Hence we must develop a means for determining whether the WI's are broken or not, this will be QAP and Noether Theorem but first let's consider the other 'global' symmetry - that of internal global symmetries

2) Global Internal Symmetries, by definition these symmetries did not involve space-time transformations and were the same at every space-time point.

These transformations are implemented by unitary operators $U(\omega)$ where $\omega^i$ is some space-time independent parameter labeling the transformation under consideration. We will consider the transformations to belong to a compact continuous Lie group $G$. For example the SU(2) group whose multiplication law is that of 2x2 unitary matrices with determinant = 1. The parameters $\omega^i$ labeling the unitary operator $U(\omega)$, which act as a representation of the group element $g(\omega) \in G$, also, as indicated, label the particular member of the abstract group we are considering. The parameters $\omega^a$ are labelled by $a = 1, \ldots, \dim G$.

(i.e. $\mathcal{U}(g(\omega))$ maps $G$ into the space of unitary operators in our $\mathcal{H}$.)

Since $\mathcal{U}(g(\omega))$ is unitary we have that

$$\mathcal{U}(g(\omega)) = e^{-i\omega^a Q^a}$$

where $Q^a$ are the generators of the (representation of the) group, the Charge operators, and they obey the Lie algebra associated with the multiplication law of the group

$$[Q^a, Q^b] = i f_{abc} Q^c$$

where $f_{abc}$ are the structure constants

of the group which characterize the groups composition laws independent of the Representation we are using (and further can be shown to be chosen as totally anti-symmetric). Since the internal symmetry generators $Q^a$ must commute with the Poincaré generators (deep theorem by Coleman & Mandula extended to SUSY by Haag, Lopuszanski, Sohnius)

The particle states of our theory can be labelled by not only mass, spin and momentum but also the eigenvalues of the commuting generators ("diagonal") of $G$. For instance if $G = SU(2)$ then $Q^a$, $a = 1, 2, 3$ obeys

$$[Q^a, Q^b] = i\epsilon_{abc} Q^c$$

and we know from our study of ∢ momentum that $Q^2 = Q_1^2 + Q_2^2 + Q_3^2$ and $Q^3$ form a CSCO for $SU(2)$ and $Q^2$ has eigenvalues $q(q+1)$ $q = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $Q^3$ has eigenvalues $\alpha = \{-q, -q+1, \ldots, q-1, q\}$. Hence we could label the states by $\{q, \alpha\}$ i.e. $|\vec{p}, \text{spin}, \text{mass}; q, \alpha \text{ in}\rangle$. The action of group transformation $\mathcal{U}(g(\omega))$ on the states gives us back a linear combination of the states (suppressing the $\vec{p}$, spin, mass labels) we have

$$\mathcal{U}(g(\omega))|q, \alpha\rangle = U(g(\omega))_{\beta\alpha}|q, \beta\rangle$$

where $U(\omega)_{\beta\alpha}$ is the $(2q+1) \times (2q+1)$ finite dimensional matrix representation of $SU(2)$.

Of course this construction is not peculiar to SU(2) and in general we can label the states of the theory by the eigenvalues of the CSCO chosen for G. Labeling these for a particular representation in sequence by $\alpha = 1, 2, \ldots$, dim of representation $= N$ we have

$$\mathcal{U}(g(\omega))\,|\,\alpha; p\ \text{in}\rangle = U(g(\omega))_{\beta\alpha}\,|\beta; p\ \text{in}\rangle$$

where $p$ is understood to be all other quantum numbers i.e. momentum, mass, spin etc. and $U^{(\omega)}_{\beta\alpha}$ is the $N$ dimensional representation matrix for $g(\omega) \in G$. As for $\mathcal{U}(\omega)$ we have that $U(\omega)_{\beta\alpha}$ can be written as

$$U(\omega)_{\beta\alpha} = \left[e^{-i\omega^a T^a}\right]_{\beta\alpha}$$

where $(T^a)_{\beta\alpha}$ are the matrix generators for the finite dimensional group representation. Note, since $\mathcal{U}^\dagger = \mathcal{U}^{-1} \Rightarrow U^\dagger = U^{-1}$ $\Rightarrow T = T^\dagger$, the $T$ are Hermitian matrices (if in addition $U^* = U^{-1} \Rightarrow U^T = U^{-1}$ orthogonal matrices then $T = T^* = T^T$)

