In order not to be too simple let's make the boson a pseudo scalar field so that the Lagrangian describing the dynamics is

$$\mathcal{L} = \frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{(m^2+a)}{2}\phi^2 - \frac{(\lambda+c)}{4!}\phi^4$$
$$+ \frac{iZ_2}{2}\bar{\Psi}\overleftrightarrow{\not{\partial}}\Psi - (M+d)\bar{\Psi}\Psi - (g+f)\bar{\Psi}\gamma_5\Psi\phi$$

Hence we have 3 field equations given by the Euler-Lagrange equations of motion

1) $$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0$$

$$= -(Z_1\partial_x^2 + (m^2+a))\phi(x) - \frac{(\lambda+c)}{3!}\phi^3(x) - (g+f)\bar{\Psi}\gamma_5\Psi(x)$$

$$= 0.$$

2) $$\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\Psi}} = 0$$

$$= (Z_2 i\not{\partial}_x - (M+d))\Psi(x) - (g+f)\gamma_5\Psi(x)\phi(x)$$

$$= 0$$

and the conjugate fermion field equation

3) $$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} = 0$$

$$= \left(i Z_2 \bar{\Psi} \overleftarrow{\not\partial}_x + (M + d) \bar{\Psi}(x) \right) + (g+f) \bar{\Psi}(x) \gamma_5 \phi(x)$$

$$= 0 .$$

The field equations plus canonical (anti-) commutation relations imply equations of motion for the Green functions.

First the canonical momenta are

$$\Pi_B \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = Z_1 \dot{\phi}(x)$$

$$\bar{\Pi}_F \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = -i Z_2 \gamma^0 \bar{\Psi}(x)$$

Hence the ETCR & ETAR are

$$\delta(x^0 - y^0) \left[\Pi_B(x), \phi(y) \right] = -i \delta^4(x-y)$$

$$\delta(x^0 - y^0) \left\{ \bar{\Pi}_F(x), \bar{\Psi}(y) \right\} = -i \delta^4(x-y)$$

Hence we have

$$\delta(x^0-y^0)\left[\dot\phi(x),\phi(y)\right] = -\frac{i}{Z_1}\delta^4(x-y)$$

$$\delta(x^0-y^0)\left\{\Psi(x),\bar\Psi(y)\right\} = \frac{1}{Z_2}\gamma^0\delta^4(x-y)$$

$$\left(\text{or } \delta(x^0-y^0)\left\{\Psi(x),\Psi^\dagger(y)\right\} = \frac{1}{Z_2}\delta^4(x-y)\right).$$

So the Green function equations of motion have the form

1)
$$-\left(Z_1\partial_x^2+(m^2+a)\right)\langle 0|T\,\phi(x)\,\phi(x_1)\cdots\bar\Psi(z_n)|0\rangle$$
$$-\frac{(\lambda+c)}{3!}\langle 0|T\,\phi^3(x)\,\phi(x_1)\cdots\bar\Psi(z_n)|0\rangle$$
$$-(g+f)\langle 0|T\,\bar\Psi(x)\gamma_5\Psi(x)\,\phi(x_1)\cdots\bar\Psi(z_n)|0\rangle$$
$$=\sum_{i=1}^{l}+i\delta^4(x-x_i)\langle 0|T\,\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\bar\Psi(z_n)|0\rangle$$

Since for T-products we had

$$\partial_x^2 T\,\phi(x)X = T\,\partial_x^2\phi(x)X + \underbrace{\sum_{i=1}^{l}\delta(x^0-x_i^0)\left[\dot\phi(x),\phi(x_i)\right]TX_{\hat i}}_{}$$
$$= -\frac{i}{Z_1}\sum_{i=1}^{l}\delta(x-x_i)\,TX_{\hat i}$$

2)

$$\{Z_2 i \not\partial_x - (M+d)\}\langle 0|T\,\Psi(x)\,\phi(x_1)\cdots\bar\Psi(z_n)|0\rangle$$

$$-(g+f)\langle 0|T\,\gamma_5\Psi(x)\phi(x)\,\phi(x_1)\cdots\bar\Psi(z_n)|0\rangle$$

$$=\sum_{i=1}^{n} + i\,\delta^4(x-z_i)\underbrace{\langle 0|T\,\phi(x_1)\cdots\Psi(z_1)\cdots\cancel{\bar\Psi(z_i)}\cdots\bar\Psi(z_n)|0\rangle}_{(-1)^{m+(i-1)}}$$

Since for T-products

$$i\not\partial_x T\,\Psi(x)\,X = T\,i\not\partial_x\,\Psi(x)\,X$$

$$\underbrace{+\sum_{i=1}^{n}\delta(x^0-z_i^0)\,i\gamma^0\,\{\Psi(x),\bar\Psi(z_i)\}\,\overbrace{T X_{\hat i}}^{(-1)^{m+(i-1)}}}$$

$$=\sum_{i=1}^{n}\frac{i}{Z_2}\delta^4(x-z_i)\,T\,X_{\hat i}\,(-1)^{m+(i-1)}$$

The $(-1)^{m+(i-1)}$ results in anti-commuting the $\Psi(x)$ all the way to the $\Psi(z_i)$ position — thus $(m+(i-1))$ interchanges with fermion fields were required.

3)

$$\langle 0|T\,\bar{\Psi}(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle\left[iZ_2\overleftarrow{\not\partial}_x+(M+d)\right]$$

$$+\,(g+f)\langle 0|T\,\bar{\Psi}(x)\gamma_5\,\phi(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$=\sum_{i=1}^{m}(-1)^{(i-1)}\,i\delta^4(x-y_i)\langle 0|T\phi(x_1)\cdots\Psi(y_1)\cdots\cancel{\Psi(y_i)}\cdots\bar{\Psi}(z_n)|0\rangle$$

Since for T-products

$$\left(T\,\bar{\Psi}(x)X\right)i\overleftarrow{\not\partial}_x=T\,\bar{\Psi}(x)\,i\overleftarrow{\not\partial}_x X$$

$$+\underbrace{\sum_{i=1}^{m}\delta(x^0-y_i^0)\,i\gamma^0(-1)^{(i-1)}\{\bar{\Psi}(x),\Psi(y_i)\}\;T X_{\hat{i}}}$$

$$=\sum_{i=1}^{m}\frac{i}{Z_2}(-1)^{(i-1)}\delta^4(x-y_i)\;T X_{\hat{i}}$$

Of course it is simpler to re-express these field equations in terms of functional derivatives on the generating functional $Z\{J,\eta,\bar{\eta}\}$. As in the purely scalar case we can multiply the above equations of motion by sources and sum to find

1) $$-\left(Z_1\partial_x^2+(m^2+a)\right)\frac{\delta}{i\delta J(x)}Z\{J,\eta,\bar{\eta}\}$$
$$-\frac{(\lambda+c)}{3!}\frac{\delta^3}{(i\delta J(x))^3}Z\{J,\eta,\bar{\eta}\}$$
$$+(g+f)\frac{\delta}{(-i\delta\eta(x))}\gamma_5\frac{\delta}{i\delta\bar{\eta}(x)}Z\{J,\eta,\bar{\eta}\}$$
$$=-J(x)Z\{J,\eta,\bar{\eta}\}$$

---

2) $$\left[Z_2 i\not{\partial}_x-(M+d)\right]\frac{\delta}{i\delta\bar{\eta}(x)}Z[J,\eta,\bar{\eta}]$$
$$-(g+f)\gamma_5\frac{\delta}{i\delta\bar{\eta}(x)}\frac{\delta}{i\delta J(x)}Z\{J,\eta,\bar{\eta}\}$$
$$=-\eta(x)Z\{J,\eta,\bar{\eta}\}$$

3)
$$\left[Z_2 i \frac{\delta}{-i\delta\eta(x)} \overleftarrow{\not{\partial}}_x + (M+d)\frac{\delta}{-i\delta\eta(x)}\right] Z[J,\eta,\bar{\eta}]$$
$$+(g+f)\frac{\delta}{-i\delta\eta(x)}\gamma_5\frac{\delta}{i\delta J(x)} Z[J,\eta,\bar{\eta}]$$
$$= +\bar{\eta}(x)\, Z[J,\eta,\bar{\eta}]$$

