allow this separation of the "coordinates" into independent and dependent without introducing unphysical, auxiliary, degrees of freedom. Hence we will derive the quantum mechanical equations of motion, i.e. the Hamiltonian in terms of the independent coordinates and momenta in the Coulomb gauge, and then show that the S-matrix is the same as if we worked in the Lorentz gauge, that is we will prove equivalence of the generating functionals (up to field re-definitions) of the two gauges.

Hence to start consider the first order formulation Lagrangian

$$\mathcal{L}_{inv} \equiv +\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{L}(\phi, D_\mu\phi)$$

$$-\frac{1}{2}F^a_{\mu\nu}\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - f_{abc}A^{b\mu}A^{c\nu}\right),$$

The Euler-Lagrange equations are given by

1) $$\frac{\partial \mathcal{L}_{inv}}{\partial F^a_{\mu\nu}} = 0 \Rightarrow \boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc} A^b_\mu A^c_\nu}$$

2) $$\frac{\partial \mathcal{L}_{inv}}{\partial A^a_\mu} - \partial_\nu \frac{\partial \mathcal{L}_{inv}}{\partial \partial_\nu A^a_\mu} = 0$$

$$\Rightarrow \quad -\partial^\nu F^a_{\mu\nu} - f_{abc} F^b_{\mu\nu} A^{c\nu} + \frac{\partial \mathcal{L}(\phi, D_\lambda \phi)}{\partial A^{a\mu}} = 0$$

But recall that $\dfrac{\partial \mathcal{L}(\phi, D_\lambda\phi)}{\partial A^{a\mu}} = \dfrac{\partial \mathcal{L}}{\partial (D^\lambda \phi)^\alpha} \dfrac{\partial (D^\lambda \phi)^\alpha}{\partial A^{a\mu}}$

however

$$\frac{\partial (D^\lambda \phi)^\alpha}{\partial A^{a\mu}} = i g^\lambda{}_\mu (T^a)_{\alpha\beta} \phi^\beta$$

and So $$\frac{\partial \mathcal{L}(\phi, D_\lambda \phi)}{\partial (D^\lambda \phi)^\alpha} \frac{\partial (D^\lambda \phi)^\alpha}{\partial A^{a\mu}} = \frac{\partial \mathcal{L}(\phi, D_\lambda \phi)}{\partial (D^\mu \phi)^\alpha} (iT^a)_{\alpha\beta} \phi^\beta$$

$$= J^a_\mu(\phi, D_\lambda \phi)$$

So we have

$$\boxed{\left[\partial^\nu F^a_{\mu\nu} - f_{abc} A^{b\nu} F^c_{\mu\nu}\right] = J^a_\mu}$$

Then of course we have the "matter" field equation

3) $$\frac{\partial \mathcal{L}_{inv}}{\partial \phi^\alpha} - \partial_\mu \frac{\partial \mathcal{L}_{inv}}{\partial \partial_\mu \phi^\alpha} = 0$$

$$= \boxed{\frac{\partial \mathcal{L}(\phi, D_\lambda \phi)}{\partial \phi^\alpha} - \partial_\mu \frac{\partial \mathcal{L}(\phi, D_\lambda \phi)}{\partial \partial_\mu \phi^\alpha} = 0}$$

Now since the Lagrangian is gauge invariant

$$\mathcal{L}_{inv}(A, F, \phi) = \mathcal{L}_{inv}(A^g, F^g, \phi^g)$$

where $$A^g_\mu(x) \equiv \mathcal{U}(g) A_\mu(x) \mathcal{U}(g)$$

$$= U(g)\partial_\mu U^{-1}(g) + U(g) A_\mu U^{-1}(g)$$

$$F^g_{\mu\nu} \equiv \mathcal{U}^{-1}(g)\, F_{\mu\nu}\, \mathcal{U}(g)$$
$$= U(g)\, F_{\mu\nu}\, U^{-1}(\omega)$$

and

$$\phi^g \equiv \mathcal{U}^{-1}(g)\, \phi\, \mathcal{U}(g).$$
$$= U(g)\, \phi$$

The field equations have the same form for A^g_μ, $F^g_{\mu\nu}$, ϕ^g; with $g(x)$ completely arbitrary. Thus there must be a relation among the equations for A_μ, $F_{\mu\nu}$ since they are not uniquely determined. Defining the adjoint representation covariant derivative as

$$(\nabla_\mu)^{ab} \equiv \partial_\mu \delta^{ab} - f_{acb} A^c_\mu$$

the $F_{\mu\nu}$ field equation yields

$$(\nabla^\nu F_{\mu\nu})^a = J^a_\mu$$

Taking the covariant derivative again yields

$$\nabla^\mu \nabla^\nu F_{\mu\nu} = \nabla^\mu J_\mu$$

$$= \tfrac{1}{2}[\nabla^\mu, \nabla^\nu] F_{\mu\nu} \qquad \text{since } F_{\mu\nu} = -F_{\nu\mu}$$

