## III) Calculational Techniques for Green functions

A) Let's begin by making some remarks about the Gell-Mann-Low perturbation expansion for the time-ordered functions in light of our discussion of $T^*$ products. Recall that we cannot go further with our axiomatic approach, & further dynamical input is needed (Microcausality, a property of all theories, contains information about forward scattering through dispersion techniques — we will not cover this topic).

We must specify the time (and therefore space — due to relativity) evolution of our system. We can do this by either giving the field equations or equivalently the Lagrangian density (i.e. Hamiltonian density). In our canonical approach to field theory we specified the Hamiltonian

$\mathcal{H}$ which we then separated into a

free part and interacting part

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$, where the free

part described the asymptotic field dynamics.

$$-i\dot{\phi}_{in}(x) = [H_{0in}(t), \phi_{in}(x)]$$

$$-i\dot{\Pi}_{in}(x) = [H_{0in}(t), \Pi_{in}(x)]$$ as well

as allowing the Hamiltonian to be diagonalized in the sense that

$$H(\phi,\Pi) = H_{0in} = H_0(\phi_{in}, \Pi_{in}).$$

Since $\{\phi_{in}, \Pi_{in}\}$ obeyed the ETCR as did $\{\phi, \Pi\}$ we assumed they were related by a unitary transformation $U^{(t)}(t)$

$$\phi_{in}(x) = U^{(t)}(t)\,\phi(x)\,U^{(t)-1}(t) \qquad (x^0 = t).$$

The time evolution operator relating interaction picture states at different times can be transformed to the Heisenberg picture & can be written in terms of the in-fields

$$U^{iP}(t,t_0) = e^{iH_0^s t}\, e^{-iH(t-t_0)}\, e^{-iH_0^s t_0}$$

but

$$U_{in}(t,t_0) = U^{iP}(0,-\infty)\,U^{iP}(t,t_0)\,U^{iP-1}(0,-\infty)$$

(Now $H\,U(0,-\infty) = U(0,-\infty)\,H_0^s$ so that

$$U_{in}(t,t_0) = e^{iHt}\,U^{ip}(0,-\infty)\,e^{-iH(t-t_0)}\,U^{-1}(0,-\infty)\,e^{-iHt_0}$$

$$= U^{(+)}(t)\,U^{(+)-1}(t_0)\;)$$

On the other hand we have that

$$U^{ip}(t,t_0) = T\,e^{-i\int_{t_0}^{t} dt''\,H_I^{ip}(t'')}$$

So that converting to in-fields ⇒

$$U_{in}(t,t_0) = T\,e^{-i\int_{t_0}^{t} dt''\,H_I^{in}(t'')}$$

Or to return to the starting point

$$i\frac{\partial}{\partial t}U_{in}(t,t_0) = H_I^{in}(t)\,U_{in}(t,t_0)\quad,\quad U_{in}(t,t) = 1$$

has the solutions

$$U_{in}(t,t_0) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\; H_I^{in}(t_1)\cdots H_I^{in}(t_n).$$

As we can see we are still able to add

terms tollin that are quasi-local function of $(t_1 \cdots t_n)$ for the $n^{th}$ term and still satisfy the Schrödinger equation. This is trivial to see instead of just

$$H_I^{in}(t_1) \cdots H_I^{in}(t_n)$$ we have for the

$n^{th}$ order term

$$H_I^{in}(t_1) \cdots H_I^{in}(t_n) + \Lambda(t_1 \cdots t_n)$$

where $\Lambda(t_1 \cdots t_n)$ contains at least one

equal time $\delta$-function $\delta(t_i - t_j)$ since

$$\int_{t_0}^{t_{j-1}} dt_j \, \delta(t_i - t_j) \, \hat{\Lambda}(t_1 \cdots t_n) = 0$$

Since $t_j < t_i$ (assuming $i < j$) in the integral. (Actually due to symmetry one can show
$\Lambda(t_1 \cdots t_n) = \delta(t_1 - t_2)\delta(t_1 - t_3) \cdots \delta(t_1 - t_n)\hat{\Lambda}(t_1)$ + finite derivatives of $\delta$'s.)

Hence we could imagine altering the time ordered product of (Hamiltonian) fields by allowing equal time delta function contributions without affecting $U(t, t_0)$. But this is precisely the type of alteration needed to go from T products to covariant T* products. Of course these additional terms in T are subject to the constraints of the generalized unitarity & microcausality equations, the only way for ~~these~~ to be satisfied is for the quasi-local ~~terms~~ effects to be described by a change in the Lagrangian (this is the miracle of field theory)

All this sounds rather abstract

but is quite relevant and of practical importance when dealing with theories that have derivative couplings: then $H_I$ depends on $\Pi$ (i.e. $\dot\phi$) not just $\phi$. In these cases we

can show that the S-matrix

is Lorentz invariant directly by

showing that

$$T e^{-i\int d^4x\, \mathcal{H}_I^{in}} = T^* e^{+i\int d^4x\, \mathcal{L}_I^{in}}$$

Of course when no derivative coupling was involved $\mathcal{H}_I = -\mathcal{L}_I$ and $T = T^*$ and from the microcausality of the in-fields Lorentz invariance followed.