Since the one particle states can be created by $a^{+}_{in\alpha}$ we have that

$$\mathcal{U}(\omega)\, a^{+}_{in\alpha}|0\rangle = U(\omega)_{\beta\alpha}\, a^{+}_{in\beta}|0\rangle$$

$$= \mathcal{U}(\omega)\, a^{+}_{in\alpha}\, \mathcal{U}^{-1}(\omega)\, \mathcal{U}(\omega)|0\rangle$$

but the vacuum is defined as the invariant state $\mathcal{U}(\omega)|0\rangle = |0\rangle$ (phase taken $= 1$)

hence $\mathcal{U}(\omega)\, a^{+}_{in\alpha}\, \mathcal{U}^{-1}(\omega) = U(\omega)_{\beta\alpha}\, a^{+}_{in\beta}$,

Thus we have

$$\mathcal{U}(\omega)\, a_{in\alpha}\, \mathcal{U}^{-1}(\omega) = U^{*}(\omega)_{\beta\alpha}\, a_{in\beta}$$

$$= U^{+}(\omega)_{\alpha\beta}\, a_{in\beta}$$

$$= U^{-1}(\omega)_{\alpha\beta}\, a_{in\beta}.$$

Fourier transforming (recall $\phi(x) \sim \int (\underset{\text{particle}}{a} e^{-ipx} + \underset{\text{anti-partic}}{b^{+}} e^{+ipx})$

we then have that the field transforms as

$$\mathcal{U}(\omega)\,\phi^\alpha_{in}(x)\,\mathcal{U}^{-1}(\omega) = U^{-1}(\omega)_{\alpha\beta}\,\phi^\beta_{in}(x)$$

That is

$$\boxed{\mathcal{U}^{-1}(\omega)\,\phi^\alpha_{in}(x)\,\mathcal{U}(\omega) = U(\omega)_{\alpha\beta}\,\phi^\beta_{in}(x)}$$

i.e. Matrix elements are invariant

$$\langle \chi'_{in}|\phi^\alpha_{in}(x)|\psi'_{in}\rangle = U(\omega)_{\alpha\beta}\langle\chi_{in}|\phi^\beta_{in}(x)|\psi_{in}\rangle$$

$$= \langle\chi_{in}|\mathcal{U}^{-1}(\omega)\,\phi^\alpha_{in}(x)\,\mathcal{U}(\omega)|\psi_{in}\rangle$$

$$\Rightarrow \quad \mathcal{U}^{-1}(\omega)\,\phi^\alpha_{in}(x)\,\mathcal{U}(\omega) = U(\omega)_{\alpha\beta}\,\phi^\beta_{in}(x)$$

---

As usual we can build up these finite transformations from infinitesimal, letting $\omega^a$ be infinitesimal and using the assumption that $\phi^\alpha$ is in the same representation as $\phi^\alpha_{in}$ we have

$$\mathcal{U}^{-1}(\omega)\,\phi^\alpha(x)\,\mathcal{U}(\omega) = \left[e^{+i\omega^a Q^a}\right]\phi^\alpha(x)\left[e^{-i\omega^a Q^a}\right]$$

$$= [1 + i\omega^a Q^a]\,\phi^\alpha(x)\,[1 - i\omega^a Q^a]$$

$$= \phi^\alpha(x) + i\omega^a[Q^a, \phi^\alpha(x)]$$

$$= U(\omega)_{\alpha\beta}\,\phi^\beta(x) = \left(e^{-i\omega^a T^a}\right)_{\alpha\beta}\phi^\beta(x)$$

$$= \phi^\alpha(x) - i\omega^a (T^a)_{\alpha\beta}\,\phi^\beta(x)$$

~~thus~~

$$\boxed{[Q^a, \phi^\alpha(x)] = -(T^a)_{\alpha\beta}\,\phi^\beta(x)}$$

$$\equiv -i(-iT^a)_{\alpha\beta}\,\phi^\beta(x)$$

$$\equiv -i\,\delta_Q^a\,\phi^\alpha(x)$$

Now consider the double commutator

$$[[Q^a, Q^b], \phi^\alpha(x)] = [Q^a, [Q^b, \phi^\alpha]] - [Q^b, [Q^a, \phi^\alpha]]$$

$$= -(T^b)_{\alpha\beta}[Q^a, \phi^\beta] + (T^a)_{\alpha\beta}[Q^b, \phi^\beta]$$

$$= (T^b)_{\alpha\beta}(T^a)_{\beta\gamma}\phi^\gamma - (T^a)_{\alpha\beta}(T^b)_{\beta\gamma}\phi^\gamma$$