but as before $[\nabla^\mu, \nabla^\nu] = i(T'^a) F^{a\mu\nu}$

where $(T'^a)_{bc} = -if_{abc}$ for the adjoint representation

Hence $[\nabla^\mu, \nabla^\nu]_{ab} F^b_{\mu\nu} = f_{cab} F^{c\mu\nu} F_{b\mu\nu}$

$$= 0.$$

So $\boxed{\nabla^\mu \nabla^\nu F_{\mu\nu} = 0}$

and, as can be checked from the ϕ^α field equations, consistent with this the current must be covariantly conserved

$$(\nabla^\mu J_\mu)^a = 0.$$

So since $\nabla^\mu J_\mu = 0 \Rightarrow$

$\nabla^\mu \nabla^\nu F_{\mu\nu} = 0$. This implies not all the A_μ^a equations are independent. Thus some of the equations are constraint equations for A_μ^a. We must have a relation between the fields to determine them uniquely. That is since $\nabla^\mu \nabla^\nu F_{\mu\nu} = 0$ we cannot invert $\nabla_\mu F^{\mu\nu} = -J^\nu$

The Euler-Lagrange equations do not determine A_μ^a; they are gauge invariant and leave $g(x)$ completely free. Hence we must specify a relation among the fields A_μ^a in order to determine the true independent degrees of freedom from the arbitrary gauge angles (artifact) dependent variables. Since the equations of motion are invariant we can choose any relation among the fields as long as it determines $g(x)$ uniquely and hence allows the A_μ^a to be determined. Of course the work then comes in expressing the dependent degrees of freedom in terms of the true independent dynamical degrees of freedom. This is easiest to accomplish in the Coulomb gauge

$$\partial_i A_i^a = 0 .$$

(Technical point: in QED the gauge transformation of the Coulomb condition $\Rightarrow \nabla^2 \omega = 0$ and hence for $\omega \xrightarrow{x\to\infty} 0$ the Poisson eq. $\Rightarrow \omega(x) = 0$, i.e. $\partial_i A_i = 0$ determined $\omega(x)$ uniquely. For non-abelian groups the gauge transformation $\Rightarrow$

$$\partial_i (\partial_i \omega^a + f_{abc}\, \omega^b A_i^c) = 0$$

or $$[\nabla^2 \delta_{ac} - f_{abc} A_i^b \partial_i] \omega^c = 0$$

Now in perturbation theory i.e. small A_i^b this operator is invertible and $\omega^c = 0$ but for large fields A_i^b it is possible that the operator has ~~non~~ zero eigenvalues and is not invertible - this is called the Gribov ambiguity - the Coulomb gauge condition does not uniquely fix the gauge angles for large ~~non~~-perturbative fields. We will ignore this problem since 1) it is known how to circumvent it 2) we could choose $A_0^a = 0$ gauge which uniquely fixes W^a even non-perturbatively.)

So let's look more closely at the Euler-Lagrange equations, some will involve time derivatives and hence represent true dynamical evolution others will not involve time derivatives and will represent constraints among the variables. We then must solve the constraints (along with the our choice of gauge) in order to eliminate the non-independent degrees of freedom from the dynamical equations. We can then express the Hamiltonian and Lagrangian in terms of these variables and hence the functional integral representation for the Green functions.

Since it is the gauge fields that present the difficulty we can wlog set $J_\mu=0$ and just treat the pure Yang-Mills theory. For $J_\mu\neq 0$ we can follow the same reasoning we will present in the $J_\mu=0$ case.

The Euler-Lagrange equations are

a) $$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc} A^b_\mu A^c_\nu$$

and

b) $$\partial^\nu F^a_{\mu\nu} - f_{abc} A^{b\nu} F^c_{\mu\nu} = 0$$

The true dynamical equations involve time derivatives and are given by (a) $\mu = 0, \nu = i$; b) $\mu = i$ above)

D1) $$\partial_0 A^a_i = F^a_{0i} + \partial_i A^a_0 + f_{abc} A^b_0 A^c_i$$

$$\boxed{\begin{aligned} &= F^a_{0i} + (\delta^{ac}\partial_i - f_{abc} A^b_i) A^c_0 \\ \partial_0 A^a_i &= F^a_{0i} + (\nabla_i)^{ab} A^b_0 \end{aligned}}$$

dynamical equation

D2) $$\boxed{\partial_0 F^a_{0i} = [\partial_j \delta^{ac} - f_{abc} A^b_j] F^c_{ji} + f_{abc} A^b_0 F^c_{0i}}$$

The momentum canonically conjugate to A^a_i is $\dfrac{\partial \mathcal{L}_{inv}}{\partial \dot{A}^a_i} = +F^a_{0i}$

However the Lagrangian $\mathcal{L}_{inv}$ is independent of $\frac{\partial}{\partial t}A_0$ and, of course, $\partial_0 F^a_{ij}$ hence they are dependent variables. We must solve their constraint equations in terms of the independent variables and substitute this in the RHS of the dynamical equations. Further we know that the gauge invariance implies that not all the F^a_{0i} can be independent and hence not all the A^a_i are independent. We use the gauge invariance to pick a gauge relation among them at equal times and it will evolve according to D1) & D2).

