Again one can construct the currents and Noether's theorem etc. — this is left as an exercise for the reader.

III.E.3) Local Internal Symmetries — Gauge Invariance Revisited.

Suppose we have a theory which is described by a Lagrangian with a global internal symmetry.

$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$ such that

$$\delta_Q^a \mathcal{L}(\varphi, \partial_\mu\varphi) = 0 \text{ and}$$

$$\delta_Q^a \varphi^\alpha(x) = -i(T^a)_{\alpha\beta}\varphi^\beta(x)$$

That is $\mathcal{L}$ is invariant under global rotations of φ^α through angles ω^a

$$\mathcal{U}^{-1}(\omega)\,\phi^\alpha(x)\,\mathcal{U}(\omega) = U(\omega)_{\alpha\beta}\,\phi^\beta(x)$$

Now suppose we ask given a globally invariant theory how does we make it invariant under local rotations. Those for which ω^a maybe different at each space-time point i.e. $\omega^a = \omega^a(x)$

Then we have that terms which do not contain spacetime derivatives and are globally invariant are also locally invariant - the space-time dependence of ω^a is no problem. However, derivatives of φ^α now transform non-covariantly under for example

$$\delta_Q(\omega)\varphi^\alpha(x) \equiv \delta_Q \varphi^\alpha(x) = \omega^a(x)\delta_Q^a \varphi^\alpha(x)$$

$$= -i\omega^a(x)(T^a)_{\alpha\beta}\varphi^\beta(x)$$

hence

$$\partial_\mu[\omega^a(x)\delta_Q^a\varphi^\alpha(x)] \equiv \delta_Q \partial_\mu \varphi^\alpha(x) = -i\partial_\mu(\omega^a(x)(T^a)_{\alpha\beta}\varphi^\beta(x))$$

$$= -i\omega^a(x)(T^a)_{\alpha\beta}\,\partial_\mu\varphi^\beta(x)$$

$$-i(\partial_\mu\omega^a(x))(T^a)_{\alpha\beta}\,\varphi^\beta(x)$$

Hence we have an extra term in the transformation law of $\partial_\mu \varphi^\alpha$ proportional to $\partial_\mu \omega^a(x)$. $\partial_\mu \varphi^\alpha$ is no longer in the same representation of G that φ^α is.

Now the global invariance of $\mathcal{L}$ implies

$$\delta_Q^a \mathcal{L}(\varphi, \partial_\mu \varphi) = 0$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi^\alpha(x)} \delta_Q^a \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \delta_Q^a \partial_\mu \varphi^\alpha(x)$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi^\alpha(x)} \delta_Q^a \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \partial_\mu \delta_Q^a \varphi^\alpha(x)$$

Under local gauge transformations however the derivative terms become

$$\delta_Q \; \mathcal{L}(\varphi, \partial_\mu \varphi)$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi^\alpha(x)} \delta_Q \varphi^\alpha(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \delta_Q \partial_\mu \varphi^\alpha(x)$$

$$= \omega^a(x) \frac{\partial \mathcal{L}}{\partial \varphi^\alpha(x)} \delta_Q^a \varphi^\alpha(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \partial_\mu \left(\omega^a(x) \delta_Q^a \varphi^\alpha(x) \right)$$

$$= \omega^a(x) \left[\frac{\partial \mathcal{L}}{\partial \varphi^\alpha(x)} \delta_Q^a \varphi^\alpha(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \partial_\mu \delta_Q^a \varphi^\alpha(x) \right]$$

$$+ \left(\partial_\mu \omega^a(x) \right) \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha(x)} \delta_Q^a \varphi^\alpha(x)$$

$$\delta_Q \mathcal{L} = \omega^a(x) \delta_Q^a \mathcal{L} + \left[\partial_\mu \omega^a(x) \right] J_a^\mu(x)$$

where $J_a^\mu(x)$ is the global symmetry Noether current $J_a^\mu(x) \equiv \frac{\partial \mathcal{L}}{\partial \partial^\mu \varphi^\alpha} \delta_Q^a \varphi^\alpha$.