$$= -[T^a, T^b]_{\alpha\beta}\,\phi^\beta$$

$$= i f_{abc}(-T^c)_{\alpha\beta}\,\phi^\beta$$

$$\Rightarrow \boxed{[T^a, T^b] = i f_{abc} T^c}$$

The $T^a$ indeed represent the Lie algebra. Since $\delta^a_Q = -iT^a$ we have

$$[\delta^a_Q, \delta^b_Q] = +f_{abc}\,\delta^c_Q .$$

---

As with the Poincaré symmetries if the dynamics of our theory is invariant under the internal symmetry transformations the S operator commutes with the unitary symmetry operators

$$S = \mathcal{U}_{(g(\omega))}\, S\, \mathcal{U}^{-1}_{(g(\omega))}$$

Using the LSZ formula for S we find this implies the global gauge invariance of our Green functions and visa versa.

$$\langle 0|T\phi^{\alpha_1}_{(x_1)}\cdots\phi^{\alpha_n}_{(x_n)}|0\rangle$$

$$= \langle 0|T\,U_{\alpha_1\beta_1}(\omega)\phi^{\beta_1}_{(x_1)}\cdots U_{\alpha_n\beta_n}\phi^{\beta_n}_{(x_n)}|0\rangle .$$

Since we can build up any finite transformation from infinitesimal variations consider $\omega^a$ infinitesimal

$$U_{\alpha\beta}(\omega) = \delta_{\alpha\beta} - i\omega^a (T^a)_{\alpha\beta}$$

(Recall: $S \sim \langle 0|T\phi^\alpha\cdots|0\rangle :\phi^\alpha_{in}\cdots:$ so that $S^{\mathcal{U}} \sim \langle 0|T\phi^\alpha\cdots|0\rangle :\phi^\beta \mathcal{U}_{\beta\alpha}^{-1\dagger}\cdots: \sim \langle 0|T\phi^\alpha\cdots|0\rangle :\phi^\beta U_{\beta\alpha}\cdots: \sim \langle 0|T\,U_{\alpha\beta}\phi^\beta\cdots|0\rangle :\phi^\alpha\cdots:$)

$$\langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n)|0\rangle$$

$$= \langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n)|0\rangle$$

$$+\sum_{i=1}^{n}\langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\left(-i\omega^a T^a\right)_{\alpha_i\beta_i}\phi^{\beta_i}(x_i)\cdots\phi^{\alpha_n}(x_n)|0\rangle$$

Since $\omega^a$ is arbitrary we obtain the (dim $G$)-global gauge group $G$ Ward Identities

$$0=\sum_{i=1}^{n}\langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\delta_Q^a\phi^{\alpha_i}(x_i)\cdots\phi^{\alpha_n}(x_n)|0\rangle$$

$$=\sum_{i=1}^{n}\langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\left(-iT^a\right)_{\alpha_i\beta_i}\phi^{\beta_i}(x_i)\cdots\phi^{\alpha_n}(x_n)|0\rangle .$$

Multiplying by sources and summing we obtain

$$0=\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{i^n}{n!}\int dx_1\cdots dx_n\,J_{\alpha_1}(x_1)\cdots J_{\alpha_n}(x_n)$$
$$\langle 0|T\,\phi^{\alpha_1}(x_1)\cdots\delta_Q^a\phi^{\alpha_i}(x_i)\cdots\phi^{\alpha_n}(x_n)|0\rangle$$

$$=\sum_{n=1}^{\infty}\frac{i^n}{(n-1)!}\int dx\,J_\alpha(x)\int dx_1\cdots dx_{n-1}\,J_{\alpha_1}(x_1)\cdots J_{\alpha_{n-1}}(x_{n-1})$$
$$\langle 0|T\,\delta_Q^a\phi^{\alpha}(x)\,\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_{n-1}}(x_{n-1})|0\rangle$$

$$0 = i\int dx\, J_\alpha(x)\, \delta_Q^a \frac{\delta}{i\delta J_\alpha(x)}\, Z[J]$$

$$= i\int dx\, J_\alpha(x) (-iT^a)_{\alpha\beta} \frac{\delta}{i\delta J_\beta(x)}\, Z[J]$$

Defining the global G gauge Ward Identity functional differential operator as

$$\hat{\delta}_Q^a \equiv \int dx\, J_\alpha(x)\, \delta_Q^a \frac{\delta}{\delta J_\alpha(x)}$$
$$= \int dx\, J_\alpha(x) (-iT^a)_{\alpha\beta} \frac{\delta}{\delta J_\beta(x)}$$

The group invariance of the Green functions and hence S-matrix is expressed as the G-WI's

$$\hat{\delta}_Q^a Z[J] = 0 .$$

~~Also~~ since $[\hat{\delta}_Q^a, \hat{\delta}_Q^b] = f_{abc} \hat{\delta}_Q^c$ (functional Diff Op),
the WI operators $\hat{\delta}_Q^a$ represent the Lie algebra on functionals.