So first let's consider the remaining Euler-Lagrange equations a) $\mu,\nu = i,j$ and b) $\mu=0$ above to yield the constraint equations:

Constraint equations

C(1) $F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i - f_{abc} A^b_i A^c_j$,

this defines F^a_{ij} in terms of A^a_i at equal times.

C(2)

$$-\left(\partial_i \delta^{ac} - f_{abc} A^b_i\right) F^c_{0i} = 0$$

or

$$\nabla_i F_{0i} = 0 \, .$$

So again not all the F^a_{0i} are independent since these equations are not independent ($\nabla_\mu \nabla_\nu F^{\mu\nu} = 0$) this is analogous to $\vec{\nabla} \cdot \vec{E} = 0$ in QED. and hence all the A^a_i cannot be treated as independent. Thus we must impose a gauge condition to relate the A^a_i. We now choose the

Coulomb gauge

$$\boxed{\partial_i A^a_i = 0}$$

Since $\mathcal{L}_{inv}$ is gauge invariant we can always do this. Hence as in QED A_i^a must be transverse. Thus the longitudinal component of the momentum canonically conjugate to A_i^a is not independent, call it F_{0i}^{aL}. Hence F_{0i}^{aL} depends upon the other degrees of freedom through the constraint equation (C2)

to see this we can always write any 3-vector in terms of its longitudinal and transverse components as

$$F_{0i}^a = F_{0i}^{aT} + F_{0i}^{aL}$$

where $\partial_i F_{0i}^a = \partial_i F_{0i}^{aL}$ i.e. $\partial_i F_{0i}^{aT} = 0$

and $\epsilon_{ijk}\partial_j F_{0k}^a = \epsilon_{ijk}\partial_j F_{0k}^{aT}$

i.e. $\epsilon_{ijk}\partial_j F_{0k}^{aL} = 0$

(i.e. $\vec{E} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{A}$ for any 3-vector)

Now we must express the dependent variables A_0^a and F_{0i}^{aL} in terms of the independent variables (recall F_{ij}^a is straightforward) which are the transverse components of A_i^a and F_{0i}^a (Coulomb gauge $\Rightarrow A_i^{aL} \equiv 0$). Towards this end we express F_{0i}^a as

$$F_{0i}^{aL} \equiv -\partial_i f^a \,;\; F_{0i}^{aT} \equiv E_i^a \,; \text{ that is } \boxed{F_{0i}^a = E_i^a - \partial_i f^a}$$

with $\partial_i E_i^a \equiv 0$, E_i^a is purely transverse

So that $\partial_i F_{0i}^a = -\nabla^2 f^a$

(i.e. the 3 dimensional projectors are

$$P_{ij}^T \equiv \delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2} \quad ; P_{ij}^L = \frac{\partial^i \partial^j}{\nabla^2}$$

$$P^T P^T = P^T \,;\; P^L P^L = P^L \,;\; P^L P^T = 0 = P^T P^L$$

$$P^T + P^L = 1.$$

So any vector
$$\begin{aligned} V^i &= \underbrace{P_{ij}^T V^j} + P_{ij}^L V^j \\ &= V_i^T + \partial_i \frac{\partial_j}{\nabla^2} V^j \\ &= V_i^T + \partial_i V^L \end{aligned}$$

That is
$$F^a_{0i} = -\partial_i f^a + E^a_i$$

where
$$-f^a = \frac{1}{\nabla^2}\partial_j F^a_{0j}$$
$$E^a_i = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right) F^a_{0j} \quad .)$$

So then E^a_i and the transverse components of A^a_i are the independent variables and they are conjugate to each other. Hence we must find A^a_0 and f^a in terms of A^{aT}_i; E^a_i. Turning to the constraint equation (C2)

$$0 = (\partial_i \delta^{ac} - f_{abc} A^b_i) F^c_{0i}$$
$$\Rightarrow \boxed{(\nabla^2 \delta^{ac} - f_{abc} A^b_i \partial_i) f^c = - f_{abc} A^b_i E^c_i}$$

Now we can formally solve this equation by introducing the Coulomb Green function $\mathcal{D}_c(x,y;A^a_i)$ such that

$$(\nabla_x^2 \delta^{ac} - f_{abc} A_i^b \partial_i^x) \mathcal{D}_C^{cd}(\vec{x}, \vec{y}; A_i^a) = \delta^{ad} \delta^3(\vec{x} - \vec{y})$$

Then we have that

$$\boxed{f^a(\vec{x}, t) = -\int d^3y \, \mathcal{D}_C^{ab}(\vec{x}, \vec{y}; A_i^a) \, f_{bcd}(A_i^c \cdot E_i^d)(\vec{y}, t)}$$

Perturbatively $\mathcal{D}_C^{ab}$ can be found by expanding in powers of A_i^b that is

in lowest order & $\mathcal{D}_C = \mathcal{D}_C^{(0)} + \mathcal{D}_C^{(1)} + \cdots$
and $\mathcal{D}_C^{(0)}$ is indep. of A_i^a
with $\nabla^2 \mathcal{D}_C^{(0)} = \delta^3(\vec{x} - \vec{y})$

$$\Rightarrow \quad \mathcal{D}_C^{(0)ab}(\vec{x}, \vec{y}) = \frac{-\delta^{ab}}{4\pi |\vec{x} - \vec{y}|} \quad \text{as}$$

in Maxwell theory.