Hence we now have converted the dynamical information contained in our theory to these functional differential equations. As we have seen from the perturbative point of view we have indirectly solved these equations by solving the operator dynamics directly in terms of our expression for the time evolution operator. The Green functions were then given in terms of the Gell-Mann-Low formula

$$\langle 0|T\phi(x_1)\cdots\phi(x_\ell)\Psi(y_1)\cdots\Psi(y_m)\bar{\Psi}(z_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$= \frac{\langle 0|T\phi_{in}(x_1)\cdots\bar{\Psi}_{in}(z_n)\, e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}{\langle 0|T\, e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}$$

where the "non-interacting" in-field dynamics is described by the free Lagrangian

$$\mathcal{L}_0^{in} = \frac{1}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \frac{1}{2}m^2\phi_{in}^2 + \frac{i}{2}\bar{\Psi}_{in}\overleftrightarrow{\not{\partial}}\Psi_{in} - M\bar{\Psi}_{in}\Psi$$

and so the interaction Lagrangian is given by

$$\mathcal{L}_I^{in} = \mathcal{L}^{in} - \mathcal{L}_0^{in}$$

$$= \frac{b_1}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \frac{a}{2}\phi_{in}^2 - \frac{(\lambda+c)}{4!}\phi_{in}^4$$

$$+ \frac{b_2}{2}\bar{\Psi}_{in}i\overleftrightarrow{\not{\partial}}\Psi_{in} - d\bar{\Psi}_{in}\Psi_{in} - (g+f)\bar{\Psi}_{in}\gamma_5\Psi_{in}\phi_{in}$$

The Feynman rules used to calculate

$\langle 0| T\, \phi(x_1) \cdots \phi(x_l)\, \Psi(y_1) \cdots \Psi(y_m)\, \overline{\Psi}(z_1) \cdots \overline{\Psi}(z_n) |0\rangle$

are given by including a F.T. factor

1) $\int \frac{d^4 p_i}{(2\pi)^4} e^{-ip_i x_i}$ for each field $\phi(x_i)$

$\int \frac{d^4 q_j}{(2\pi)^4} e^{-iq_j y_j}$ for each field $\Psi(y_j)$

$\int \frac{d^4 \overline{q}_k}{(2\pi)^4} e^{+i\overline{q}_k z_k}$ for each field $\overline{\Psi}(z_k)$

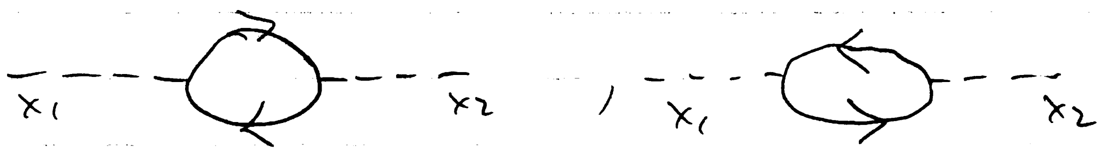
2) Draw all topologically distinct (*) graphs with l - ϕ lines with momentum p_i going out of graph, m - Ψ lines directed out of graph with momentum q_j flowing out of graph and n - $\overline{\Psi}$ lines directed into graph with momentum $\overline{q}_k$ flowing into graph and with vertices described below.

3) Include a factor of δ_{mn} and an energy momentum conserving delta function for each connected subgraph $(2\pi)^4 \delta^4(\overline{q} \cdots - q \cdots - p \cdots$

4) Label the momentum routing through the graph with E-M conservation at each vertex. First let external momentum flow through the lines, then pick an internal loop momentum for each independent loop in the graph, each lines internal momentum is the sum of the loop momenta flowing through the line.

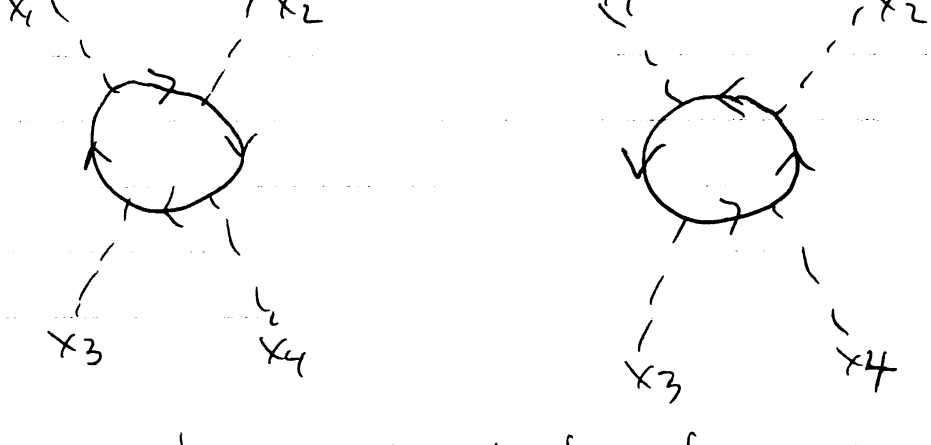
(*) Note on topologically distinct:

1)



are not topologically distinct, they only contribute once ~~not~~ twice to Feynman diagram expansion

2)



These are topologically distinct, since

they come from two distinct sets of contractions in the Gell-Mann-Low expansion via Wick's Theorem — they both contribute to our Feynman diagram expansion.

The time ordered function is then given by

$$\langle 0 | T\, \phi(x_1) \cdots \bar{\Psi}(z_n) | 0 \rangle$$

$$= \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 \bar{q}_n}{(2\pi)^4}\, e^{-ip_i x_i - iq_j y_j + i\bar{q}_k z_k}$$

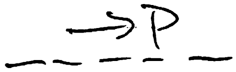
$$\times \sum_{\Gamma \in G_I^{(\ell,m,n)}}' \alpha(\Gamma)\, (2\pi)^4 \delta^4(\bar{q}_{11} + \cdots - q_{11} - \cdots - p_{11} \cdots)\, \delta_{m_1 n_1}$$

$$\cdots (2\pi)^4 \delta^4(\bar{q}_{A1} + \cdots - q_{A1} \cdots - p_{A1} \cdots)\, \delta_{m_A n_A} \times$$

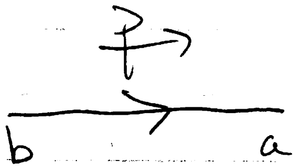
$$\times \int \frac{d^4 k_1}{(2\pi)^4} \cdots \frac{d^4 k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p, q, \bar{q}, k)$$

where the Feynman integrand I_Γ for graph Γ is made according to the following correspondences:

1) Each line corresponds to the Factor

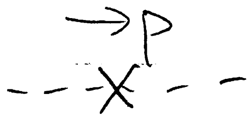
a) 

$$\frac{i}{p^2-m^2+i\epsilon}$$

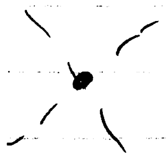
b) 

$$\left(\frac{i}{\not p-m+i\epsilon}\right)_{ab}$$

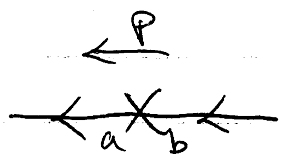
2) The lines are joined by the vertices which correspond to the factor $i I_r$

a) 

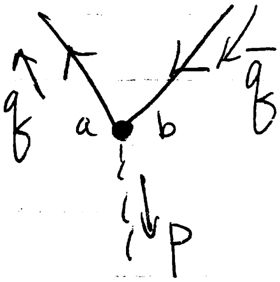
$i(b_1 p^2 - a)$

b) 

$-i(\lambda + c)$

c) 

$i(\not p_{ab} b_2 - d\,\delta_{ab})$

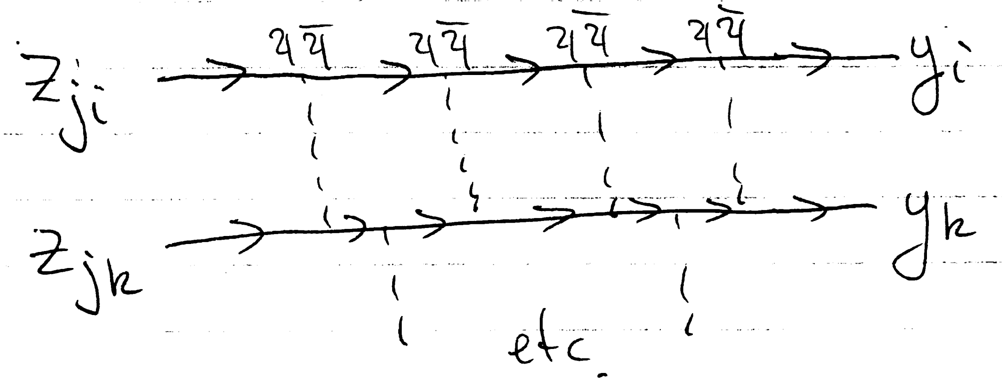
d) 

$-i(g+f)(\gamma_5)_{ab}$

3) A factor of (-1) is included for each closed Fermion loop

4) An overall $(-1)^P$ factor for the signature of the permutation that maps the $(y_1 \ldots y_m z_1 \ldots z_n)$ into the open fermion lines of the graph

always $(\psi\bar{\psi})^n$ in between z_{ji} and y_i



So we have

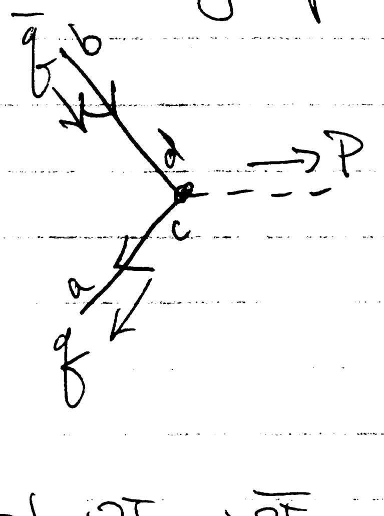
$$(y_1, \ldots y_m, z_1, \ldots z_n) \xrightarrow{P} (y_1, z_{j1}; y_2, z_{j2}; \ldots ; y_m, z_{jm})$$

if the number of interchanges required by P is odd then $(-1)^P = -1$; if even then $(-1)^P = +1$.