However if derivative coupling occurs $\mathcal{H}_I \neq -\mathcal{L}_I$ and the non-Lorentz invariant terms in the difference $\mathcal{H}_I + \mathcal{L}_I$ cancel the non-Lorentz invariance of $Te^{-i\int dx \mathcal{H}_I^{in}}$; in order to obtain the expression in terms of $T^*$.

In order to show this explicitly let's consider the $\phi^4$ theory but imagine we have more general kinetic energy, mass & interaction terms. We might expect this since as we already discussed

$\phi(x) \xrightarrow[x^0 \to -\infty]{LSZ} Z^{+1/2} \phi_{in}(x)$ so that

certainly the normalization of the kinetic energy terms could involve a wavefunction renormalization Z factor. At the same time we might find radiative corrections to the mass parameter and coupling constant so that the most general self-interacting scalar field theory is described by the Lagrangian density (now in terms of the renormalized field $\phi_R = Z^{-1/2}\phi$ &c)

we re-label $\phi_R \to \phi$ in what follows

$$\mathcal{L} = \frac{Z}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{(m^2+\delta m^2)}{2}\phi^2 - \frac{(\lambda+\delta\lambda)}{4!}\phi^4$$

where $Z = 1+b$, $\delta m^2 = a$, $\delta\lambda = c$ with $a, b, c$ all "counter-terms" that begin in order $\lambda$ and higher.
$m$ is the renormalized mass as $\lambda$ is the renormalized coupling constant.

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ .

Now $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \; - \frac{1}{2}m^2\phi^2$

and $\mathcal{L}_I = \frac{b}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}a\phi^2 - \frac{(\lambda+c)}{4!}\phi^4$

we have derivative coupling in the first term. The momentum is given by

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = Z\dot{\phi} .$$

So the Hamiltonian becomes

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L}$$

$$= \frac{1}{2Z}\Pi^2 + \frac{Z}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{(m^2+a)}{2}\phi^2 + \frac{(\lambda+c)}{4!}\phi^4 .$$

However $\mathcal{H}_0^{in}$ is given by the free part of $\mathcal{H}$

$$\mathcal{H}_0^{in} = \frac{1}{2}\Pi_{in}^2 + \frac{1}{2}\vec{\nabla}\phi_{in}\cdot\vec{\nabla}\phi_{in} + \frac{m^2}{2}\phi_{in}^2 .$$

Hence the interaction Hamiltonian becomes

$$\mathcal{H}_I^{in} = \mathcal{H}(\Pi_{in},\phi_{in}) - \mathcal{H}_0^{in}$$

$$= \frac{1}{2}\frac{(1-Z)}{Z}\Pi_{in}^2 + \frac{(Z-1)}{2}\vec{\nabla}\phi_{in}\cdot\vec{\nabla}\phi_{in}$$

$$+ \frac{a}{2}\phi_{in}^2 + \frac{\lambda+c}{4!}\phi_{in}^4$$

But we see this is not $\mathcal{L}_I$ in fact using $\Pi_{in} = \dot{\phi}_{in}$ we find (Recall $-i\dot{\phi}_{in} = [H_0^{in}, \phi_{in}] = -i\Pi_{in}$)

$$\boxed{\mathcal{H}_I^{in} = -\mathcal{L}_I^{in} + \frac{(1-Z)^2}{2Z}\dot{\phi}_{in}\dot{\phi}_{in}}$$

$\mathcal{H}_I^{in}$ differs from $-\mathcal{L}_I^{in}$ by <u>non-covariant</u> terms!! However the T-operator is not covariant either since time derivatives are involved. These two properties will combine to produce an invariant S-operator. Alternately we can use the quasi-local operators in the definition of U to transform to the $T^*$ operator which is covariant and in the process $Te^{-i\int_{-\infty}^{\infty} d^4x \mathcal{H}_I^{in}}$ becomes $T^* e^{+i\int_{-\infty}^{+\infty} d^4x \mathcal{L}_I^{in}}$ with $\mathcal{L}_I^{in}$ also covariant.