Hence even if $\mathcal{L}$ is globally invariant $\delta_Q^a \mathcal{L} = 0$ we have that

$\delta_Q(\omega)\mathcal{L} = \left[\partial_\mu \omega^a(x) \right] J_a^\mu(x)$ is not locally invariant

As you well know, in order to restore invariance of the Lagrangian we must introduce new degrees of freedom in our theory to compensate for the change in $\mathcal{L}$. These are the gauge fields (or Yang-Mills fields). In order to determine the character of these fields we consider what is necessary for symmetry restoration. In particular we know that if $\partial_\mu \varphi^\alpha(x)$ transforms locally as it does globally then $\mathcal{L}$ was constructed to be invariant.

Hence we must modify the derivative to the covariant derivative

$$(D_\mu \varphi)^\alpha = D_\mu^{\alpha\beta} \varphi^\beta(x)$$

$$\equiv \partial_\mu \varphi^\alpha(x) + i A_\mu^a(x) (T^a)_{\alpha\beta} \varphi^\beta(x).$$

where $A_\mu^a(x)$ are the Yang-Mills fields - they are 4-vectors since the derivative must be and since there are dim $\mathcal{G}$ angles of rotation w^a that violate the gauge invariance we must introduce dim $\mathcal{G}$ gauge fields to compensate for these transformations - hence the superscript "a" on A^a_μ.

The gauge transformation properties of A^a_μ are determined from the requirement that $(D_\mu\varphi)^\alpha$ transforms locally just as $\partial_\mu\varphi^\alpha$ and hence φ^α did globally. Then ~~since~~

$\mathcal{L}(\varphi,\partial_\mu\varphi)$ is globally invariant

$\mathcal{L}(\varphi, D_\mu\varphi)$ will be locally invariant!

Hence

$$\delta_Q(\omega)(D_\mu\varphi)^\alpha \equiv -i\omega^a(x)(T^a)_{\alpha\beta}(D_\mu\varphi)^\beta$$

$$= \partial_\mu \delta_Q(\omega)\varphi^\alpha(x) + iA^a_\mu(x)(T^a)_{\alpha\beta}\delta_Q(\omega)\varphi^\beta(x)$$

$$+ i\left(\delta_Q(\omega)A^a_\mu(x)\right)(T^a)_{\alpha\beta}\varphi^\beta(x).$$

expanding the RHS we have

$$= -i\,\omega^a(x)\,(T^a)_{\alpha\beta}\,\partial_\mu\varphi^\beta(x) - i(\partial_\mu\omega^a)(T^a)_{\alpha\beta}\varphi^\beta(x)$$
$$+ iA_\mu^a(T^a)_{\alpha\beta}\left(-i\omega^b(T^b)_{\beta\gamma}\right)\varphi^\gamma(x))$$
$$+ i\left(\delta_Q(\omega)A_\mu^a(x)\right)(T^a)_{\alpha\beta}\varphi^\beta(x).$$

$$= -i\,\omega^a(x)(T^a)_{\alpha\beta}\Big[\underbrace{\partial_\mu\varphi^\beta(x) + iA_\mu^b(T^b)_{\beta\gamma}\varphi^\gamma(x)}_{=(D_\mu\varphi)^\beta}\Big]$$
$$- \omega^a(T^a)_{\alpha\beta}A_\mu^b(T^b)_{\beta\gamma}\varphi^\gamma$$
$$- i\,\partial_\mu\omega^a(T^a)_{\alpha\beta}\varphi^\beta + A_\mu^a\omega^b(T^aT^b)_{\alpha\gamma}\varphi^\gamma$$
$$+ \left(\delta_Q(\omega)A_\mu^a(x)\right)(iT^a)_{\alpha\beta}\varphi^\beta$$

$\Rightarrow$

$$\left(\delta_Q(\omega)A_\mu^a(x)\right)(iT^a)_{\alpha\beta}\varphi^\beta$$
$$= +\partial_\mu\omega^a(iT^a)_{\alpha\beta}\varphi^\beta$$
$$+ \omega^aA_\mu^b\left(T^aT^b - T^bT^a\right)_{\alpha\gamma}\varphi^\gamma$$
$$= \left[\partial_\mu\omega^c + f_{abc}\,\omega^aA_\mu^b\right](iT^c)_{\alpha\beta}\varphi^\beta$$