5) Of course $\alpha(\Gamma)$ is the usual symmetry number for graph Γ calculated by going from G-M-L expansion and Wick's theorem to graph. Since we exclude vacuum bubbles in $G^{(\ell,m,n)}$ the $\alpha(\Gamma)$ is the same as the ϕ^4 model case.

Examples

1) $\langle 0|T\,\phi(x)\,\Psi_a(y)\,\bar{\Psi}_b(z)|0\rangle$ lowest order contribution has one graph



So

$$\langle 0|T\,\phi(x)\,\Psi_a(y)\,\bar{\Psi}_b(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4\bar{q}}{(2\pi)^4}\, e^{-ipx-iqy+i\bar{q}}$$

$$\times (2\pi)^4\delta^4(\bar{q}-p-q)\left(\frac{i}{p^2-m^2+i\epsilon}\right)\left(\frac{i}{\not{q}-M+i\epsilon}\right)_{ac}\left[-i(g+f)\gamma_5\right]_{cd} \times$$

$$\times\left(\frac{i}{\not{\bar{q}}-M+i\epsilon}\right)_{db}$$

2) Pseudo-Scalar self energy in second order has a contribution given below

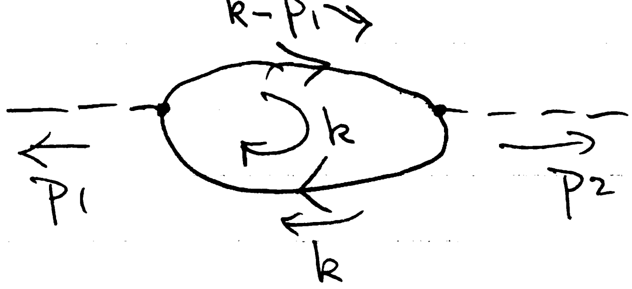
$$\langle 0|T\,\phi(x_1)\,\phi(x_2)|0\rangle$$

$$= \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-ip_1x_1-ip_2x_2}(2\pi)^4\delta^4(p_1+p_2)$$

$$\times\left(\frac{i}{p_1^2-m^2+i\epsilon}\right)\left(\frac{i}{p_2^2-m^2+i\epsilon}\right)(-1)\int\frac{d^4k}{(2\pi)^4}\times\left(-i(g+f)\right)^2$$

$$\times \mathrm{Tr}\left[\gamma_5\frac{i}{\not{k}-\not{p}_1-M}\gamma_5\frac{i}{\not{k}-M}\right]$$

closed Fermion loop!

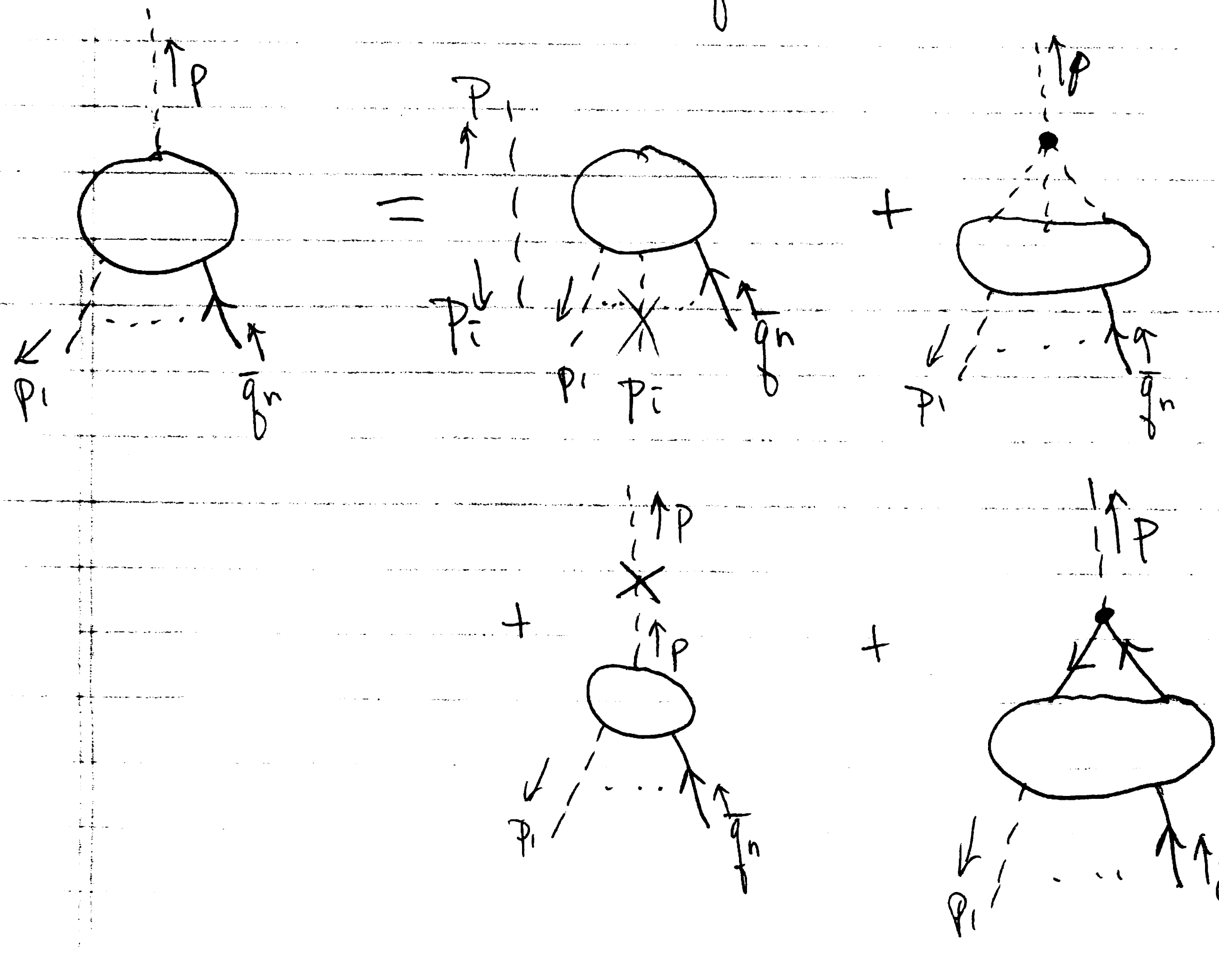


Using the perturbation expansion in terms of Feynman diagrams we can now explicitly check that the Green functions' equations of motion are satisfied.

First consider the graphical alternatives for the Green function

$\langle 0|T\,\phi(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$ focusing

on the $\phi(x)$ line's options



Hence the general Feynman integrand consists of a sum of 4 terms

$$I_r = \frac{i}{p^2-m^2} I_r^{\hat{i}} + \frac{-i(\lambda+c)}{3!} \frac{i}{p^2-m^2} I_r^{\phi^3}$$

$$+(-ia+ibp^2)\frac{i}{p^2-m^2} I_r$$

$$-i(g+f)\frac{i}{p^2-m^2} I_r^{\bar{\Psi}\gamma_5\Psi}$$

Thus if we consider $(\partial_x^2+m^2)$ acting on $\langle 0|T\phi(x)\phi(x_1)\ldots\bar{\Psi}(z_n)|0\rangle$ the propagators will cancel with the $(-p^2+m^2)$ factor leaving a $(-i)$. Hence in coordinate space this corresponds to the following Green functions

$$(\partial_x^2+m^2)\langle 0|T\phi(x)\phi(x_1)\ldots\bar{\Psi}(z_n)|0\rangle$$

$$= -i\sum_{i=1}^{l}\delta^4(x-x_i)\langle 0|T\phi(x_1)\ldots\cancel{\phi(x_i)}\ldots\bar{\Psi}(z_n)|0\rangle$$

$$-\frac{(\lambda+c)}{3!}\langle 0|T\phi^3(x)\phi(x_1)\ldots\bar{\Psi}(z_n)|0\rangle$$

$$-a\langle 0|T\phi(x)\phi(x_1)\ldots\bar{\Psi}(z_n)|0\rangle$$

$$-b\partial_x^2 \langle 0|T\phi(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$-(g+f)\langle 0|T\bar{\Psi}(x)\gamma_5\Psi(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