So far so good, but how do we know if the Green functions obey a WI or not? We need a tool to determine whether the generating functional satisfies the WI functional differential equation. The tool will be the Schwinger action principle. In order to derive the action principle we must know something about the dynamics of the theory — that is suppose we are given the Lagrangian $\mathcal{L}$ describing the dynamics of fields $\varphi^\alpha$. Then we can represent the generating functional by means of our functional integral. Since we are talking about global symmetries so far there is no confusion in how to define the integral

$$Z[J] = \frac{\int [d\varphi^\alpha] \cdot e^{i\int dx [\mathcal{L}(\varphi^\alpha) + J_\alpha \varphi^\alpha]}}{\int [d\varphi^\alpha]\, e^{i\int dx \mathcal{L}(\varphi^\alpha)}}$$

We would like to check if $Z[J]$ obeys the global G WI, to see if this is the case we make a change of integration variables that is just the form of our gauge transformati—

Let

$$\varphi'^{\alpha} \equiv U(\omega)_{\alpha\beta}\,\varphi^{\beta}$$

Since $U(\omega)$ is unitary

$$[d\varphi'^{\alpha}] = (\det U)[d\varphi^{\alpha}] = [d\varphi^{\alpha}]$$

For infinitesimal $\omega$ we have

$$\varphi'^{\alpha} = \varphi^{\alpha} + \omega^{a}\delta_{Q}^{a}\varphi^{\alpha} = \varphi^{\alpha} - i\omega^{a}(T^{a})_{\alpha\beta}\varphi^{\beta}$$

Hence, defining the variation of the Lagrangian as

$$\omega^{a}\delta_{Q}^{a}\mathcal{L} \equiv \mathcal{L}(\varphi + \omega^{a}\delta_{Q}^{a}\varphi) - \mathcal{L}(\varphi)$$

we have

$$Z[J] = \frac{\int [d\varphi'^{\alpha}]\, e^{i\int dx\,[\mathcal{L}(\varphi'^{\alpha}) + J_{\alpha}\varphi'^{\alpha}]}}{\int [d\varphi^{\alpha}]\, e^{i\int dx\,\mathcal{L}(\varphi)}}$$

$$= \frac{\int [d\varphi^\alpha]\, e^{i\int dx \left[\mathcal{L}(\varphi^\alpha) + \overset{\omega^a}{\vee}\delta^a_Q \mathcal{L}(\varphi^\alpha) + J_\alpha \varphi^\alpha + J_\alpha \omega^a \delta^a_Q \varphi\right.}}{\int [d\varphi^\alpha]\, e^{i\int dx\, \mathcal{L}(\varphi^\alpha)}}$$

$$= \frac{\int [d\varphi^\alpha]\, e^{i\int dx [\mathcal{L}(\varphi^\alpha) + J_\alpha \varphi^\alpha]}\; e^{i\int dx [\omega^a \delta^a_Q \mathcal{L}(\varphi^\alpha) + J_\alpha \omega^a \delta^a_Q}}{\int [d\varphi^\alpha]\, e^{i\int dx\, \mathcal{L}(\varphi^\alpha)}}$$

$$= \frac{\int [d\varphi^\alpha] \left(1 + i\omega^a \int dx \left(\delta^a_Q \mathcal{L}(\varphi^\alpha) + J_\alpha \delta^a_Q \varphi^\alpha\right)\right) \times e^{i\int dx [\mathcal{L}(\varphi^\alpha) + J_\alpha \varphi^\alpha]}}{\int [d\varphi^\alpha]\, e^{i\int dx\, \mathcal{L}(\varphi^\alpha)}}$$

$$= Z[J] + \frac{\int [d\varphi^\alpha]\, i\omega^a \int dx \left(\delta^a_Q \mathcal{L} + J_\alpha \delta^a_Q \varphi^\alpha\right) e^{i\int dx (\mathcal{L} + J_\alpha \varphi^\alpha}}{\int [d\varphi^\alpha]\, e^{i\int dx\, \mathcal{L}(\varphi^\alpha)}}$$

$$= Z[J],$$

Then we obtain the relation

$$-i\omega^a \int dx\, J_\alpha(x)\, \delta_Q^a \frac{\delta}{i\delta J_\alpha(x)} Z[J]$$

$$= i\omega^a \int dx\, \delta_Q^a \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right] Z[J]$$