To be specific recall that

$$T\dot{\phi}_{in}(x)\dot{\phi}_{in}(y) = \partial_0^x \partial_0^y T\phi_{in}(x)\phi_{in}(y) - i\delta^4(x-y)$$

The T-operator is non-covariant. We can define the $T^*$ operator by (Wick T-operator)

$$T^* \overset{\circ}{\phi}_{in}(x)\, \overset{\circ}{\phi}_{in}(y) \equiv \partial_0^x \partial_0^y\, T \phi_{in}(x)\, \phi_{in}(y)$$

Hence

$$T^* \overset{\circ}{\phi}_{in}(x)\, \overset{\circ}{\phi}_{in}(y) = T\, \overset{\circ}{\phi}_{in}(x)\, \overset{\circ}{\phi}_{in}(x) + i\delta^4(x-y)$$

$T$ and $T^*$ (the Dyson & Wick T-operator) differ by the contact (quasi-local) term $i\delta^4(x-y)$.

Hence if we consider the S operator

$$T e^{-\int d^4x\, \mathcal{H}_I^{in}} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dz_1 \cdots dz_n \left(\frac{1-Z}{2Z}\right)^n$$

$$T\, \overset{\circ}{\phi}{}^2_{in}(z_1) \cdots \overset{\circ}{\phi}{}^2_{in}(z_n)\, e^{+i\int dx \left[\mathcal{L}_I^{in}(x) - \frac{(1-Z)}{2} \overset{\circ}{\phi}{}^2_{in}\right]}$$

we can imagine replacing $T$ by $T^*$ with no changes <u>except</u> when two $\overset{\circ}{\phi}_{in}$ fields have been contracted. (Above we are assuming Wick ordering so that we can ignore the self contractions for convenience)

But when pairs of $\phi_{in}$ contract we just get $\phi_{in}^2$ back again; i.e.

$$\left(\frac{1-z}{2z}\right)^2 \int dz_1 dz_2 \, T \, \phi_{in}^2(z_1) \, \phi_{in}^2(z_2) \cdots$$

$$= \left(\frac{1-z}{2z}\right)^2 \int dz_1 dz_2 \, T^* \, \phi_{in}^2(z_1) \, \phi_{in}^2(z_2) \cdots$$

$$- i\left(\frac{1-z}{2z}\right)^2 \int dz_1 dz_2 \, \phi_{in}(z_1) \, \phi_{in}(z_2) \, 2\delta(z_1 - z_2) \cdots$$

$$= \left(\frac{1-z}{2z}\right)^2 \int dz_1 dz_2 \, T^* \, \phi_{in}^2(z_1) \, \phi_{in}(z_2) \cdots$$

$$- 2i\left(\frac{1-z}{2z}\right)^2 \int dz_1 \, T^* \, \phi_{in}(z_1)^2 \cdots$$

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And so on, we just reproduce the same $\phi_{in}^2$ interaction terms with a combinatoric factor. So in general we have

$$T e^{-i\int d^4x \mathcal{H}_I^{in}}$$

$$= \sum_{l=0}^{\infty} \frac{(-i)^l}{l!} \int dz_1 \cdots dz_l \, C_l \, T^* \phi_{in}^2(z_1) \cdots \phi_{in}^2(z_l) \times$$

$$\times e^{+i\int dx \left[\mathcal{L}_I^{in}(x) - \frac{(1-Z)}{2} \dot{\phi}_{in}^2(x)\right]}$$

where for each $n \geq l$ there is a contribution to the $C_l$ term that comes from the number of ways we can ~~do~~ take the $(n-l) =$ (even integer) time derivatives on the $\theta$-functions to get ETCR of $[\dot{\phi}_{in}, \phi_{in}]$; Then removing $2(n-l)$ fields leaving $l$ vertices. So

$$C_l = l! (-i)^{-l} \sum_{n=l}^{\infty} \frac{(-i)^n}{n!} \left(\frac{1-Z}{2Z}\right)^n \underbrace{\frac{n!}{(n-l)!\, l!}} \underbrace{B_{n,l}} \underbrace{(-i)^{n-l}}_{\#\text{ of ETCR.}}$$

# of ways to pick $(n-l)$ vertices out of $n$ (i.e. pick out $(n-l)$ derivatives from $n$ terms)

# of ways to commute the $(n-l)$ pairs of fields to leave the $l$ vertices (i.e. # of ways to take $(n-l)$ derivatives)

$$C_\ell = \left(\frac{1-z}{2}\right)^\ell \sum_{n=\ell}^{\infty} \frac{(-i)^{(n-\ell)2}}{(n-\ell)!} \left(\frac{1-z}{2}\right)^{n-\ell} \left(\frac{1}{z}\right)^n B_{n,\ell}$$

$$= \left(\frac{1-z}{2}\right)^\ell \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1-z}{2z}\right)^m B_{m+\ell,\ell}\right] \frac{1}{z^\ell}$$

Now

$B_{n,\ell}$ = # of ways to take $(n-\ell)$ derivatives and commute $[\phi, \phi\phi]$. The commutator gives $2^{n-\ell}$ terms since $\phi^2$ is at each vertex.