Then $\nabla_x^2 \delta^{ac} \mathcal{D}_C^{(1)cd} - f_{abc} A_i^b \partial_i^x \mathcal{D}_C^{(0)cd} = 0$

$$\Rightarrow \quad \mathcal{D}_C^{(1)ab}(\vec{x}, \vec{y}; A) = \int d^3z \, \mathcal{D}_C^{(0)ac}(\vec{x}, \vec{z}) \left(f_{cde} A_i^d(\vec{z}) \partial_i^z\right) \mathcal{D}_C^{(0)eb}(\vec{z}, \vec{y})$$

etc. So we have

$$\boxed{\mathcal{D}_C^{ab}(\vec{x},\vec{y},A_i^a) = \frac{-\delta^{ab}}{4\pi|\vec{x}-\vec{y}|} + \int d^3z \frac{1}{4\pi|\vec{x}-\vec{z}|} f_{acb} A_i^c(z)\partial_i^z \frac{1}{4\pi|\vec{z}-\vec{y}|} + \cdots}$$

Hence we have f^a in terms of the independent variables.

Next we need an equation for A_0^a. Taking the divergence of (DI) we find

$$\partial_0\partial_i A_i^a = \partial_i F_{0i}^a + (\delta^{ac}\nabla^2 - f_{abc}A_i^b\partial_i)A_0^c$$

Now we use the Coulomb gauge condition

$$\partial_i A_i^a = 0$$

and the decomposition of F_{0i}^a; $\partial_i F_{0i}^a = -\nabla^2 f^a$

to obtain

$$\boxed{\nabla^2 f^a = (\nabla^2\delta^{ac} - f_{abc}A_i^b\partial_i)A_0^c}$$

Again we can solve this by using the Coulomb gauge Green function $\mathcal{D}_C$; yielding

$$\boxed{A_0^a(\vec{x},t) = \int d^3y\, \mathcal{D}_C^{ab}(\vec{x},\vec{y},A_i^a)\nabla_y^2 f^b(\vec{y},t)}$$

So considering $\mathcal{D}_C$ an integral operator we can write these solutions in a short hand way as

$$\boxed{\begin{aligned} f^a &= \mathcal{D}_C^{ab}(f_{bcd}A_i^c E_i^c) \\ A_0^a &= \mathcal{D}_C^{ab}(\nabla^2 f^b) \end{aligned}}$$

Mission accomplished we have expressed the dependent variables in terms of the independent variables. We may now construct the Hamiltonian in terms of the independent variables alone, as well as the Lagrangian.

Hence $\mathcal{H} = E_i^a \dfrac{\partial A_i^a}{\partial t} - \mathcal{L}_{inv}$

with $\mathcal{L}_{inv}$ now written in terms of E_i^a, A_i^a only.

That is using the Euler-Lagrange eq.'s

$$\mathcal{L}_{inv} = \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \tfrac{1}{2}F^a_{\mu\nu}(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - f_{abc}A^{b\mu}A^{c\nu})$$

$$= -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

$$= +\tfrac{1}{2}F^a_{0i}F^a_{0i} - \tfrac{1}{4}F^a_{ij}F^a_{ij}$$

we have the second order formulation.
Now we use the constraints to write

$$\boxed{\begin{aligned} F^a_{ij} &= \partial_i A^a_j - \partial_j A^a_i - f_{abc}A^b_i A^c_j \\ &\equiv \epsilon_{ijk}B^a_k \end{aligned}}$$

Hence $F^a_{ij}F^a_{ij} = \epsilon_{ijk}\epsilon_{ijl}B^a_k B^a_l = 2B^a_i B^a_i$

and
$$F^a_{0i}F^a_{0i} = (E^a_i - \partial_i f^a)(E^a_i - \partial_i f^a)$$

So

$$\boxed{\mathcal{L}_{inv} = \tfrac{1}{2}(E^a_i - \partial_i f^a)(E^a_i - \partial_i f^a) - \tfrac{1}{2}B^a_i B^a_i}$$

Where we have E^a_i is transverse

$B_i^a = B_i^a(A_j^b)$ is a function of transverse A_j^b

and $f^a = \mathcal{D}_c^{ab}(f_{bcd} A_i^c E_i^c)$

is a function of E_i and A_i^c also.