Since $\omega^a$ is arbitrary we have

$$\boxed{\int dx\, J_\alpha(x)\, \delta_Q^a \frac{\delta}{\delta J_\alpha(x)} Z[J] = -i \int dx\, \delta_Q^a \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right] Z[J]}$$

Or in terms of the WI functional differential operators

$$\boxed{\delta_Q^a Z[J] = -i \int dx \left(\delta_Q^a \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right]\right) Z[J]}$$

This is the Schwinger Quantum Action Principle.

Thus we obtain the variation of the generative functional in terms of the variation of the action inserted in the generating functional. Hence the Green functions are G-invariant if the Lagrangian is G-invariant.

Since $\omega^a \delta^a_Q \mathcal{L}(\varphi) = \mathcal{L}(\varphi + \omega^a \delta^a_Q \varphi) - \mathcal{L}(\varphi)$

we can simply check if the Lagrangian is gauge invariant by replacing each field with the gauge transformed field and see if $\mathcal{L}$ stays the same.

If we have an invariance of $\mathcal{L}$ we recall that Noether's theorem told us that there was a conserved current associated with the symmetry. If the symmetry is broken then the current is not conserved. Hence the Noether current conservation equation should be another (equivalent) way to probe the invariances (or lack of invariances) of a theory.

Since the action principle tells us it is the variation of the Lagrangian that controls the transformation properties of the Green functions let's investigate it more closely — remember we are now dealing with classical quantities — functions of the functional integration variables!

$$\delta^a_Q \mathcal{L}(\varphi^\alpha) = \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} \delta^a_Q \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \delta^a_Q \partial_\mu \varphi^\alpha$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} \delta^a_Q \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \partial_\mu \delta^a_Q \varphi^\alpha$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} \delta^a_Q \varphi^\alpha + \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \delta^a_Q \varphi^\alpha \right]$$

$$- \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \right) \delta^a_Q \varphi^\alpha$$

So as usual we obtain Noether's Identity

$$\boxed{\delta^a_Q \mathcal{L}(\varphi^\alpha) = \partial_\mu J^\mu_a + \left( \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} \right) \delta^a_Q \varphi^\alpha}$$

as an algebraic Identity.

where

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \delta^a_Q \varphi^\alpha$$ is the

Noether current

and $$\frac{\partial^E \mathcal{L}}{\partial \varphi^\alpha} = \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha}$$ is

the Euler-Lagrange derivative.

We can then integrate this expression to obtain

$$\delta^a_Q \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right](x) Z[J]$$

$$= \partial^x_\mu J^\mu_a\left[\frac{\delta}{i\delta J_\alpha}\right](x) + \left(\frac{\partial^E \mathcal{L}}{\partial \varphi^\alpha}\right)\left[\frac{\delta}{i\delta J^\alpha}\right]\left(\delta^a_Q \frac{\delta}{i\delta J_\alpha}\right) Z[J]$$

But we recall that the Euler-Lagrange equations are just the equations of motion for $Z[J]$ — this use of dynamics converts the algebraic identity into a theorem

$$\frac{\partial^{\varepsilon}\mathcal{L}}{\partial\varphi^{\alpha}}\left[\frac{\delta}{i\delta J^{\alpha}}\right](x)\, Z\{J\} = - J_{\alpha}(x)\, Z\{J\}$$

Thus we obtain the statement of Noether's Theorem for the Generating Functional

$$\boxed{\begin{aligned}&\partial_{\mu} J^{\mu}_{a}\left[\frac{\delta}{i\delta J_{\alpha}}\right](x)\, Z\{J\}\\ &= -iJ_{\alpha}(x)\,\delta^{a}_{Q}\frac{\delta}{\delta J_{\alpha}(x)}\, Z\{J\}\\ &\qquad + \delta^{a}_{Q}\mathcal{L}\left[\frac{\delta}{i\delta J_{\alpha}}\right](x)\, Z\{J\}\end{aligned}}$$