Hence $B_{n,\ell} = 2^{n-\ell} \frac{(n-1)!}{(\ell-1)!}$

Since the $(n-\ell)$ derivatives can act on $(n-1)(n-2)\cdots(n-(n-\ell))$ terms

$$= \frac{(n-1)!}{(\ell-1)!} .$$

So we find

$$C_\ell = \left(\frac{1-z}{2}\right)^\ell \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1-z}{2z}\right)^m 2^m \frac{(m+\ell-1)!}{(\ell-1)!}\right] \frac{1}{z^\ell}$$

But

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(n+m-1)!}{(n-1)!} x^m$$

let $x = \left(\frac{1-z}{z}\right) \Rightarrow z = \frac{1}{1+x}$

$\Rightarrow$

$$z^\ell = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(m+\ell-1)!}{(\ell-1)!} \left(\frac{1-z}{z}\right)^m$$

So we have simply that

$$\boxed{C_\ell = \left(\frac{1-z}{2}\right)^\ell}$$

Hence

$$T e^{-i\int dx\, \mathcal{H}_I^{in}} = \sum_{l=0}^{\infty} \frac{(-i)^l}{l!}\left(\frac{1-Z}{2}\right)^l \times$$

$$\times T^* \phi_{in}^2(z_1)\cdots\phi_{in}^2(z_l)\, e^{+i\int dx\left[\mathcal{L}_I^{in} - \frac{(1-Z)}{2}\phi_{in}^2(x)\right]}$$

So

$$\boxed{T e^{-i\int d^4x\, \mathcal{H}_I^{in}} = T^* e^{+i\int d^4x\, \mathcal{L}_I^{in}}}$$

The desired result.

Of course the above detailed combinatoric argument gives no insight as to what cancellation is occurring. Hence consider the alternate graphical proof.

Now we will only get a problem in replacing $T$ by $T^*$ when the $\phi(z_i)\,\phi(z_j)$ commute this can happen any # of times for each such vertex in $H_I$ :

graphically we can add to each line
with one vertex
—×— any # —×— + —××— + —×××— + . . .

where we replace each $T$ by $T^*$ and each additional × above the first represents a commutator term – since each commutator has a choice of 2 fields in each vertex the factor of 2 cancels the $\frac{1}{2}$ of the vertex, also each $(-i)$ from the commutator cancels the $(-i)$ of the vertex to give $(-1)$ So we find $T($—×— + —××— + —×××— . . .$)$

$= T^*($═×═$)$

where

$$\text{---}\!\times\!\text{---} + \text{---}\!\times\!\times\!\text{---} + \cdots = -\frac{i}{2}\left(\frac{1-Z}{Z}\right)\left[1-\left(\frac{1-Z}{Z}\right)+\left(\frac{1-Z}{Z}\right)^2 - \left(\frac{1-Z}{Z}\right)^3+\cdots\right]_V$$

$$= -\frac{i}{2}\left(\frac{1-Z}{Z}\right)\frac{1}{1+\left(\frac{1-Z}{Z}\right)} T^*\int dx\, \phi_i^2(x) \quad T^*\int d^4x\, \phi_i(x)^2$$

$$= -\frac{i}{2}\left(\frac{1-Z}{\cancel{Z}}\right)\frac{\cancel{Z}}{\cancel{Z}+1-\cancel{Z}} T^*\int dx\, \phi_i^2(x)$$

$$= -\frac{i}{2} T^*\int dx\, \phi_i^2(x)\,(1-Z)$$

So we find

$$T e^{-i\int d^4x\, \mathcal{H}_I}$$

$$= T^* e^{-i\int d^4x\left[\frac{(1-Z)}{2}\dot\phi_i\dot\phi_i - \frac{(1-Z)}{2}\vec\nabla\phi_i\cdot\vec\nabla\phi_i + \frac{a}{2}\phi_i^2 + \frac{\lambda+c}{4!}\phi_i^4\right]}$$

$$= T^* e^{+i\int d^4x\left[\frac{(Z-1)}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{a}{2}\phi_i^2 - \frac{\lambda+c}{4!}\phi_i^4\right]}$$

$$= T^* e^{+i\int d^4x\, \mathcal{L}_I(\phi_i)}$$

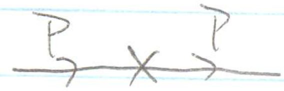
So in what follows we will always be using the covariant time ordered product $T^*$ but we will drop the $*$ superscript and just call it $T$.

Recall all derivatives move in and out of $T^*$ without any covariance difficulties.