Combining similar terms and recalling that $1+b = Z_1$ we find the first field equation

1) $$[Z_1\partial_x^2+(m^2+a)]\langle 0|T\phi(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$+\frac{(\lambda+c)}{3!}\langle 0|T\phi^3(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

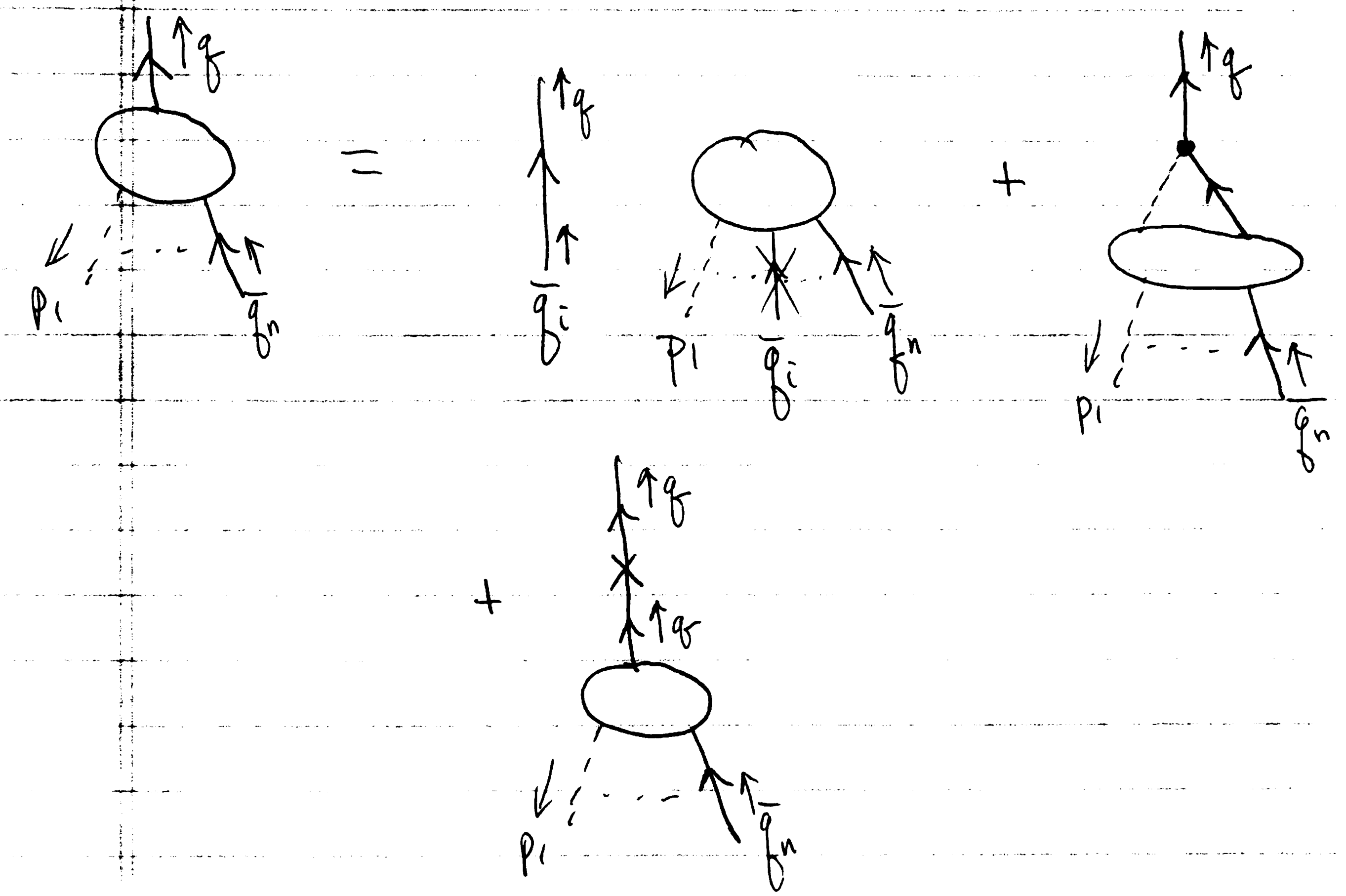
$$+(g+f)\langle 0|T\bar{\Psi}(x)\gamma_5\Psi(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$= -i\sum_{i=1}^{\ell} \delta^4(x-x_i)\langle 0|T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\bar{\Psi}(z_n)|0\rangle$$

as found on p. -334-.

Next consider the graphical alternatives for the Green function

$\langle 0|T\Psi(x)\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$ focusing on the $\Psi(x)$ line's options



Thus the general Feynman integrand has 3 contributing classes of graphs

$$I_\Gamma = \frac{i}{\not q - M} I_\Gamma^{\hat{i}} (-1)^{m+(i-1)} + \frac{i}{\not q - M}(-i(g+f)) I_\Gamma^{(\gamma_5 \tau \phi)}$$

$$+ \frac{i}{\not q - M}(i\not q b_2 - id) I_\Gamma$$

Hence multiplying $\langle 0|T\,\psi(x)\,\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle$ by $(i\not{\partial}_x - M)$ results in a factor of $(\not{p}-M)$ which cancels the propagators leaving a factor of i. So in coordinate space these momentum space Feynman integrands correspond to

$$(i\not{\partial}_x - M)\langle 0|T\,\psi(x)\,\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle$$

$$= \sum_{i=1}^{n} i\delta^4(x-z_n)(-1)^{m+(i-1)}\langle 0|T\,\phi(x_1)\cdots\cancel{\bar{\psi}(z_i)}\cdots\bar{\psi}(z_n)|0\rangle$$

$$+(g+f)\langle 0|T\,\gamma_5\psi(x)\phi(x)\;\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle$$

$$-ib_2\not{\partial}_x\langle 0|T\,\psi(x)\,\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle$$

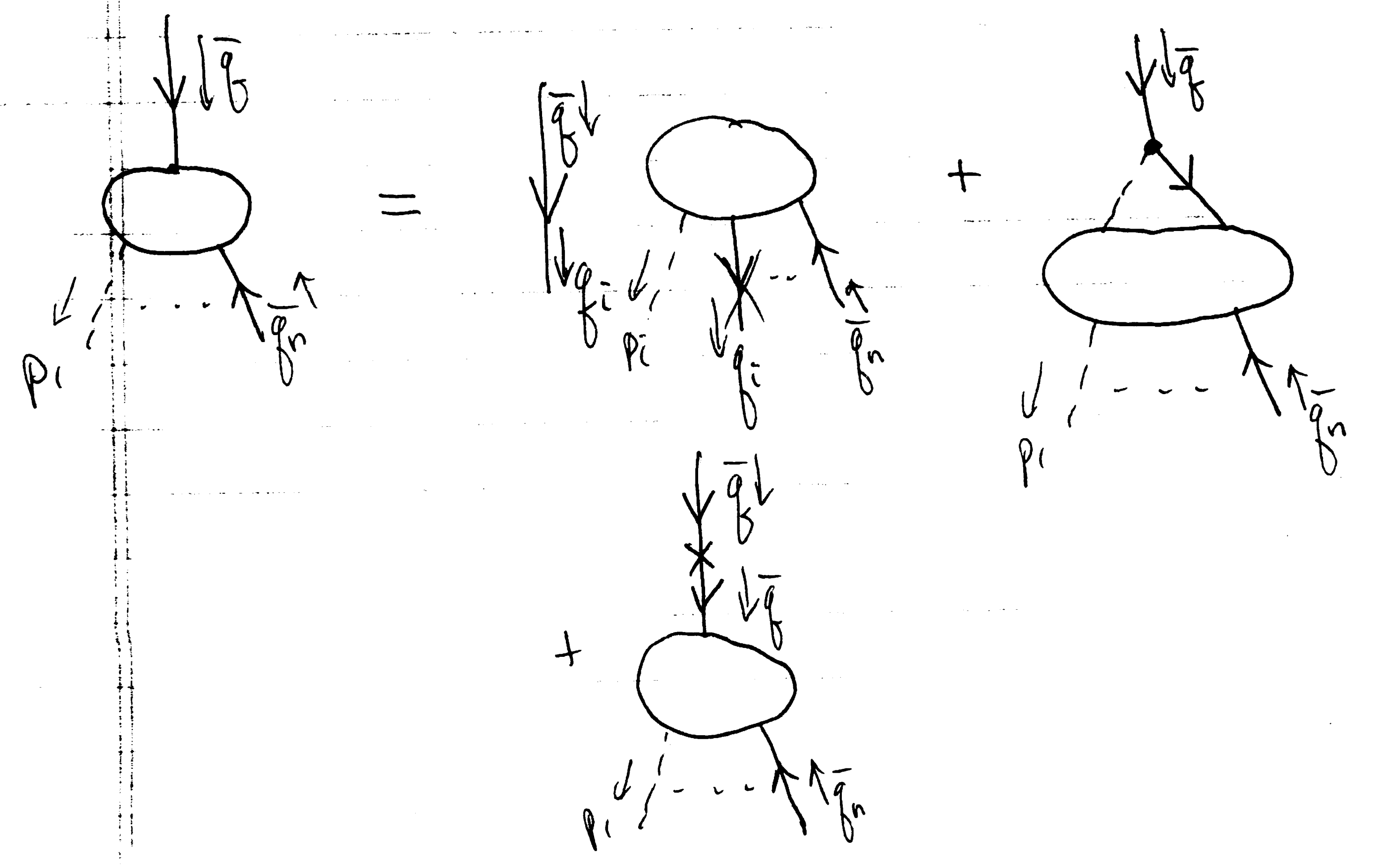
$$+\,d\langle 0|T\,\psi(x)\,\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle$$