Hence $\mathcal{L}_{NW}$ is written in terms of the independent variables

Thus

$$\mathcal{L} = E_i^a \frac{\partial A_i^a}{\partial t} - \frac{1}{2}(E_i^a - \partial_i f^a)(E_i^a - \partial_i f^i) + \frac{1}{2} B_i^a B_i^a$$

Now from the Euler-Lagrange eq. (D1)

$$\frac{\partial A_i^a}{\partial t} = \partial_0 A_i^a = F_{0i}^a + (\partial_i \delta^{ac} - f_{abc} A_i^b) A_0^c$$

using the definition for F_{0i}^a and constraint equation for A_0^a

$$= E_i^a - \partial_i f^a + (\partial_i \delta^{ac} - f_{abc} A_i^b)\mathcal{D}_c^{cd}(\nabla^2 f^d$$

$$\partial_0 A_i^a = E_i^a - [\partial_i \delta^{ad} - (\partial_i \delta^{ac} - f_{abc} A_i^b)\mathcal{D}_c^{cd} \nabla^2] f^d$$

As a check since A_i^a is transverse and E_i^a is transverse the [ ] must be transverse.

$$\partial_i[\partial_i\delta^{ad}-(\partial_i\delta^{ac}+f_{abc}A_i^b)\mathcal{D}_c^{cd}\nabla^2]$$
$$=[\nabla^2\delta^{ad}-(\nabla^2\delta^{ac}+f_{abc}A_i^b\partial_i)\mathcal{D}_c^{cd}\nabla^2]$$

but the Coulomb Green function is defined by
$$(\nabla^2\delta^{ac}+f_{abc}A_i^b\partial_i)\mathcal{D}_c^{cd}=\delta^{ad}\mathbb{1}$$

hence $\partial_i[\;]=0$ and is transverse.

So

$$\int d^3x\, E_i^a\partial_0 A_i^a=\int d^3x\Big[E_i^aE_i^a-E_i^a\partial_i f^a+E_i^a\partial_i\mathcal{D}_c^{ad}\nabla^2 f^d-f_{abc}E_i^aA_i^b\times\mathcal{D}_c^{cd}\nabla^2 f^d\Big]$$

Now integrating by parts and throwing away surface terms and using the fact that E_i^a is transverse $\partial_iE_i^a=0$ we have

$$\int d^3x\, E_i^a\partial_0 A_i^a=\int d^3x\big[E_i^aE_i^a-f_{abc}E_i^aA_i^b\mathcal{D}_c^{cd}\nabla^2 f^d.$$

Now using the constraint eq. (C2)

$$\left(\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i \cdot\right) f^c = f_{abc} A_i^b E_i^c$$

we have

$$- f_{abc} E_i^a A_i^b \mathcal{D}_c^{cd} \nabla^2 f^d$$

$$= -\left(\nabla^2 \delta^{c\cancel{f}} - f_{cef} A_i^e \partial_i\right) f^f \mathcal{D}_c^{cd} \nabla^2 f^d$$

$$= -\left(\nabla^2 f^c\right) \mathcal{D}_c^{cd} \nabla^2 f^d + f_{cef} A_i^e (\partial_i f^f) \mathcal{D}_c^{cd} \nabla^2 f^d$$

So integrating by parts using $\partial_i A_i^e = 0$

$$-\int d^3x\, f_{abc} E_i^a A_i^b \mathcal{D}_c^{cd} \nabla^2 f^d$$

$$= \int d^3x - \left[f^c \left(\nabla^2 \mathcal{D}_c^{cd}\right) \nabla^2 f^d + f^f \left(f_{cef} A_i^e \partial_i \mathcal{D}_c^{cd} \right) \times \nabla^2 f^d \right]$$

$$= \int d^3x - \left[f^a \left(\nabla^2 \delta^{ad} - f_{abc} A_i^b \partial_i \right) \mathcal{D}_c^{cd} \right] \nabla^2 f^d$$

Using the Coulomb Green function eq. $\Rightarrow$

$$= \int d^3x - \left(f^a \nabla^2 f^a \right) = + \int d^3x (\partial_i f^a)(\partial_i f^a)$$

Hence

$$\boxed{\int d^3x\, E_i^a\, \partial_0 A_i^a = \int d^3x \left[E_i^a E_i^a + (\partial_i f^a)(\partial_i f^a) \right]}$$

So the Hamiltonian written in terms of the independent degrees of freedom, becomes after another integration by parts

$$\boxed{H = \int d^3x \left[\tfrac{1}{2} E_i^a E_i^a + \tfrac{1}{2} B_i^a B_i^a + \tfrac{1}{2} (\partial_i f^a)(\partial_i f^a) \right]}$$

This is similar to the Hamiltonian we found in Coulomb gauge QED where $\int d^3x\, \frac{1}{2} (\partial_i f^a)^2$ is like the instantaneous Coulomb interaction which occurred there.

Finally we would like to apply our path integral representation of the generating functional to this case. Since we have been treating A_i^a and E_i^a as independent fields in our 1st order formalism we must integrate over both of them with the usual exponential of the action in terms of them times i. We also include a source term for A_i^a (we could include a E_i^a source term but there is no need to do so) thus

$$Z[j_i^a] \equiv \int [dE_i^a][dA_i^a]\, e^{i\int d^4x\,[\mathcal{L}_{inv} - j_i^a A_i^a]}$$

$$= \int [dE_i^a][dA_i^a]\, e^{i\int d^4x\,[\frac{1}{2}E_i^a E_i^a + \frac{1}{2}\partial_i f^a \partial_i f^a - \frac{1}{2}B_i^a B_i^a - j_i^a A_i^a]}$$