Hence the Gell-Mann-Low formula is expressed in terms of $T^*$ operators now and is given by

$$\boxed{\begin{aligned}&\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= \frac{\langle 0|T\,\phi_{in}(x_1)\cdots\phi_{in}(x_n)\,e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}{\langle 0|T\,e^{+i\int d^4x\,\mathcal{L}_I^{in}(x)}|0\rangle}\end{aligned}}$$

Our same graphical rules apply with the addition of vertices for $\frac{a}{2}\phi_{in}^2$ & $\frac{b}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in}$

 $= -ia + ibp^2$

&  $= -i(\lambda + c)$

We can now apply this G-M-L perturbative expansion to formally express the generating functional as

$$Z[J] = \langle 0 | T e^{i\int dx\, J(x)\phi(x)} | 0 \rangle$$

$$= \frac{\langle 0 | T e^{+i\int d^4x \left[\mathcal{L}_I^{in}(x) + J(x)\phi_{in}(x)\right]} | 0 \rangle}{\langle 0 | T e^{+i\int d^4x\, \mathcal{L}_I^{in}(x)} | 0 \rangle}$$

Since

$$\mathcal{L}_I^{in} = \frac{b}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \frac{1}{2}a\phi_{in}^2 - \frac{(\lambda+c)}{4!}\phi_{in}^4$$

we can imagine expressing each

factor of a field by $\frac{\delta}{i\delta J(x)}$ since when this operates on $e^{i\int dz J(z)\phi_{in}(z)}$ it just brings down $\phi_{in}(x)$!

So the GML formula becomes

$$Z[J] = \langle 0|T e^{i\int dx \mathcal{L}_I\left(\frac{\delta}{i\delta J}\right)} e^{i\int dy J(y)\phi_{in}(y)}|0\rangle_{NVB}$$

where we have excluded vacuum bubbles.

Now the interaction lagrangian no longer involves field operators and can be brought out of the VEV

$$Z[J] = e^{i\int d^4x \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)} \langle 0|T e^{i\int dy J(y)\phi_{in}(y)}|0\rangle$$

(where it is understood we exclude vac. bubbles, that is $Z[J]\Big|_{J=0} = 1$.)

But $\phi_{in}$ are free fields; hence we know how to calculate $\langle 0 | T\, e^{i\int dy\, J(y)\phi_{in}(y)} | 0 \rangle$ from Wick's Theorem

$$\langle 0 | T\, e^{i\int dx\, J(x)\phi_{in}(x)} | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n\, J(x_1) \cdots J(x_n) \times$$

$$\times \langle 0 | T\, \phi_{in}(x_1) \cdots \phi_{in}(x_n) | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{\substack{(1 \ldots n) \xrightarrow{P} (i_1, j_1) \ldots (i_{n/2}, j_{n/2}) \\ i_1 > i_2 \cdots > i_{n/2} \\ i_a > j_a.}} \prod_{a=1}^{n/2} \int dx_{i_a} dx_{j_a}\, J(x_{i_a}) \Delta_F(x_{i_a} - x_{j_a}) J(x_{j_a})$$

Since the integration variables are dummy variables we get equivalent terms in the sum. The number of terms is just the # of permutations of $(1 \ldots n)$ into $(i_1, j_1) \ldots (i_{n/2}, j_{n/2})$ with $i_a > j_a$, $i_a > i_{a+1}$. That is the # of ways of contracting $\phi_{in}(x_1) \cdots \phi_{in}(x_n)$ into pairs. This is

$$= \frac{\overbrace{n(n-1)\cdots(n-\frac{n}{2}+1)}^{\#\text{ of } i_a\text{'s}}\ \overbrace{(n-\frac{n}{2})\cdots 1}^{\#\text{ of } j_a\text{'s}}}{\underbrace{(\frac{n}{2})!}_{i_a > i_{a+1}}\ \underbrace{2^{n/2}}_{i_a > j^a}}.$$

$$= \frac{n!}{(\frac{n}{2})!\, 2^{n/2}}$$

Hence

$$\langle 0 | T e^{i\int dx\, J(x)\, \phi_{in}(x)} | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n/2}}{(\frac{n}{2})!} \left[ \frac{1}{2} \int dx\, dy\, J(x) \Delta_F(x-y) J(y) \right]^{n/2}$$

$$= e^{-\frac{1}{2}\int dx\, dy\, J(x) \Delta_F(x-y) J(y)} \equiv Z_{in}[J]$$

Thus we have the expansion of $Z[J]$ in terms of functional derivatives

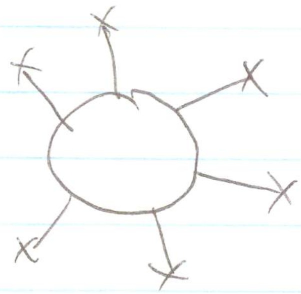
$$\boxed{\begin{aligned} Z[J] &= e^{i\int dx\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)} Z_{in}[J] \\ &= e^{i\int dx\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)} e^{-\frac{1}{2}\int dx\,dy\, J(x)\Delta_F(x-y)J(y)} \end{aligned}}$$