Combining similar terms we obtain the second of our Green function equations of motion with $Z_2 = 1+b_2$

$$[Z_2 i \not\partial_x - (M+d)]\langle 0|T\,\Psi(x)\,\phi(x_1)\cdots\overline{\Psi}(z_n)|0\rangle$$
$$-(g+f)\langle 0|T\,\gamma_5\Psi(x)\,\phi(x)\,\phi(x_1)\cdots\overline{\Psi}(z_n)|0\rangle$$
$$=\sum_{i=1}^{n} i(-1)^{m+(i-1)}\,\delta^4(x-z_n)\langle 0|T\,\phi(x_1)\cdots\cancel{\overline{\Psi}(z_i)}\cdots\overline{\Psi}(z_n)|0\rangle$$

in agreement with the field equation on page -290-.

Finally we consider the graphical alternatives for $\langle 0|T\,\Psi(x)\,\phi(x_1)\cdots\overline{\Psi}(z_n)|0\rangle$



These 3 classes of Feynman diagrams have the form

$$I_\Gamma = \frac{i}{\not{q}-M}(-1)^i I_\Gamma^{\hat{i}} - i(g+f) I_\Gamma^{\phi\bar{\Psi}\gamma_5} \frac{i}{\not{q}-M}$$

$$+ I_\Gamma [i\not{q} b_2 - id] \frac{i}{\not{q}-M}$$

Care must be taken with (-1) factors, $I_\Gamma^{\phi\bar{\Psi}\gamma_5}$ is understood to have $\phi\bar{\Psi}\gamma_5$ in the position where $\bar{\Psi}$ was originally, so no extra (-1) appears. Similarly for the last term. The first $(-1)^i$ results from moving $\bar{\Psi}(x)$ to the right of $\bar{\Psi}(y_i)$. The coordinate space contributions are obtained after multiplying by $(i\overleftarrow{\not{\partial}}_x + M)$ on the right, which in momentum space yields $(-\not{q}+M)$ and so leaves a $(-i)$ factor when cancelling the propagator. Hence we obtain

$$\langle 0|T\,\bar{\Psi}(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle(i\overleftarrow{\not{\partial}}_x + M)$$

$$= \sum_{i=1}^{m_f} i(-1)^{i-1}\,\delta^4(x-y_i)\langle 0|T\,\phi(x_1)\cdots\cancel{\bar{\Psi}(y_i)}\cdots\bar{\Psi}(z_n)|0\rangle$$

$$-(g+f)\langle 0|T\,\phi(x)\bar{\Psi}(x)\gamma_5\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$-\langle 0|T\,\bar{\Psi}(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle\left[ib_2\not\partial_x+d\right]$$

Combining similar terms and recalling that $Z_2=1+b_2$ we obtain the last field equation as found in page -291-

$$\langle 0|T\,\bar{\Psi}(x)\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle\left[Z_2 i\overleftarrow{\not\partial}_x+(M+d)\right]$$

$$+(g+f)\langle 0|T\,\phi(x)\bar{\Psi}(x)\gamma_5\,\phi(x_1)\cdots\bar{\Psi}(z_n)|0\rangle$$

$$=\sum_{i=1}^{m_1} i(-1)^{i+1}\delta^4(x-y_i)\langle 0|T\,\phi(x_1)\cdots\cancel{\bar{\Psi}(y_i)}\cdots\bar{\Psi}(z_n)|0\rangle$$

So we have perturbatively verified the Green functions equations of motion. We expected the result since the Gell-Mann-Low perturbation expansion was derived by using the operator solution to the dynamics in the form of the time evolution operator.

Certainly we would like to solve the dynamics of our field theory more generally. To do this we turn our attention to the equations of motion that follow for the generating functional $Z[J,\eta,\bar\eta]$. (Again, once we have perturbatively verified the Green function equations of motion, the functional differential equations of motion for $Z[J,\eta,\bar\eta]$ are also perturbatively verified). As in the bosonic case we would like to find a general solution to these equations of motion by means of functionally Fourier transforming. Our result will be as before

$$Z[J,\eta,\bar\eta] = \frac{\int[d\varphi][d\psi][d\bar\psi]\, e^{i\int dx\,[\mathcal{L}(x)+J\varphi+\bar\eta\psi+\bar\psi\eta]}}{\int[d\varphi][d\psi][d\bar\psi]\, e^{i\int dx\,\mathcal{L}(x)}}$$

The generating functional is given by a sum over all field configurations each contribution being weighted by $e^{i(\text{Action}) + \text{sources}}$.

The new aspect here being the integral over fermion field configurations. The $\varphi(x)$ were <u>classical</u> functions of space-time but now the $\psi(x)$ are anti-commuting classical functions of space-time. We must discuss in more detail what we mean by sums over such functions, as we did for derivatives with respect to Grassmann variables.

So on to Fermion Functional Integration

Functional Integration for Fermions

Finally we desire to solve our dynamics more generally by Functional F.T.:

Recall working with our concrete model, that the E-L equations are

$$E[\bar{\psi}](y)\, Z[J,\eta,\bar{\eta}] = -\eta(y) Z$$

$$E[\psi](y)\, Z = +\bar{\eta}(y) Z$$

$$E[\phi](x)\, Z = -J(x) Z$$

explicitly written

1) $$\left[iZ_2 \not{\partial}_y - (M+d)\right]\frac{\delta}{i\delta\bar{\eta}(y)} Z - (g+f)\gamma_5 \frac{\delta^2}{i\delta\bar{\eta}(y)\, i\delta J(y)} Z = -\eta(y) Z$$

2) $$\left[iZ_2 \frac{\delta}{-i\delta\eta(y)}\overleftarrow{\not{\partial}}_y + (M+d)\frac{\delta}{-i\delta\eta(y)}\right] Z + (g+f)\frac{\delta}{-i\delta\eta(y)}\gamma_5 \frac{\delta}{i\delta J(y)} Z = +\bar{\eta}(y) Z$$

3) $$\left[-Z_1 \partial_x^2 - (m^2+a)\right]\frac{\delta}{i\delta J(x)} Z - \frac{\lambda+c}{3!}\frac{\delta^3}{(i\delta J(x))^3} Z - (g+f)\frac{\delta}{i\delta\eta(x)}\gamma_5 \frac{\delta}{i\delta\bar{\eta}(x)} Z = -J(x) Z$$

We now desire to F.T. Z wrt. $\eta, \bar{\eta}, J$

So we need to consider integrals of the form

$$\int [d\varphi][d\psi][d\bar{\psi}]\, \tilde{Z}[\varphi,\psi,\bar{\psi}]\, e^{i\int dx\,[J\varphi + \bar{\eta}\psi + \bar{\psi}\eta]}$$

As before we define the integrals by expanding $\varphi, \psi, \bar{\psi}$ in terms of complete orthonormal sets of functions

$$\varphi(x) = \sum_{\alpha=1}^{\infty} \varphi_\alpha f_\alpha(x)$$

$$\sum_\alpha f_\alpha(x) f_\alpha(x') = \delta^4(x-x')\ ,\ \int d^4x\, f_\alpha(x) f_\beta(x) = \delta_{\alpha\beta}$$

next we must expand the Fermi (grassmann valued) functions

contains spin coordinate also (s) A so to speak

$$\psi_\alpha(x) = \sum_{A=1}^{\infty} \psi^A \quad U^A_\alpha(x)$$

$$\bar{\psi}_\alpha(x) = \sum_{A=1}^{\infty} \psi^{*A} \quad \bar{U}^A_\alpha(x)$$

$U^A(x)$ is a complete set of 4-comp spinor <u>functions</u> $(\bar{U}^A = U^{\dagger A}\gamma_0)$

while ψ^A are the <u>anti-commuting</u> complex # coefficients

Further

ψ^A & $\psi^{*A} \equiv \bar{\psi}^A$ are indep. since $\psi_\alpha, \bar{\psi}_\alpha$ are indep.