$$\boxed{\begin{aligned} &= \int [dE_i^a][dA_i^a]\, e^{i\int d^4x\,[E_i^a \partial_0 A_i^a - \frac{1}{2}E_i^a E_i^a - \frac{1}{2}B_i^a B_i^a - \frac{1}{2}(\partial_i f^a)(\partial_i f^a) - j_i^a A_i^a]} \\ &Z[j_i^a] = \int [dE_i^a][dA_i^a]\, e^{i\int d^4x\,[E_i^a \partial_0 A_i^a - \mathcal{H} - j_i^a A_i^a]} \end{aligned}}$$

where we see that this is written in the form of the phase space path integral, that is an integral over the momenta and the conjugate coordinates of the exponential of i times the Lagrangian. and since A_i^a is transverse so is j_i^a

$$\partial_i j_i^a = 0.$$

(Aside: Recall for scalar field theory

$$Z[J]=\int[d\varphi]\, e^{i\int d^4x[\mathcal{L}(\varphi,\partial_\mu\varphi)+J\varphi]}$$

Now if $\mathcal{L}$ is quadratic in $\partial_\mu\varphi$ i.e. $\mathcal{L}=\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-V(\varphi)$ then we can introduce a gaussian integral over the momentum π i.e.

$$\int[d\pi]\, e^{i\int d^4x[\pi\dot\varphi-\frac{1}{2}\pi^2]}=e^{+\frac{i}{2}\int d^4x\,\dot\varphi\dot\varphi}$$

So

$$e^{i\int d^4x[\mathcal{L}+J\varphi]}=e^{i\int d^4x[\frac{1}{2}\dot\varphi^2-\frac{1}{2}\vec\nabla\varphi\cdot\vec\nabla\varphi-V(\varphi)+J\varphi]}$$

$$=\int[d\pi]\, e^{i\int d^4x[\pi\dot\varphi-\frac{1}{2}\pi^2-\frac{1}{2}\vec\nabla\varphi\cdot\vec\nabla\varphi-V(\varphi)+J\varphi]}$$

$$=\int[d\pi]\, e^{i\int d^4x[\pi\dot\varphi-\mathcal{H}(\pi,\varphi)+J\varphi]}$$

hence

$$\boxed{Z[J]=\int[d\pi][d\varphi]\, e^{i\int d^4x[\pi\dot\varphi-\mathcal{H}(\pi,\varphi)+J\varphi]}}$$

This can be derived from first principles directly and is the most general form for the path integral quantization of a theory i.e. this is true for all Hamiltonians written in terms of the independent dynamical degrees of freedom and their conjugate momenta.)

Now we would like to re-express the integral in various ways. ~~Before~~ doing this let it be made explicit that A_i^a and E_i^a are transverse by putting a superscript T on them so.

$$\boxed{Z[j_i^a] = \int [dE_i^{aT}][dA_i^{aT}] \times \\ \times e^{i\int d^4x \left[E_i^{aT}\partial_0 A_i^{aT} - \mathcal{H}(E_i^{aT}, A_i^{aT}) - j_i^a A_i^{aT}\right]}}$$

where it is understood that B_i and f^a in $\mathcal{H}$ depend on A_i^{aT} and E_i^{aT} through their constraint equations.

It is difficult to integrate over the transversely constrained fields, so we will relax the constraint by integrating over all fields and putting δ functionals for the transverse condition.

Then we introduce the dummy variable E^{aL}

$$\int [dE_i^{aT}] = \int [dE_i^{aT}][dE^{aL}]\,\delta[E^{aL}]$$

where we introduce 3 independent components of the field E_i^a by

$$E_i^a \equiv \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) E_j^{aT} + \frac{\partial_i}{\nabla^2} E^{aL}$$

Then $\boxed{\partial_i E_i^a = E^{aL}}$ the longitudinal part of E_i^a

So

$$\int [dE_i^{aT}] = \int [dE_i^a]\,\delta[\partial_i E_i^a]\,J$$

where J is the Jacobian in going from an integral over the three variables $\{E_i^{aT}, E^{aL}\}$ to the three independent variables $\{E_i^a\}$. So we have that

$$[dE_i^a] = \left|\frac{\delta E_i^a}{\delta(E_j^{bT}, E^{bL})}\right| [dE_i^{aT}][dE^{aL}]$$

$$= \det\left(\frac{\vec{\nabla}}{\nabla^2}\right) [dE_i^{aT}][dE^{aL}]$$

Since $\det\left(\frac{\vec{\nabla}}{\nabla^2}\right)$ is just an ∞ constant independent of the fields it cancels from our normalization factors and can be ignored. We can do the same for A_i^{aT} hence

$$\boxed{Z[j_i^a] = \int [dE_i^a][dA_i^a]\delta[\partial_i E_i^a]\delta[\partial_i A_i^a] \times e^{i\int d^4x\left[E_i^a \partial_0 A_i^a - \frac{1}{2}E_i^a E_i^a - \frac{1}{2}B_i^a B_i^a - \frac{1}{2}(\partial_i f^a)(\partial_i f^a) - j_i^a A_i^a\right]}}$$

Thus we have obtained the generating functional for the Yang-Mills theory in the Coulomb gauge. We could use their formula to develop a perturbation expansion in the strength of A_i^a i.e. rescale A_i^a & E_i^a by g, the gauge coupling constant, then we could expand in g. Of course the Green functions will not be Lorentz covariant and hence the Lorentz invariance of the S-matrix will not be obvious (manifest) until all contributions to a particular order in perturbation theory are summed up. The Coulomb gauge Feynman Rules are not a useful set of rules in which to calculate. Their use is in the fact that we have rigorously derived the functional integral representation for $Z[j_i^a]$ from first principles in the Coulomb gauge. Since the S-matrix is gauge invariant and Lorentz invariant we should be able to find a more Lorentz invariant form for the generating functional.