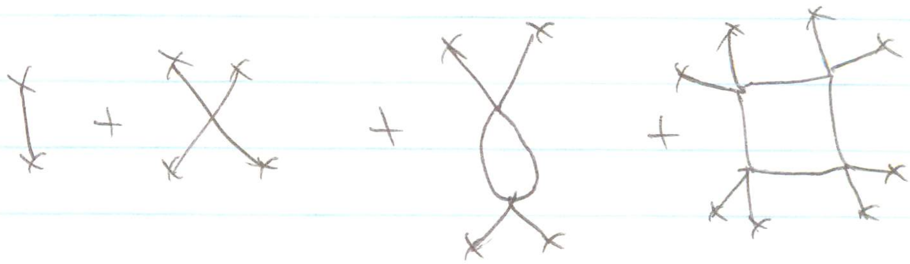
(where again it is understood we are to exclude vacuum bubbles $\left[e^{i\int dx\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)} Z_{in}[J]\right]\Big|_{J=0}$)

Clearly this reproduces our Wick's theorem expansion of the G-M-L formula. We have the interaction vertices connected by the propagators. Of course in practice if you want to evaluate $Z[J]$ you turn to our Feynman diagram expansion for $\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle$ and multiply by $\frac{i^n}{n!}\int dx_1 \cdots dx_n\, J(x_1)\cdots J(x_n)$.

But this expression often is useful for expressing properties of the generating functional & therefore Green functions.

That $Z[J]$ can be written as a sum of Feynman diagrams is often expressed by

$Z[J] = \sum_{\Gamma}$ 

= 

$+ \cdots$

where the $\longrightarrow\!\times$ cross on the end of external lines indicates that we are integrating over $\int dx$ with $J(x)$ attached.

So

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n \, J(x_1) \dots J(x_n) \times$$

$$\times \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_n}{(2\pi)^4} e^{-ip_i x_i} \sum_{\Gamma \in G^n} \alpha(\Gamma) (2\pi)^4 \delta^4(\textstyle\sum' p) \dots$$

$$\dots (2\pi)^4 \delta(\textstyle\sum' p) \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_{m(\Gamma)}}{(2\pi)^4} I_\Gamma(p,k) \; .$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_n}{(2\pi)^4} \tilde{J}(-p_1) \dots \tilde{J}(-p_n) \times$$

$$\times \sum_{\Gamma \in G^n} \alpha(\Gamma) (2\pi)^4 \delta^4(\textstyle\sum' p) \dots (2\pi)^4 \delta^4(\textstyle\sum' p) \times$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_{m(\Gamma)}}{(2\pi)^4} I_\Gamma(p,k) ,$$

with $I_\Gamma$ the Feynman integrand made as usual from the $\phi^4$ Feynman rules.

In particular the field equation imply differential equations of motion for the time ordered functions which then can be written as functional differential equations for the generating functional $Z[J]$.

The Euler-Lagrange equations of motion for $\phi(x)$ are given by

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} = 0 \equiv \frac{\partial^\varepsilon \mathcal{L}}{\partial^\varepsilon \phi}$$

$$\frac{\partial^\varepsilon \mathcal{L}}{\partial^\varepsilon \phi} = -\left(Z\partial^2 + (m^2+a)\right)\phi(x) - \frac{(\lambda+c)}{3!}\phi^3(x) = 0.$$

Now we can consider the implication of this for the time ordered functions

$$\langle 0 | T\left[-\left(Z\partial_x^2 + (m^2+a)\right)\phi(x) - \frac{\lambda+c}{3!}\phi^3(x)\right]\phi(x_1)\cdots\phi(x_n)|0\rangle$$

Since we are using $T^*$ products here

we can bring the $\partial^2$ in or out of the $T^*$ with no effect. However, we cannot when the $\partial^2$ is inside $T^*$ directly use the field equation on $\phi$ to set the above equal to zero. This is because we are not expanding the $T^*$ simply in terms of the step function times fields, there are also contact terms (quasi-local terms). However these contact terms are exactly those that arise from moving the $\partial^2$ through the $T$ operator's step functions

i.e. in the case of free in-fields

$$\partial_x^2 T^* \phi_{in}(x) \phi_{in}(y) = T^* \partial_x^2 \phi_{in}(x) \phi_{in}(y)$$

$$= T\, \partial_x^2 \phi_{in}(x) \phi_{in}(y) - i\delta^4(x-y)$$

So $(\partial_x^2 + m^2) T^* \phi_{in}(x) \phi_{in}(y)$

$$= T \underbrace{(\partial_x^2 + m^2) \phi_{in}(x)}_{\to 0} \phi_{in}(y) - i\delta^4(x-y)$$

Hence we can set the Euler-Lag. operator inside the $T^*$ operator to the T operator's contact terms. Thus we obtain

$$\boxed{\begin{aligned}&\langle 0|T\left[-(Z\partial_x^2+(m^2+\alpha))\phi(x)-\frac{\lambda+c}{3!}\phi^3(x)\right]\phi(x_1)\cdots\phi(x_n)|0\rangle\\ &=\sum_{i=1}^{n}+i\delta^4(x-x_i)\langle 0|T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)|0\rangle\end{aligned}}$$

