III c) Besides models that involve bosonic fields, we are also interested in time ordered functions involving fermion fields, $\psi(x)$ and $\bar{\psi}(x)$

We know from our perturbative analysis of the time evolution operator that the Green functions are given by the Gell-Mann-Low formula just as in the purely bosonic case

$$\langle 0|T\,\phi(x_1)\cdots\phi(x_\ell)\,\psi(y_1)\cdots\psi(y_m)\,\bar{\psi}(z_1)\cdots\bar{\psi}(z_n)|0\rangle$$

$$= \frac{\langle 0|T\,\phi_{in}(x_1)\cdots\bar{\psi}_{in}(z_n)\,e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}{\langle 0|T\,e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}$$

where, of course the time ordering symbol above refers to the T^* operator and interchange of fermion operators under the time ordered symbol picks up a minus sign for each permutation required to return to the original order.

As before we can introduce the generating functional for these Green functions

$$Z[J,\eta,\bar\eta] \equiv \langle 0|T e^{i\int dx[J\phi + \bar\eta\psi + \bar\psi\eta]}|0\rangle$$

$$= \sum_{l,m,n=0}^{\infty} \frac{i^{l+m+n}}{l!\,m!\,n!} \int dx_1 \cdots dx_l\, dy_1 \cdots dy_m\, dz_1 \cdots dz_n$$

$$J(x_1)\cdots J(x_l)\,\bar\eta(y_m)\cdots\bar\eta(y_1)\,\times$$

$$\times \langle 0|T\phi(x_1)\cdots\phi(x_l)\psi(y_1)\cdots\psi(y_m)\bar\psi(z_1)\cdots\bar\psi(z_n)|0\rangle \times$$

$$\times\, \eta(z_n)\cdots\eta(z_1)$$

where $J(x)$ is an ordinary real function while $\eta(x)$ and $\bar\eta(x)$ are anti-commuting, complex spinor functions i.e. mappings of space-time into an (infinite dimensional) Grassmann algebra.

Thus $\{\eta(x),\eta(y)\} = 0 = \{\bar\eta(x),\eta(y)\}$.

(Note the anti-commutivity of the fermion sources follows from the anti commutivity of the fermion fields i.e.

$$\int dx_1 dx_2\, \eta(x_1)\eta(x_2) \langle 0| T\, \psi(x_1)\psi(x_2)\, \underline{X}\, |0\rangle$$

$$= \int dx_1 dx_2\, \eta(x_2)\, \eta(x_1) \langle 0| T\, \psi(x_2)\, \psi(x_1)\, \underline{X}\, |0\rangle$$

1) if η are ~~real~~ (complex) functions then $\eta(x_2)\eta(x_1) = \eta(x_1)\,\eta(x_2)$

$$\Rightarrow \quad = \int dx_1 dx_2\, \eta(x_1)\eta(x_2)\, \langle 0| T\, \psi(x_2)\psi(x_1)\, \underline{X}\, |0\rangle$$

$$\Rightarrow T\left[\psi(x_1), \psi(x_2)\right] = 0$$

2) if η are anti-commuting functions $\eta(x_2)\,\eta(x_1) = -\,\eta(x_1)\,\eta(x_2)$ then $\Rightarrow$

$$= -\int dx_1 dx_2\, \eta(x_1)\eta(x_2)\, \langle 0| T\, \psi(x_2)\psi(x_1)\, \underline{X}\, |0\rangle$$

as required $T\{\psi(x_1), \psi(x_2)\} = 0$

Thus reversing the discussion we see that

$$T\{\psi(x_1), \psi(x_2)\}\, \underline{X} = 0 \;\Rightarrow\; \{\eta(x_1), \eta(x_2)\} = 0\,.)$$

Hence we must be cautious with moving $\eta, \bar{\eta}, \psi, \bar{\psi}$ through each other since they all anti-commute ($\{\eta, \psi\} = 0$ etc, also),
So we have

$$Z[J,\eta,\bar{\eta}] = \sum_{l,m,n=0}^{\infty} \frac{i^{l+m+n}}{l!\,m!\,n!} \int dx_1 \cdots dz_n\, J(x_1)\cdots J(x_l)$$

$$[-\eta(z_n)][-\eta(z_{n-1})]\cdots[-\eta(z_1)]\,\bar{\eta}(y_m)\cdots\bar{\eta}(y_1)\times$$

$$\times \langle 0|T\,\phi(x_1)\cdots\phi(x_l)\,\psi(y_1)\cdots\psi(y_m)\,\bar{\psi}(z_1)\cdots\bar{\psi}(z_n)|0\rangle$$