Towards this end we now would like to introduce as dummy variables the fields we eliminated by their constraint equations.

First in the above integral we have that

$$f^a = -\mathcal{D}_c^{ab}\left(f_{bcd} A_i^c E_i^c\right)$$

So we can introduce the dummy variable f^a by the integral

$$1 = \int [df^a]\, \delta[f^a + \mathcal{D}_c^{ab} f_{bcd} A_i^c E_i^c]$$

Alternatively we can use the constraint equation f^a obeys p.-535-

$$[\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i] f^c = -f_{abc} A_i^b E_i^c$$

and the Green function equation

$$[\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i]\, \mathcal{D}_c^{cd} = \delta^{ad}$$

to write the δ-functional as

$$\delta[(\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i) f^c + f_{abc} A_i^b E_i^c]$$

$$= \delta[(\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i)[f^c + \mathcal{D}_c^{cd} f_{def} A_i^e E_i^f]]$$

$$= \frac{1}{\det[\nabla^2 \delta^{ac} - f_{abc} A_i^b \partial_i]}\, \delta[f^a + \mathcal{D}_c^{ab} f_{bcd} A_i^c E_i^d]$$

Thus

$$1=\int[df^a](\det M_c)\,\delta[(\nabla^2\delta^{ac}-f_{abc}A_i^b\partial_i)f^c+f_{abc}A_i^bE_i^c]$$

where

$$M_c^{ac}(x,y)\equiv(\nabla^2\delta^{ac}-f_{abc}A_i^b(x)\partial_i^x)\delta^4(x-y)$$

Hence we have

$$Z[j_i^a]=\int[dE_i^a][dA_i^a][df^a](\det M_c)\times$$

$$\times\delta[\partial_iE_i^a]\delta[\partial_iA_i^a]\delta[(\nabla^2\delta^{ac}-f_{abc}A_i^b\partial_i)f^c+f_{abc}A_i^bE_i^c]$$

$$\times e^{i\int d^4x[E_i^a\partial_0A_i^a-\frac{1}{2}(E_i^aE_i^a+B_i^aB_i^a+(\partial_if^a)(\partial_if^a)) - j_i^aA_i^a]}$$

We next would like to change variables from E_i^a to F_{0i}^a So define

$$F_{0i}^a\equiv E_i^a-\partial_if^a$$

Then

$$[dE_i^a][df^a]\,\delta[\partial_i E_i^a]\,\delta[(\nabla^2\delta^{ac} - f_{abc}A_i^b\partial_i)f^c + f_{abc}A_i^b E_i^c]$$

$$= [dF_{0i}^a][df^a]\,\delta[\partial_i F_{0i}^a + \nabla^2 f^a]\times$$
$$\times\,\delta[\nabla^2 f^a + f_{abc}A_i^b \underbrace{(E_i^c - \partial_i f^c)}_{F_{0i}^c}]$$

$$= [dF_{0i}^a][df^a]\,\delta[\partial_i F_{0i}^a + \nabla^2 f^a]\,\delta[\nabla^2 f^a + f_{abc}A_i^b F_{0i}^c]$$
$$= [dF_{0i}^a][df^a]\,\delta[\partial_i F_{0i}^a - f_{abc}A_i^b F_{0i}^c]\times$$
$$\times\,\delta[\nabla^2 f^a + f_{abc}A_i^b F_{0i}^c]$$

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in addition the exponent of the integral becomes

$$\int d^4x\,[E_i^a\partial_0 A_i^a - \tfrac{1}{2}(E_i^a E_i^a + B_i^a B_i^a + (\partial_i f^a)(\partial_i f^a)]$$

$$= \int d^4x\,[F_{0i}^a\partial_0 A_i^a + \partial_i f^a\partial_0 A_i^a - \tfrac{1}{2}B_i B_i$$
$$- \tfrac{1}{2}(F_{0i}^a F_{0i}^a) + \partial_i f^a E_i^a]$$

Also using the definition of B_i^a p. -539- we have

$$\frac{1}{2} B_i^a B_i^a = \frac{1}{4}\left(\partial_i A_j^a - \partial_j A_i^a - f_{abc} A_i^b A_j^c\right)^2$$

The integral becomes

$$Z[j_i^a] = \int [dF_{oi}^a][dA_i^a][df^a]\,\delta[\nabla^2 f^a + f_{abc} A_i^b F_{oi}^c]$$
$$\times (\det M_c) \times \delta[\partial_i F_{oi}^a - f_{abc} A_i^b F_{oi}^c]\,\delta[\partial_i A_i^a]$$
$$\times e^{i\int d^4x \left[F_{oi}^a \partial_0 A_i^a - \frac{1}{2}(F_{oi}^a F_{oi}^a) - \frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a - f_{abc} A_i^b A_j^c)^2 - j_i^a A_i^a\right]}$$

where we used integration by parts and transversality of E_i^a, A_i^a, i.e. the δ-functionals $\ni \partial_i E_i^a = 0 = \partial_i A_i^a$!