So $U^A_\alpha(x)$ are orthogonal & complete

$$\int d^4x \ \overline{U}^A(x) U^B(x) = \delta^{AB} = \int d^4x \ \overline{U}^A(x)_\alpha U^B_\alpha(x)$$

and $\sum_A{}' U^A(x)\overline{U}^A(x') = \delta^4(x-x')\mathbb{1} \Rightarrow \sum_A{}' U^A_\alpha(x)\overline{U}^A(x')_\beta$

$$= \delta^4(x-x')\delta_{\alpha\beta}$$

Then we define

$$\int [d\psi][d\overline{\psi}] F[\psi,\overline{\psi}]$$

$$= \lim_{m,n\to\infty} \int \underbrace{d\psi^m \cdots d\psi^1 \ d\overline{\psi}^1 \cdots d\overline{\psi}^n}_{\text{standard ordering of Grassmann integrals}} F(\psi^1,\ldots,\psi^m;\overline{\psi}^1,\ldots,\overline{\psi}^n)$$

where we must define integration over eles. of the Grassmann algebra ψ^i & $\overline{\psi}^i$

We desire the integral to be similar to the bosonic infinite integral $\int_{-\infty}^{+\infty} d\varphi_\alpha$

This integral is translation invariant and so we demand our integral over Grassmann elts. to be translation invariant.

$$\int d\eta^A f(\eta^A) = \int d\eta^A f(\eta^A + \xi^A)$$ for each A & similarly for $\bar{\eta}^{\dot{A}}$

Since $\eta^A \eta^A = 0$ (no sum on A)

$f(\eta^A) = f_0 + f_1 \eta^A$ only — the expansion terminates. So translation invariance $\Rightarrow$

$$\int d\eta^A (f_0 + f_1 \eta^A) = \int d\eta^A (f_0 + f_1 \eta^A + f_1 \xi^A)$$

$$\Rightarrow \int d\eta^A f_1 \xi^A = 0 = \underbrace{f_1 \xi^A}_{\eta^A \text{ indep. constant}} \int d\eta^A$$

$$\Rightarrow \boxed{\int d\eta^A (\text{const}) = 0.}$$

So the only possibility for a nontrivial definition of the integral is

$$\int d\eta^A \eta^A = \# = C$$

and we might as well define that constant to be 1

So $$\boxed{\int d\psi^A \psi^B = \delta^{AB}}$$

but this just looks like differentiation

$$\frac{\partial}{\partial \psi^A} \psi^B = \delta^{AB}$$

So
$$\int d\psi^A \psi^B = \frac{\partial}{\partial \psi^A} \psi^B = \delta^{AB}$$
integration = differentiation for Grassmann variables

& similarly $\int d\bar{\psi}^A \bar{\psi}^B = \frac{\partial}{\partial \bar{\psi}^A} \bar{\psi}^B = \delta^{AB}$

Once again we consider the Gaussian and find a closed Result for the integral

$$\int [d\psi][d\bar{\psi}]\, e^{+\int d^4x\, \bar{\psi}(x)\psi(x)} = \int [d\psi][d\bar{\psi}]\, e^{+\int d^4x \sum_{A,B} \bar{U}^A_{(x)} U^B_{(x)} \bar{\psi}^A \psi^B}$$

$$= \lim_{m,n\to\infty} \int d\psi^m \ldots d\psi^1\, d\bar{\psi}^1 \ldots d\bar{\psi}^n\, e^{+\sum_{A=1}^{m} \bar{\psi}^A \psi^A}$$

$$= \lim_{m\to\infty} \left(\int d\psi^1 d\bar{\psi}^1\, e^{+\bar{\psi}^1\psi^1}\right) \ldots \left(\int d\psi^m d\bar{\psi}^m\, e^{+\bar{\psi}^m\psi^m}\right)$$

$$= \lim_{m\to\infty} \left(\int d\psi^1 d\bar{\psi}^1 (1 + \bar{\psi}^1\psi^1)\right)^m = 1$$

So

$$\boxed{\int [d\psi][d\bar\psi]\, e^{\int d^4x\, \bar\psi(x)\psi(x)} = 1}$$

Using the translation invariance we have

$$1 = \int [d\psi][d\bar\psi]\, e^{\int dx\, (\bar\psi+\bar\eta)(\psi+\eta)}$$

$$\Rightarrow \quad e^{-\bar\eta\eta} = \int [d\psi][d\bar\psi]\, e^{\int dx\{\bar\psi\psi + \bar\eta\psi + \bar\psi\eta\}}$$

Letting $\eta \to i\eta$ & $\bar\eta \to i\bar\eta$ in this $\eta, \bar\eta$ identity (treat $\eta, \bar\eta$ as indep.) $\Rightarrow$

$$\boxed{\int [d\psi][d\bar\psi]\, e^{\int dx[\bar\psi\psi + i\bar\eta\psi + i\bar\psi\eta]} = e^{+\bar\eta\eta}}$$

Finally we would like to change variables for the Grassmann integrals

Here we will find a difference compared to the boson case since the integrals are like derivatives

Suppose

$$\psi'_\alpha(x) = \int \not{D}_{\alpha\beta}(x,y)\,\psi_\beta(y)\,d^4y$$

(assume $\not{D}$ is indep. of $\psi, \bar\psi$, but we don't have to)

Similarly

$$\bar\psi'_\alpha(x) = \int dy\,\bar\psi_\beta(y)\,\bar{\not{D}}_{\beta\alpha}(y,x)$$

& $\bar{\not{D}}(y,x) \equiv \gamma^0 \not{D}^\dagger(y,x)\gamma^0$ $\quad \left(\not{D}^\dagger_{\beta\alpha}(y,x) = \not{D}^*_{\alpha\beta}(x,y)\right)$

Expanding the functions

$$\sum_A \psi'^A U^A_\alpha(x) = \int dy\,\not{D}_{\alpha\beta}(x,y)\sum_B \psi^B U^B_\beta(y)$$

Mult. by $\int \bar{U}^B_\alpha(x)\,d^4x \Rightarrow$

$$\psi'^B = \int dx\,dy\,\bar{U}^B_\alpha(x)\,\not{D}_{\alpha\beta}(x,y)\,U^C_\beta(y)\,\psi^C$$

So define

$$\boxed{D^{AB} \equiv \int dx\,dy\,\bar{U}^A_\alpha(x)\,\not{D}_{\alpha\beta}(x,y)\,U^B_\beta(y)}$$

yields

$$\psi'^A = \sum_B D^{AB}\,\psi^B$$

Now

$$\int d\psi'^A = \frac{\partial}{\partial \psi'^A} = \frac{\partial \psi^B}{\partial \psi'^A}\,\frac{\partial}{\partial \psi^B}$$

$$= \frac{\partial \psi^B}{\partial \psi'^A}\int d\psi^B$$

But

$$\psi^B = \sum_A D^{-1\,BA}\,\psi'^A$$

where $D^{-1\,BA} D^{AC} = \delta^{BC}$

So $\dfrac{\partial \psi^B}{\partial \psi'^A} = D^{-1\,BA}$

So

$$\int d\psi'^A = D^{-1\,BA}\int d\psi^B$$

or more directly

$$\int d\psi^B = \frac{\partial}{\partial \psi^B} = \frac{\partial \psi'^A}{\partial \psi^B}\,\frac{\partial}{\partial \psi'^A} = D^{AB}\int d\psi'^A$$

So

$$\boxed{\begin{aligned}\int d\psi'^A &= \int d\psi^B D^{-1BA} \\ \text{or} \quad \int d\psi^B &= \int d\psi'^A D^{AB}\end{aligned}}$$

(Recall for Bosons we have the Jacobian not the inverse Jacobian $\int [d\varphi'] = \int (\det K)[d\varphi]$)

Now

$$\int [d\psi'] = \int d\psi'^m \dots d\psi'^1$$

$$= \int d\psi^{B_m} d\psi^{B_{m-1}} \dots d\psi^{B_1} D^{-1 B_m m} D^{-1 B_{m-1} m-1} \dots D^{-1 B_1 1}$$

But since the $d\psi^i$ anti-commute

$$d\psi^{B_m} \dots d\psi^{B_1} = d\psi^m \dots d\psi^1 \quad \epsilon^{B_1 \dots B_m}$$

where $\epsilon^{B_m \dots B_1}$ is the m^{th} rank totally anti-symm. tensor in m dimensions.