We first remark that if the Lagrangian is invariant then if $\delta^a_Q \mathcal{L} = 0$ then

$$\partial_\mu J^\mu_a \, Z[J] = -i \, \delta^a_Q(x) \, Z[J]$$

where $\delta^a_Q = \int dx \, \delta^a_Q(x)$

$$\delta^a_Q(x) = J_\alpha(x) \, \delta^a_Q \frac{\delta}{\delta J_\alpha(x)}$$

The divergence of the current is the <u>local</u> gauge variation of the Green functions. Further the <u>local</u> WI operators $\delta^a_Q(x)$ represent the <u>current operator</u> algebra

$$[\delta^a_Q(x), \delta^b_Q(y)] = \delta(x-y) \, f_{abc} \, \delta^c_Q(x).$$

Secondly we can integrate Noether's theorem over $\int d^4x$.

$$\int dx\, \partial_\mu J^\mu_a(x) Z[J] = -i\int dx\, \delta^a_Q(x)\, Z[J]$$

$$+\int dx\, \delta^a_Q \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right](x)\, Z[J]$$

Throwing away surface terms the LHS vanishes (hence) we recover the Schwinger Quantum Action Principle from Noether's Theorem

$$\boxed{\delta^a_Q Z[J] = -i\int dx\, \delta^a_Q \mathcal{L}\left[\frac{\delta}{i\delta J_\alpha}\right](x)\, Z[J]}$$

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Finally, although we have been considering the action principle & Noether's theorem for global internal symmetries we can obtain their validity for global space-time symmetries as well.

Consider for example the variation of the Lagrangian under space-time translations

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x) \equiv -i\delta^\mu_P\phi(x)$$

hence we consider the variation of $\mathcal{L}(\varphi)$

$$\delta^\mu_P\mathcal{L} = \partial_\nu\left[\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}\delta^\mu_P\varphi\right] + \left(\frac{\partial^\varepsilon\mathcal{L}}{\partial\varphi}\delta^\mu_P\varphi\right)$$

Since we are always interested in translationally invariant theories we manifestly construct $\mathcal{L}$ so that it is a scalar thus its intrinsic variation is

$$a_\mu\delta^\mu_P\mathcal{L} = \mathcal{L}(\varphi + a_\mu\partial^\mu\varphi) - \mathcal{L}(\varphi)$$

$$\equiv a_\mu\partial^\mu\mathcal{L}$$

Hence we have the identity

$$\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}\partial^\mu\varphi - g^{\mu\nu}\mathcal{L}\right] = \left(\frac{\partial^\varepsilon\mathcal{L}}{\partial\varphi}\delta^\mu_P\varphi\right)$$

Defining the Energy-Momentum Tensor as

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

we have the identity

$$\boxed{\partial_\mu T^{\mu\nu} = \frac{\partial^E \mathcal{L}}{\partial \varphi} \delta^\nu_P \varphi}$$

Inserting this in $Z[J]$ we obtain

$$\partial_\mu T^{\mu\nu}\left[\frac{\delta}{i\delta J}\right](x)\, Z[J] = \frac{\partial^E \mathcal{L}}{\partial \varphi}\left[\frac{\delta}{i\delta J}\right](x)\, \delta^\nu_P \frac{\delta}{i\delta J(x)} Z[J]$$

Using the Euler-Lagrange equations of motion we find

$$\boxed{\partial_\mu T^{\mu\nu}\left[\frac{\delta}{i\delta J}\right](x)\, Z[J] = +i\, J(x)\, \delta^\nu_P \frac{\delta}{\delta J(x)} Z[J]}$$

(Note if $\delta_P^\mu \mathcal{L} = \partial^\mu \mathcal{L} + \Delta^\mu \mathcal{L}$ i.e. $\mathcal{L}$ not a scalar, then we would have breaking terms here also)

Integrating over $x$ and throwing away surface terms we obtain the Translation invariance WI

$$\delta_P^\nu Z[J] = 0 .$$

Similarly one can consider Lorentz transformations to obtain the X momentum tensor conservation eq.

$$\partial_\mu M^{\mu\nu\rho}\left[\frac{\delta}{i\delta J}\right](x) Z[J] = iJ(x)\delta_M^{\nu\rho} \frac{\delta}{\delta J(x)} Z[J]$$

and upon integration the Lorentz WI

$$\delta_M^{\mu\nu} Z[J] = 0 .$$