Now f^a appears only in the δ-functional hence we can do the f^a integral to obtain simply $\frac{1}{\det \nabla^2}$. As usual this is a ~~field~~ independent(ls) constant that ~~cancels~~ in the normalization; hence we ignore it.

So we have

$$Z[j_i^a] = \int [dF_{0i}^a][dA_i^a](\det M_c)\,\delta[\partial_i A_i^a] \times$$

$$\times\, \delta[\partial_i F_{0i}^a - i f_{abc} A_i^b F_{0i}^c] \times$$

$$\times\, e^{i\int d^4x \left[F_{0i}^a \partial_0 A_i^a - \frac{1}{2}(F_{0i}^a F_{0i}^a) - \frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a - f_{abc} A_i^b A_j^c)^2 - j_i^a A_i^a\right]}$$

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finally we can represent the second functional δ by means of a Functional Fourier transform with integration variable A_0^a

$$\delta[\partial_i F_{0i}^a + f_{abc} A_i^b F_{0i}^c] = \int [dA_0^a]\, e^{i\int d^4x \left[A_0^a(\partial_i F_{0i}^a - f_{abc} \times A_i^b F_{0i}^c)\right]}$$

integrating by parts this becomes

$$= \int [dA_0^a]\, e^{i\int d^4x\, F_{0i}^a(-\partial_i A_0^a - f_{abc} A_0^b A_i^c)}$$

in addition we can write the $(B_i^a)^2$ term in terms of a Gaussian integral over dummy variable F_{ij}^a

$$\int [dF_{ij}^a]\, e^{i\int d^4x \left[\frac{1}{4}F_{ij}^a F_{ij}^a - \frac{1}{2}F_{ij}^a(\partial_i A_j^a - \partial_j A_i^a - f_{abc} A_i^b A_j^c)\right]}$$

$$= e^{i\int d^4x \left[-\frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a - f_{abc}A_i^b A_j^c)^2\right]}$$

Putting this altogether we obtain

$$Z[j_i^a] = \int [dF_{0i}^a][dF_{ij}^a][dA_i^a][dA_0^a](\det M_c)\,\delta[\partial_i A_i^a] \times$$

$$\times e^{i\int d^4x \left[-\frac{1}{2}F_{0i}^a F_{0i}^a + \frac{1}{4}F_{ij}^a F_{ij}^a - \frac{1}{2}F_{ij}^a(\partial_i A_j^a - \partial_j A_i^a - f_{abc} A_i^b A_j^c) + F_{0i}^a(\partial_0 A_i^a - \partial_i A_0^a - f_{abc} A_0^b A_i^c) - j_i^a A_i^a\right]}$$

Grouping terms this becomes

$$\boxed{Z[j_i^a] = \int [dF_{\mu\nu}^a][dA_\mu^a](\det M_c)\,\delta[\partial_i A_i^a] \times e^{i\int d^4x \left[+\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2}F_{\mu\nu}^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{abc}A_\mu^b A_\nu^c) - j_i^a A_i^a\right]}}.$$

Now this is just the exponent of the action with the additional δ and det. With the understanding that $j_0^a = 0$ we can write this as

$$\boxed{Z[j_\mu^a] = \int [dA_\mu^a][dF_{\mu\nu}^a] (\det M_c)\, \delta[\partial_i A_i^a] \times e^{i\int d^4x \left[\mathcal{L}_{inv}(A,F) + j_\mu^a A^{a\mu}\right]}}$$

Since M_c is independent of F and $\mathcal{L}_{inv}(A,F)$ is Gaussian in F we can perform the $F_{\mu\nu}^a$ integral to obtain the second order formulation Lagrangian in the exponent

$$\boxed{Z[j_\mu^a] = \int [dA_\mu^a] (\det M_c)\, \delta[\partial_i A_i^a] \times e^{i\int d^4x \left[\mathcal{L}_{inv}(A) + j_\mu^a A^{a\mu}\right]}}$$

where
$$\mathcal{L}_{inv}(A) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc} A^b_\mu A^c_\nu$

Thus we have derived the Faddeev-Popov form of the functional integral representation of the generating functional in the Coulomb gauge.

Note that the gauge variation of the Coulomb gauge condition is

$$\begin{aligned}\delta_Q(\omega)\, \partial_i A^a_i &= \partial_i (\partial_i \omega^a + f_{abc} \omega^b A^c_i) \\ &= (\nabla^2 \delta^{ac} - f_{abc} A^b_i \partial_i)\omega^c \\ &= \int d^4y\, M^{ac}_c(x,y;A)\,\omega^c(y)\end{aligned}$$

This is just the operator necessary to guarantee that only one A^a_μ from each gauge orbit is chosen by the Coulomb gauge condition. Hence we have obtained the same integral representation for Z as we did in our intuitive approach