T-operator here not $T^*$

Since $\partial_x^2 T\phi(x)\phi(x_1)\;\phi(x_n)$

$$=T\partial_x^2\phi(x)\phi(x_1)\cdots\phi(x_n)+\sum_{i=1}^{n}\delta(x^0-x_i^0)T\;\phi(x_1)\cdots[\dot\phi(x),\phi(x_i)]\cdots\phi(x_n)$$

but $\delta(x^0-y^0)[\dot\phi(x),\phi(y)]=-\frac{i}{Z}\delta^4(x-y)$ Since $\Pi=Z\dot\phi$

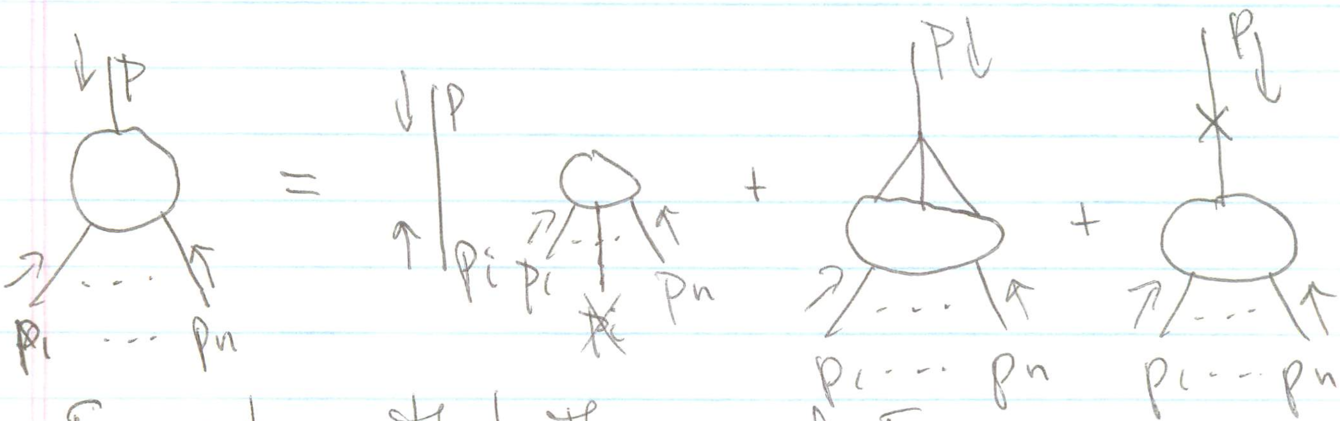
So

$$Z\partial_x^2T\phi(x)\phi(x_1)\cdots\phi(x_n)=TZ\partial_x^2\phi(x)\phi(x_1)\cdots\phi(x_n)$$

$$-i\sum_{i=1}^{n}\delta^4(x-x_i)T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)$$

We can verify this equation of motion in our perturbative expansion for the time ordered functions also

Consider the graphical contributions to $\langle 0 T\, \phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$




So we have that the general Feynman integrand is a sum of 3 types of terms

$$I_\Gamma = \frac{i}{p^2-m^2} I_\Gamma^{\hat{i}} + \frac{-i(\lambda+c)}{3!}\frac{i}{p^2-m^2} I_\Gamma^{\phi^3}$$

$$+(-ia+ibp^2)\frac{i}{p^2-m^2} I_\Gamma$$

Then if we consider $(\partial_x^2+m^2)$ acting on $\langle 0|T\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$ we see that the propagators will cancel and leave $(-i)$ in the coordinate space relation

$$(\partial_x^2+m^2)\langle 0|T\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$= -i\sum_{i=1}^{n}\delta^4(x-x_i)\langle 0|T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)|0\rangle$$

$$-\frac{(\lambda+c)}{3!}\langle 0|T\phi^3(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$-a\langle 0|T\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$-b\partial_x^2\langle 0|T\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

Recalling that $1+b=Z$ we have

$$\left[Z\partial_x^2+(m^2+a)\right]\langle 0|T\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$+\frac{(\lambda+c)}{3!}\langle 0|T\phi^3(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$= -i\sum_{i=1}^{n}\delta^4(x-x_i)\langle 0|T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)|0\rangle$$

Hence we verify that indeed the perturbative solution for $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ obeys the Green function equation of motion which followed from the field dynamics.

This is of course expected since we constructed the solution to the dynamics of the field theory,

The time evolution operator, directly from Schrödinger's equation, then we used it to derive the GML perturbative solution for the Green functions. Of course this is a perturbative solution, we could ask for a more general approach to solving the field dynamics as represented by the time ordered function equations of motion given above. This leads to the functional integral representation for the Green function.