Thus we can obtain the $(l+m+n)$ point function by functionally differentiating

$$\boxed{\begin{aligned}&\langle 0|T\,\phi(x_1)\cdots\phi(x_l)\,\psi(y_1)\cdots\psi(y_m)\,\bar{\psi}(z_1)\cdots\bar{\psi}(z_n)|0\rangle \\ &= \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_l)}\,\frac{\delta}{i\delta\bar{\eta}(y_1)}\cdots\frac{\delta}{i\delta\bar{\eta}(y_m)}\times \\ &\times\frac{\delta}{-i\delta\eta(z_1)}\cdots\frac{\delta}{-i\delta\eta(z_n)}\,Z[J,\eta,\bar{\eta}]\Big|_{J=\eta=\bar{\eta}=0}\end{aligned}}$$

(Note: Lorentz covariance implies $\# \text{ of } \psi = \# \text{ of } \bar{\psi}$ or else $\langle 0|T \cdots |0\rangle = 0$ $\forall$ cases $m = n$ above

and we have

$$\langle 0|T\,\phi(x_1)\cdots\bar{\psi}(z_n)|0\rangle = \frac{\delta^l}{i^l\,\delta J(x_1)\cdots\delta J(x_l)}\,\frac{\delta^{2n}}{\delta\bar{\eta}(y_1)\cdots\delta\eta(z_n)}$$

$$\times Z\{J,\eta,\bar{\eta}\}\Big|_{J=\eta=\bar{\eta}=0},$$

The $+i$ and $-i$ factors cancel from the fermion derivatives.)

Remarks

1) We defined the bosonic functional derivative through the functional Taylor series

$$\delta Z[J] = Z[J+\delta J] - Z[J] = \int \frac{\delta Z[J]}{\delta J(y)}\,\delta J(y)\,dy$$

with $\dfrac{\delta}{\delta J(x)}\cdot\dfrac{\delta}{\delta J(y)} = +\dfrac{\delta}{\delta J(y)}\dfrac{\delta}{\delta J(x)}$

The anti-commuting Fermion derivatives

$$\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\eta(y)} = -\frac{\delta}{\delta\eta(y)}\frac{\delta}{\delta\eta(x)}\,, \text{ etc.}$$

are defined similarly by the Taylor expansion involving Grassmann variables

$$\delta Z[\eta,\bar\eta] = Z[\eta+\delta\eta,\ \bar\eta+\delta\bar\eta] - Z[\eta,\bar\eta]$$
$$= \int \frac{\delta Z[\eta,\bar\eta]}{\delta\eta(y)}\,\delta\eta(y)\,dy + \int \frac{\delta Z[\eta,\bar\eta]}{\delta\bar\eta(y)}\,\delta\bar\eta(y)\,dy$$

For $Z = \eta + \bar\eta$ we have

$$\frac{\delta\eta(x)}{\delta\eta(y)} = \delta^4(x-y) \quad ; \quad \frac{\delta\bar\eta(x)}{\delta\bar\eta(y)} = \delta^4(x-y)$$

η & $\bar\eta$ are considered independent so

$$\frac{\delta\eta}{\delta\bar\eta} = 0 = \frac{\delta\bar\eta}{\delta\eta}\,.$$ Note that $\eta, \bar\eta$ are spinors with indices running from 1 to 4 (we ~~are~~ using Dirac fields ($\alpha = 1, 2$, $\dot\alpha = 1, 2$ if Weyl fields))

So $\frac{\delta \eta_a(x)}{\delta \eta_b(y)} = \delta_{ab}\delta^4(x-y)$, $\frac{\delta \bar{\eta}_a(x)}{\delta \bar{\eta}_b(y)} = \delta_{ab}\delta^4(x-y)$

is what is meant and $\frac{\delta}{\delta \eta}\bar{\eta} = -\bar{\eta}\frac{\delta}{\delta \eta}$ etc.