But $\epsilon^{B_1 \dots B_m} (D^{-1})^{B_m m} \dots (D^{-1})^{B_1 1}$

$$= + \det D^{-1}$$

So

$$\boxed{\int [d\psi'] = \int [d\psi] \,(+\det D^{-1})}$$

Similarly

$$\bar\psi'_\alpha(x) = \sum_A' \bar\psi'^A \,\bar U^A_\alpha(x) = \int dy \sum_C' \bar\psi^C \,\bar U^C_\beta(y)\,\bar{\not D}_{\beta\alpha}(y,x)$$

Mult. by $\int dx\, U^B_\alpha(x)$

$$\Rightarrow \quad \bar\psi'^B = \int dx\,dy \sum_C' \bar\psi^C \,\bar U^C_\beta(y)\,\bar{\not D}_{\beta\alpha}(y,x)\,U^B_\alpha(x)$$

So $$\bar D^{CB} \equiv \int dx\,dy\, \bar U^C_\beta(y)\,\bar{\not D}_{\beta\alpha}(y,x)\,U^B_\alpha(x)$$

Plus indeed

$$(D^\dagger)^{AB} = D^{*BA}$$

$$= \int dx\,dy \left(\bar U^B_\alpha(x)\,\not D_{\alpha\beta}(x,y)\,U^A_\beta(y)\right)^*$$

$$= \int dx\,dy\, \left(U^B_\alpha(x)\,\gamma^0\right) \not D^\dagger_{\beta\alpha}(y,x)\,(\gamma^0\gamma^0)_{\beta\gamma}\, U^{A*}_\gamma(y)$$

$$= \int dx\,dy\, \bar U^A_\beta(y)\,(\gamma^0 \not D^\dagger(y,x)\gamma^0)_{\beta\alpha}\, U^B_\alpha(x)$$

$$= \int dx\,dy\, \bar U^A_\beta(y)\,\bar{\not D}(y,x)_{\beta\alpha}\, U^B_\alpha(x)$$

So

$$\boxed{(D^{+})^{AB} = \overline{D}^{AB}}$$

So

$$\overline{\psi}'^{B} = \overline{\psi}^{C}\,\overline{D}^{CB}$$

$$\overline{\psi}'^{A} = \overline{\psi}^{B}\,\overline{D}^{BA}$$

Now

$$\int d\overline{\psi}'^{A} = \frac{\partial}{\partial\overline{\psi}'^{A}} = \frac{\partial\overline{\psi}^{B}}{\partial\overline{\psi}'^{A}}\frac{\partial}{\partial\overline{\psi}^{B}}$$

$$= \frac{\partial\overline{\psi}^{B}}{\partial\overline{\psi}'^{A}}\int d\overline{\psi}^{B}$$

But

$$\overline{\psi}^{B} = \overline{\psi}'^{A}\;\overline{D}^{-1AB}$$

$$\boxed{\frac{\partial\overline{\psi}^{B}}{\partial\overline{\psi}'^{A}} = \overline{D}^{-1AB}}$$

So

$$\int d\overline{\psi}'^{A} = \int \overline{D}^{-1AB}\,d\overline{\psi}^{B}$$

or

$$\int d\overline{\psi}^{B} = \frac{\partial\overline{\psi}'^{A}}{\partial\overline{\psi}^{B}}\frac{\partial}{\partial\overline{\psi}'^{A}} = \int \overline{D}^{BA}\,d\overline{\psi}'^{A}$$

So

$$\boxed{\begin{aligned}\int d\bar{\psi}'^{A} &= \int \bar{D}^{-1\,AB}\, d\bar{\psi}^{B} \\ \int d\bar{\psi}^{B} &= \int \bar{D}^{BA}\, d\bar{\psi}'^{A}\end{aligned}}$$

Now

$$\int [d\bar{\psi}'] = \int d\bar{\psi}'^{1} \dots d\bar{\psi}'^{n}$$

$$= \int d\bar{\psi}^{B_1} \dots d\bar{\psi}^{B_n}\, \bar{D}^{-1\,1B_1} \dots \bar{D}^{-1\,nB_n}$$

But

$$d\bar{\psi}^{B_1} \dots d\bar{\psi}^{B_n} = d\bar{\psi}^{1} \dots d\bar{\psi}^{n}\, \epsilon^{B_1 \dots B_n}$$

and

$$\epsilon^{B_1} \dots \epsilon^{B_n}\, \bar{D}^{-1\,1B_1} \dots \bar{D}^{-1\,nB_n}$$

$$= + \det \bar{D}^{-1}$$

So

$$\boxed{\int [d\bar{\psi}'] = \int [d\bar{\psi}] (\det \bar{D}^{-1})}$$

Then

$$\boxed{\int [d\psi'][d\bar{\psi}'] = \int [d\psi][d\bar{\psi}]\left[+\det\left[(D\bar{D})^{-1}\right]\right]}$$

In general $\gamma^0 \not{D}^\dagger \gamma^0 = \not{D}$ so that

$$\int [d\psi'][d\bar{\psi}'] = \int [d\psi][d\bar{\psi}]\left[\det D^2\right]^{-1}$$

Recall for Bosons $\int [d\varphi_\alpha'] = \int [d\varphi_\alpha][\det K]$

just $\det K$ not $(\det K)^{-1}$ as for fermions —

(This is due to the fermion loops giving (-1))

Notice $\not{D}(x,y) = \not{d}_{xy}$ & $U^A(x) = U_{Ax}$

we have

$$D^{AB} = \bar{U}_{Ax}\not{d}_{xy}U_{By}$$

$D = \bar{U} \not{d} U^T$

but $\bar{U}_{Ax} U_{Bx} = \delta^{AB} \Rightarrow \bar{U} U^T = 1$

$\Rightarrow \det D = \det(\bar{U} \not{d} U^T) = \det(\not{d})$

So for

$$\psi' = \not{d}\psi \quad , \bar{\psi}' = \bar{\psi}\not{d}$$

we have

$$\boxed{\int [d\not{d}\psi][d\bar{\psi}\not{d}] = \int [d\psi][d\bar{\psi}][\det(\not{d}\not{d})]^{-1}}$$

So we can evaluate integrals of the form

$$\int [d\psi][d\bar{\psi}]\, e^{\int dx\,dy\,[\bar{\psi}(x)\not{D}(x,y)\psi(y)] + i\int dx(\bar{\eta}\psi + \bar{\psi}\eta)}$$

(One approach is to assume $\not{D} = \not{D}^{1/2}\not{D}^{1/2}$

then consider

$$= \int [d\psi][d\bar{\psi}]\, e^{\int dx\,dy\,dz\,[\bar{\psi}(x)\bar{\not{D}}^{1/2}(x,z)\not{D}^{1/2}(z,y)\psi(y)]}$$

$$\times\, e^{i\int dx(\bar{\eta}\not{D}^{-1/2}\not{D}^{1/2}\psi + \bar{\psi}\bar{\not{D}}^{*1/2}\bar{\not{D}}^{-1/2}\eta)}$$

$$= \int [d\psi][d\bar\psi]\, e^{\bar\psi \not{d}^{1/2} \not{d}^{1/2}\psi + i\bar\eta \not{d}^{-1/2}\not{d}^{1/2}\psi + i\bar\psi\not{d}^{1/2}\not{d}^{-1/2}\eta}$$

Now let $\psi' = \not{d}^{1/2}\psi$; $\bar\psi' = \bar\psi\not{d}^{1/2}$

$\Rightarrow$

$$= \int [d\psi'][d\bar\psi'][\det(\not{d}^{1/2}\not{d}^{1/2})]\, e^{\bar\psi'\psi' + i\bar\eta\not{d}^{-1/2}\psi' + i\bar\psi'\not{d}^{-1/2}\eta}$$

$$= (\det(\not{d}))\, e^{+\bar\eta\not{d}^{-1/2}\not{d}^{-1/2}\eta}$$

$$= [\det(\not{d})]\, e^{+\bar\eta\not{d}^{-1}\eta}$$

$$= [\det(\not{d})]\, e^{\int dx\,dy\, \bar\eta(x)\not{D}^{-1}(x,y)\eta(y)}$$

So

$$\boxed{\int [d\psi][d\bar\psi]\, e^{\int dx\,dy\,\bar\psi(x)\not{D}(x,y)\psi(y) + i\int dx(\bar\eta(x)\psi(x) + \bar\psi(x)\eta(x))} = [\det(\not{d})]\, e^{\int dx\,dy\,\bar\eta(x)\not{D}^{-1}(x,y)\eta(y)}}$$

More to the point we consider the first expression directly and change ψ variables only

$$= \int [d\psi][d\bar\psi]\, e^{\bar\psi \not{D} \psi + i(\bar\eta \psi + \bar\psi \eta)}$$

$$= \int [d\psi][d\bar\psi]\, e^{\bar\psi (\not{D}\psi) + i(\bar\eta \not{D}^{-1})(\not{D}\psi) + i\bar\psi\eta}$$