Hence we find the gauge transformation of the Yang-Mills field

$$\boxed{\delta_Q(\omega) A^a_\mu(x) \equiv \partial_\mu \omega^a(x) + f_{abc}\, \omega^b(x) A^c_\mu(x)}$$

and the covariant derivative transformation is assumed

$$\boxed{\delta_Q(\omega)(D_\mu\varphi)^\alpha = -i\omega^a(x)(T^a)_{\alpha\beta}(D_\mu\varphi)^\beta .}$$

Hence we have that the globally invariant Lagrangian with all derivative replaced by covariant derivatives is locally gauge invariant

$$\delta_Q(\omega)\mathcal{L}(\varphi, D_\mu\varphi) = 0 .$$

In terms of quantum fields we have that

$$[Q(\omega), \phi^\alpha(x)] \equiv -i\delta_Q(\omega)\phi^\alpha(x)$$

$$= -\omega^a(x)(T^a)_{\alpha\beta}\phi^\beta(x)$$

where $Q(\omega) = \omega^a(x)\, Q^a$ with our "global" Q^a charge
and

$$[Q(\omega), A^a_\mu(x)] \equiv -i\delta_{Q(\omega)} A^a_\mu(x)$$
$$= -i(\partial_\mu \omega^a(x) + f_{abc}\, \omega^b(x) A^c_\mu(x)).$$

quantum field now

hence

$$[Q(\omega), (D_\mu\phi)^\alpha] = -\omega^a(x) (T^a)_{\alpha\beta} (D_\mu\phi)^\beta$$

where

$$(D_\mu\phi)^\alpha = \partial_\mu \phi^\alpha(x) + i A^a_\mu(x) (T^a)_{\alpha\beta}\phi^\beta(x).$$

and $\mathcal{L}(\phi, D_\mu\phi)$ describes the gauge invariant dynamics of ϕ in the presence of A.

For finite gauge transformations we have that

$$\mathcal{U}^{-1}(\omega(x))\, \phi^\alpha(x)\, \mathcal{U}(\omega(x)) = U(\omega(x))_{\alpha\beta}\, \phi^\beta(x)$$
$$= (e^{-i\omega^a(x) T^a})_{\alpha\beta}\, \phi^\beta(x)$$

hence

$$\mathcal{U}^{-1}(\omega)\, \partial_\mu \phi \;\; \mathcal{U}(\omega) = \partial_\mu\left[\left(e^{-i\omega^a(x) T^a}\right)_{\alpha\beta} \phi^\beta(x)\right]$$

$$= (\partial_\mu U)\;\; \phi \;+ U\;\partial_\mu \phi$$

$$= (\partial_\mu U) U^{-1} U \phi + U \partial_\mu \phi$$

but $U U^{-1} = 1 \;\Rightarrow\; \partial_\mu U\, U^{-1} + U \partial_\mu U^{-1} = 0$

So

$$\mathcal{U}^{-1}(\omega)\, \partial_\mu \phi\, \mathcal{U}(\omega) = U \partial_\mu \phi - U \partial_\mu U^{-1} U \phi$$

$$= U\left[\partial_\mu \phi + (U^{-1}\partial_\mu U)\phi\right]$$

Thus

$$\mathcal{U}^{-1}(\omega)\, D_\mu \phi\, \mathcal{U}(\omega) \equiv U D_\mu \phi$$

$$= \mathcal{U}^{-1}(\omega)\left(\partial_\mu \phi + A_\mu \phi\right)\mathcal{U}(\omega)$$

where we define $\boxed{(A_\mu)_{\alpha\beta} \equiv A_\mu^a (iT^a)_{\alpha\beta}}$.