Minus signs are important!!

2) Given the generating functional $Z[J,\eta,\bar{\eta}]$ we can express the S-operator in terms of the in-fields as before, where now the Axiom 3 asymptotic condition for fermions is

Axiom 3: $\exists$ a covariant Heisenberg field $\psi(x)$ which obeys the asymptotic condition

$$\lim_{t\to-\infty} \langle \chi | \begin{matrix} b_\alpha \\ d_\alpha \end{matrix}(t) | \phi \rangle = Z_2^{1/2} \langle \chi | \begin{matrix} b_\alpha^{in} \\ d_\alpha^{in} \end{matrix} | \phi \rangle$$

$$\lim_{t\to+\infty} \langle \chi | \begin{matrix} b_\alpha \\ d_\alpha \end{matrix}(t) | \phi \rangle = Z_2^{1/2} \langle \chi | \begin{matrix} b_\alpha^{out} \\ d_\alpha^{out} \end{matrix} | \phi \rangle$$

where $\psi^{in}_{out}(x)$ are the free asymptotic in- & out- fields.

They are described by the in- & out-Dirac Lagrangians

$$\mathcal{L}_{in} = \frac{i}{2} \bar{\psi}_{in} \overleftrightarrow{\not\partial} \psi_{in} - M \bar{\psi}_{in} \psi_{in}$$

$$\mathcal{L}_{out} = \frac{i}{2} \bar{\psi}_{out} \overleftrightarrow{\not\partial} \psi_{out} - M \bar{\psi}_{out} \psi_{out}$$

Up to an irrelevant total divergence these are equivalent to

$$\mathcal{L}_{in} = \bar{\psi}_{in} (i\not\partial - M) \psi_{in}$$

$$\mathcal{L}_{out} = \bar{\psi}_{out} (i\not\partial - M) \psi_{out} .$$

The fields can be expanded in terms of a complete set of normalizable Dirac equation wave packets $U_\alpha(x)$, $V_\alpha(x)$

$$\psi_{in}(x) = \sum_\alpha{}' [b_\alpha^{in} U_\alpha(x) + d_\alpha^{in\dagger} V_\alpha(x)]$$

(for plane waves

$$= \int d^3k \sum_{s=1}^{2} [b_s^{in}(\vec{k}) U_{ks}(x) + d_s^{in\dagger}(\vec{k}) V_{ks}(x)]$$

with $U_{ks}(x) = \frac{1}{(2\pi)^3 2\omega_k} u^{(s)}(\vec{k}) e^{-ikx}$

The smeared interpolating fields are defined analogously by

$$b_\alpha(t) \equiv \int d^3x\, \overline{U}_\alpha(x)\gamma^0 \psi(x)$$

$$d^\dagger_\alpha(t) \equiv \int d^3x\, \overline{V}_\alpha(x)\gamma^0 \psi(x) \quad \text{etc.}$$

Axiom 4 now states microcausality in terms of anti-commutation relations

Microcausality $\{\psi(x), \overline{\psi}(y)\} = 0$ for $(x-y)^2 < 0$

Since observables are at least bilinear in $\psi, \overline{\psi}$

This converts to space-like commutivity for them.

The S-operator can then be found by using the fermion fields LSZ Reduction formula (Homework)

$$\boxed{S = :e^{\int d^4x\left[\phi_{in}(x) Z_1^{-1/2}(\partial_x^2+m^2)\frac{\delta}{\delta J(x)} + \bar{\psi}_{in}(x) Z_2^{-1/2}(i\overrightarrow{\not\partial}_x - M)\frac{\delta}{\delta \bar{\eta}(x)} + \frac{\delta}{\delta \eta(x)} Z_2^{-1/2}(-i\overleftarrow{\not\partial}_x - M)\psi_{in}(x)\right]} : Z[J,\eta,\bar{\eta}]}$$

In order to be concrete we can pick a particular model involving the interaction of fermions and bosons in which to work. For simplicity let's choose a single scalar field ϕ interacting with a single Dirac fermion Ψ via a Yukawa interaction