Now let $\psi' = \not{D}\psi$, $\bar\psi' = \bar\psi$ since they are independent

$$= \int [d\psi'][d\bar\psi'] \,[\det \not{D}]\, e^{\bar\psi'\psi' + i(\bar\eta\not{D}^{-1})\psi' + i\bar\psi'\eta}$$

$$= [\det \not{D}]\, e^{+\bar\eta \not{D}^{-1} \eta}$$

$$\boxed{= [\det \not{D}]\, e^{+\int dx\,dy\, \bar\eta(x) \not{D}^{-1}(x,y) \eta(y)} = \int [d\psi][d\bar\psi]\, e^{+\int dx\,dy\, \bar\psi(x)\not{D}(x,y)\psi(y) + i\int dx (\bar\eta(x)\psi(x) + \bar\psi(x)\eta(x))}}$$

Finally we introduce the Functional Fourier Transform:

$$Z[\eta,\bar\eta] = \int [d\psi][d\bar\psi]\, e^{i\int dx\,\{\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)\}}\,\tilde Z[\psi,\bar\psi]$$

$$= \lim_{m,n\to\infty}\int d\psi^m\cdots d\psi^1 d\bar\psi^1\cdots d\bar\psi^n\, e^{i\sum_A^m(\bar\eta_A\psi_A+\bar\psi_A\eta_A)}\,\tilde Z(\psi_1\ldots\psi_m,\bar\psi_1\ldots\bar\psi_n)$$

The inverse formula is

$$\tilde Z[\psi,\bar\psi] = \int [d\eta][d\bar\eta]\, e^{-i\int dx\{\bar\eta\psi+\bar\psi\eta\}}\,Z[\eta,\bar\eta]$$

$$\Rightarrow \delta[\eta-\eta'] = \lim_{m\to\infty} i\delta(\eta^m-\eta'^m)\cdots i\delta(\eta^1-\eta'^1)$$

$$\delta[\bar\eta-\bar\eta'] = \lim_{n\to\infty}(-i)\delta(\bar\eta^n-\bar\eta'^n)\cdots(-i)\delta(\bar\eta^1-\bar\eta'^1)$$

HW: Show that

$$\delta(\eta^A-\eta'^A) = \eta^A-\eta'^A$$

$$\delta(\bar\eta^A-\bar\eta'^A) = \bar\eta^A-\bar\eta'^A.$$

and so

$$\delta[\eta-\eta'] = \int [d\bar{\psi}]\, e^{i\int dx\, [\bar{\psi}(x)][\eta(x)-\eta'(x)]}$$

$$\delta[\bar{\eta}-\bar{\eta}'] = \int [d\psi]\, e^{i\int dx\, (\bar{\eta}(x)-\bar{\eta}'(x))\, \psi(x)} .$$

So finally we can F.T. our 3 equations of motion for $Z[J, \eta, \bar{\eta}]$

$\Rightarrow$

1) $$\left\{ [i Z_2 \not{\partial}_y - (M+d)](+\psi(y)) - (g+f)\gamma_5 \psi(y)\, \phi(y) \right\} \tilde{Z}[\varphi, \psi, \bar{\psi}] = -i \frac{\delta}{\delta \bar{\psi}(y)} \tilde{Z}$$

2) $$\left\{ [i Z_2 \bar{\psi}(y) \overleftarrow{\not{\partial}}_y + (M+d)\bar{\psi}(y)] + (g+f)\, \bar{\psi}(y)\gamma_5 \varphi(y) \right\} \tilde{Z} = -i \frac{\delta}{\delta \psi(y)} \tilde{Z}$$

3) $$\left\{ [-Z_1 \partial_x^2 - (m^2+c)]\varphi(x) - \frac{\lambda+c}{3!}\varphi^3(x) - (g+f)\, \bar{\psi}(x)\gamma_5 \psi(x) \right\} \tilde{Z} = -i \frac{\delta}{\delta \varphi(x)} \tilde{Z}$$

The first 2 eq. $\Rightarrow$

$$\tilde{Z} = e^{+i\int dy\left[\bar{\psi}(y)\left(\frac{i}{2}Z_2\overleftrightarrow{\not\partial}_y - (M+d)\right)\psi(y) - (g+f)\bar{\psi}(y)\gamma_5\psi(y)\varphi(y)\right]} \times \hat{Z}[\varphi]$$

eq. 3. $\Rightarrow$

$$\hat{Z}\{\varphi\} = \frac{1}{N}e^{-i\int dx\left[\frac{1}{2}\{\varphi(x)(Z_1\partial_x^2 + (m^2+a))\varphi(x)\} + \frac{(\lambda+c)}{4!}\varphi^4(x)\right]}$$

So,

$$\tilde{Z}[\varphi,\psi,\bar{\psi}] = \frac{1}{N}e^{+i\int dx\left[Z_2\frac{i}{2}\bar{\psi}\overleftrightarrow{\not\partial}\psi - (M+d)\bar{\psi}\psi - (g+f)\bar{\psi}\gamma_5\psi\varphi + Z_1\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(m^2+a)\varphi^2 + \frac{(\lambda+c)}{4!}\varphi^4\right]}$$

$$= \frac{1}{N}e^{i\int dx\,\mathcal{L}[\varphi,\psi,\bar{\psi}]}$$

$\Rightarrow$

$$\boxed{Z[J,\eta,\bar{\eta}] = \frac{1}{N}\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right][d\psi][d\bar{\psi}]\,e^{i\int d^4x\left[\mathcal{L}[\varphi,\psi,\bar{\psi}] + J\varphi + \bar{\psi}\eta + \bar{\eta}\psi\right]}}$$

with $Z[0,0,0] = 1 \Rightarrow N = \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right][d\psi][d\bar{\psi}]\,e^{i\int dx\,\mathcal{L}[\varphi,\psi,\bar{\psi}]}$.

Again we can find a perturbative solution

$$Z[J,\eta,\bar\eta] = \frac{1}{N} e^{i\int dx\, \mathcal{L}_I\left[\frac{\delta}{i\delta J}, \frac{\delta}{-i\delta\eta}, \frac{\delta}{i\delta\bar\eta}\right]}$$

$$\times \underbrace{\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right][d\psi][d\bar\psi]\, e^{i\int dx\left\{\mathcal{L}_{in}[\varphi,\psi,\bar\psi] + J\varphi + \bar\eta\psi + \bar\psi\eta\right\}}}_{\equiv Z_{in}[J,\eta,\bar\eta] \text{ with } Z_{in}[0,0,0]=1}$$

But we have

$$Z_{in} = \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{-\frac{i}{2}\int dx\,\varphi(x)(\partial_x^2+m^2)\varphi(x) + i\int dx\, J\varphi}$$

$$\times \int[d\psi][d\bar\psi]\, e^{i\int dx\left\{\bar\psi[i\not\partial - M]\psi + \bar\eta\psi + \bar\psi\eta\right\}}$$

So 1) $K(x,y) = i(\partial_x^2 + m^2)\delta^4(x-y)$

$$\Rightarrow \quad K^{-1}(x,y) = \Delta_F(x-y)$$

&

2) $\not D = i(i\not\partial_x - M)\delta^4(x-y)$

and

$$\int d^4y\, \not D(x,y)\not D^{-1}(y,z) = \delta^4(x-z)$$

$$= i(i\not\partial_x - M)\not D^{-1}(x,z)$$

$\Rightarrow$

$$(i\not\partial_x - M)\not{D}^{-1}(x,z) = -i\,\delta^4(x-z)$$

$$\Rightarrow \quad \not{D}^{-1}(x,z) = -S_F(x-z)$$

So

$$\boxed{Z_{in}[J,\eta,\bar\eta] = e^{-\frac{1}{2}\int dx\,dy\, J(x)\Delta_F(x-y)J(y)} \times e^{-\int dx\,dy\,\bar\eta(x) S_F(x-y)\eta(y)}}$$

(Alt. proof of Wick's Thm.)

This just reproduces our GML expansion.

$$Z[J,\eta,\bar\eta] = e^{i\int dx\,\mathcal{L}_I} Z_{in}[J,\eta,\bar\eta]$$
$$= e^{i\int dx\,\mathcal{L}_I} \langle 0|T e^{i\int dx[J\phi_{in} + \bar\eta\psi_{in} + \bar\psi_{in}\eta]}|0\rangle$$
$$= \langle 0|T e^{i\int dx[J\phi + \bar\psi\eta + \bar\eta\psi]}|0\rangle$$

In addition we have that $Z[J,\eta,\bar\eta]$ obeys E-L eq. by construction. So much for fermions.