$$= \mathcal{U}^{-1}(\omega)\partial_\mu\phi\,\mathcal{U}(\omega) + \mathcal{U}^{-1}(\omega)A_\mu\mathcal{U}(\omega)\,\mathcal{U}^{-1}(\omega)\phi\,\mathcal{U}(\omega)$$

$$\equiv U(\omega)D_\mu\phi$$

$$= U(\omega)\partial_\mu\phi + U(\omega)(U^{-1}\partial_\mu U)\phi$$
$$+ \mathcal{U}^{-1}(\omega)A_\mu\mathcal{U}(\omega)\,U(\omega)\phi$$

$$= U(\omega)(\partial_\mu\phi + A_\mu\phi) - U(\omega)A_\mu\phi$$
$$+ U(U^{-1}\partial_\mu U)\phi + \mathcal{U}^{-1}(\omega)A_\mu\mathcal{U}(\omega)U(\omega)\phi$$

$$= U(\omega)D_\mu\phi$$
$$+ \left[\mathcal{U}^{-1}(\omega)A_\mu\mathcal{U}(\omega) - (U\partial_\mu U^{-1}) - UA_\mu U^{-1}\right]\times$$
$$\times U(\omega)\phi$$

Hence

$$\boxed{\mathcal{U}^{-1}(\omega)A_\mu\mathcal{U}(\omega) = U\partial_\mu U^{-1} + UA_\mu U^{-1}}$$

$$= U(\omega)\partial_\mu U^{-1}(\omega) + U(\omega)A_\mu U^{-1}(\omega)$$

<u>Check</u>: for $\omega^a(x)$ infinitesimal $\mathcal{U}(\omega) = e^{-iQ(\omega)}$

$$[Q(\omega), A_\mu] = -i\left[i\partial_\mu \omega^a (T^a) \right.$$
$$\left. - i\omega^a [T^a, A_\mu] \right]$$

So

$$i(T^a)_{\alpha\beta} [Q(\omega), A^a_\mu] = i(T^a)_{\alpha\beta} (-i\partial_\mu \omega^a)$$
$$- \omega^b i A^c_\mu [T^b, T^c]_{\alpha\beta}$$
$$= i(T^a)_{\alpha\beta} [-i\partial_\mu \omega^a - i f_{abc} \omega^b A^c_\mu]$$

$\Rightarrow$

$$[Q(\omega), A^a_\mu] = -i[\partial_\mu \omega^a + f_{abc} \omega^b A^c_\mu] \checkmark .$$

Also check that $[\delta_Q(\omega'), \delta_Q(\omega)] = \delta_Q(f_{abc} \omega'^b \omega^c)$

Since we have introduced an additional degree of freedom we can ask if there are additional invariant polynomials we may add to the Lagrangian (through dimension 4) hence giving the dynamics of the gauge fields.

Since A_μ^a transforms inhomogeneously (it is a connection) no simple powers of it are invariant. Since D_μ is covariant we can imagine that $D_\mu D_\nu - D_\nu D_\mu$ is covariant (∥ displacement of ∥ displacement)

So consider

$$[D_\mu, D_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$$

$$\boxed{= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]}$$

$$\equiv i(T^a)\, F_{\mu\nu}^a = \boxed{F_{\mu\nu}}$$

where $F_{\mu\nu}$ is the anti-symmetric, covariant field strength tensor.

using $[T^a, T^b] = i f_{abc} T^c$ we have

$$\boxed{F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{abc} A_\mu^b A_\nu^c}$$

We can check that $F_{\mu\nu}^a$ transforms in the adjoint representation of the group

$$\mathcal{U}^{-1}(\omega) F_{\mu\nu}\, \mathcal{U}(\omega) = \partial_\mu (U A_\nu U^{-1} + U \partial_\nu U^{-1})$$

$$- \partial_\nu (U A_\mu U^{-1} + U \partial_\mu U^{-1})$$

$$+ [U A_\mu U^{-1} + U \partial_\mu U^{-1},\ U A_\nu U^{-1} + U \partial_\nu U^{-1}]$$

$$= U(\partial_\mu A_\nu - \partial_\nu A_\mu) U^{-1} + \partial_\mu U \cancel{U^{-1} U} A_\nu U^{-1}$$

$$+ U A_\nu \cancel{U^{-1} U} \partial_\mu U^{-1} + \partial_\mu U \partial_\nu U^{-1} + U \partial_\mu \partial_\nu U^{-1}$$