III) B.) In order to derive a more general solution to the infinite set of Green function differential equations

$$[Z\partial_x^2 + (m^2+a)]\langle 0|T\,\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$+\frac{(\lambda+c)}{3!}\langle 0|T\,\phi^3(x)\,\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$= -i\sum_{i=1}^{n}\delta^4(x-x_i)\langle 0|T\,\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)|0\rangle$$

it is convenient to recast them into the form of a functional differential equation for the generating functional $Z[J]$.

So multiplying by $\frac{i^n}{n!} J(x_1)\cdots J(x_n)$ and integrating we have

$$\left[Z\partial_x^2 + (m^2+a)\right] \frac{i^n}{n!}\int dx_1\cdots dx_n\, J(x_1)\cdots J(x_n)$$

$$\langle 0|T\phi(x)\,\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$+\frac{(\lambda+c)}{3!}\frac{i^n}{n!}\int dx_1\cdots dx_n\, J(x_1)\cdots J(x_n)\langle 0|T\phi^3(x)\,\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$= -i\,\frac{i^n}{n!}\sum_{i=1}^{n}\int dx_1\cdots dx_n\,\delta(x-x_i)J(x_i)\,J(x_1)\cdots \cancel{J(x_i)}\cdots J(x_n)$$

$$\langle 0|T\phi(x_1)\cdots\cancel{\phi(x_i)}\cdots\phi(x_n)|0\rangle$$

Since the integration variables are dummy variables we have $n$ of the same terms on the RHS

$$= -i\,\frac{i^n}{n!}J(x)\,n\int dx_1\cdots dx_{n-1}\,J(x_1)\cdots J(x_{n-1})$$

$$\langle 0|T\phi(x_1)\cdots\phi(x_{n-1})|0\rangle$$

where we re-labelled the variables from $x_1 \cdots x_{n-1}$ and used the symmetry of $\langle 0|T\phi(x_1)\cdots\phi(x_{n-1})|0\rangle$

$$= \frac{i^{n-1}}{(n-1)!}J(x)\int dx_1\cdots dx_{n-1}\,J(x_1)\cdots J(x_{n-1})$$

$$\langle 0|T\phi(x_1)\cdots\phi(x_{n-1})|0\rangle$$

Summing over $n=0$ to $\infty$ we find

$$[Z\partial_x^2 + (m^2+a)]\langle 0|T\phi(x)\, e^{i\int dy\, J(y)\phi(y)}|0\rangle$$
$$+ \frac{(\lambda+c)}{3!}\langle 0|T\phi^3(x)\, e^{i\int dy\, J(y)\phi(y)}|0\rangle$$
$$= J(x)\langle 0|T\, e^{i\int dy\, J(y)\phi(y)}|0\rangle$$

But recall the definition of the time ordered function generating functional

$$Z[J] \equiv \langle 0|T e^{i\int dy\, J(y)\phi(y)}|0\rangle .$$

Further

$$\langle 0|T\phi(x)\, e^{i\int dy\, J(y)\phi(y)}|0\rangle = \frac{\delta}{i\delta J(x)} Z[J]$$

and $$\langle 0|T\phi^3(x)\, e^{i\int dy\, J(y)\phi(y)}|0\rangle = \frac{\delta^3}{i^3\delta J(x)^3} Z[J]$$

( Often we write the inserted generating functionals as

$$\phi(x)\, Z[J] \equiv \langle 0|T\phi(x)\, e^{i\int dy\, J(y)\phi(y)}|0\rangle$$
$$\phi^3(x)\, Z[J] \equiv \langle 0|T\phi^3(x) \quad '' \quad |0\rangle$$

$$\mathcal{L} = Z\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m^2+a)\phi^2 - \frac{(\lambda+c)}{4!}\phi^4$$

$$\Rightarrow \quad Z\partial^2\phi + (m^2+a)\phi + \frac{(\lambda+c)}{3!}\phi^3 = 0$$

Or in general

$$B_1(x_1)\cdots B_\ell(x_\ell)\, Z[J] \equiv \langle 0|T\, B_1(x_1)\cdots B_\ell(x_\ell)\, e^{i\int d^4y\, J(y)\phi(y)}|0\rangle$$

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Hence we have converted the infinite sequence of Green function field equations into a functional differential equation for $Z[J]$

$$\boxed{\left[Z\partial_x^2 + (m^2+a)\right]\frac{\delta}{i\delta J(x)}Z[J] + \frac{(\lambda+c)}{3!}\frac{\delta^3}{i^3\delta J(x)^3}Z[J] = J(x)\,Z[J]}$$

In general we solve a DE. by Fourier transforming. Since $Z[J]$ is a function of a infinite # of variables $\{J(x_1), J(x_2), \ldots\}$, that is $J(x_i)$ at each space-time lattice site, we must perform an infinite # of Fourier transforms — that is a functional Fourier transform. To do this we must introduce the idea of a functional integral, a $\infty$ product of integrals!