$$- \partial_\nu U \boxed{U^{-1} U} A_\mu U^{-1} - U A_\mu \cancel{U^{-1} U} \partial_\nu U^{-1}$$

$$- \cancel{\partial_\nu U \partial_\mu U^{-1}} - U \partial_\nu \partial_\mu U^{-1}$$

$$+ U A_\mu U^{-1} U A_\nu U^{-1} - U A_\nu U^{-1} U A_\mu U^{-1}$$

$$+ U A_\mu \cancel{U^{-1} U} \partial_\nu U^{-1} - U \partial_\nu \boxed{U^{-1} U} A_\mu U^{-1}$$

$$+ U \partial_\mu \cancel{U^{-1}} U A_\nu U^{-1} - U A_\nu \cancel{U^{-1} U} \partial_\mu U^{-1}$$

$$+ U \partial_\mu U^{-1} U \partial_\nu U^{-1} - \underbrace{U \partial_\nu U^{-1} U \partial_\mu U^{-1}}_{= +\partial_\nu U U^{-1} U \partial_\mu U^{-1}}$$

$$= U[\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]] U^{-1}$$

Hence

$$\boxed{\mathcal{U}^{\dagger}(\omega)\, F_{\mu\nu}\, \mathcal{U}(\omega) = U(\omega)\, F_{\mu\nu}\, U^{-1}(\omega)}$$

For infinitesimal ω^a

$$[Q(\omega), F_{\mu\nu}] = -\omega^a_{(x)} [T^a, F_{\mu\nu}]$$

That is

$$\boxed{[Q(\omega), F^a_{\mu\nu}] = f_{abc}\, \omega^b_{(x)} F^c_{\mu\nu}(x)}$$

$$\equiv -i\delta_{Q(\omega)} F^a_{\mu\nu}$$

Since $F_{\mu\nu}$ is an anti-symmetric tensor $F_\mu{}^\mu = 0$ hence we can only make a Lorentz invariant by squaring $F_{\mu\nu}$

$$\boxed{[Q(\omega), F^a_{\mu\nu} F^{a\mu\nu}] = 0}$$

Thus $F^a_{\mu\nu} F^{a\mu\nu}$ is locally gauge invariant.

Since A^μ is dimension 1 in mass units $F^{\mu\nu}$ is dim. 2 so F^2 dim 4. There are no other gauge invariant dimension 4 or less terms we can make. (Note: $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ allows $F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$ to be invariant but it is a total space-time divergence as well as being odd under parity.) Thus the most general locally gauge invariant Renormalizable Lagrangian is given by

$$\boxed{\mathcal{L}_{inv} \equiv \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}$$

$$\boxed{\delta_Q(\omega)\mathcal{L}_{inv} = 0}$$

with $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc}A^b_\mu A^c_\nu$

This is called the second order formalism since we have $F^a_{\mu\nu}$ implicitly understood in terms of A^a_μ i.e. A^a_ν is the only independent field. Instead we could incorporate the definition of $F^a_{\mu\nu}$ as a field equation that is treat $F^a_{\mu\nu}$ as another independent field.

The Lagrangian is

$$\boxed{\begin{aligned}\mathcal{L}'_{inv} \equiv {} & \mathcal{L}(\phi, D_\mu \phi) + \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\ & - \tfrac{1}{2} F^a_{\mu\nu} \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - f_{abc} A^{b\mu} A^{c\nu} \right)\end{aligned}}$$

This is called the first order formalism

Note the Euler-Lagrange field equations for $F^a_{\mu\nu}$ $\Rightarrow$

$$\frac{\partial \mathcal{L}'_{inv}}{\partial F^a_{\mu\nu}} = 0 = + F^a_{\mu\nu} - \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc} A^b_\mu A^c_\nu \right)$$

$$\Rightarrow \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{abc} A^b_\mu A^c_\nu$$

Substituting this into $\mathcal{L}'_{inv}$ we find

$$\mathcal{L}'_{inv} = \mathcal{L}(\phi, D_\mu \phi) - \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \mathcal{L}_{inv}$$

The second order formulation.