field theory under consideration. However before embarking on this let's make some final comments about our LSZ formulae and time ordered products.

Remarks:

1) $S^{-1} = S^{\dagger}$, S is a unitary operator.
since $\langle {}_{out}^{in} | {}_{out}^{in} \rangle = 1 \Rightarrow \sum_f |S_{fi}|^2 = 1 \Rightarrow S^{-1} = S^{\dagger}$!

In order to show this we must exploit the generalized unitarity formula for the time ordered functions.

Besides the time ordered product of fields

$$T\,\phi(x_1)\cdots\phi(x_n) \equiv \sum_P \theta(x^0_{i_1} - x^0_{i_2})\,\theta(x^0_{i_2} - x^0_{i_3})\cdots\theta(x^0_{i_{n-1}} - x^0_{i_n})\,\phi(x_{i_1})\cdots\phi(x_{i_n})$$

we can define the anti-time ordered product denoted by $\bar{T}$

$$\overline{T}\,\phi(x_1)\cdots\phi(x_n) \equiv \sum_P \theta(x^0_{i_n}-x^0_{i_{n-1}})\cdots\theta(x^0_{i_2}-x^0_{i_1})\,\phi(x_{i_1})\cdots\phi(x_{i_n}),$$

The fields are ordered from left to right by increasing time.

The complex conjugate of the time ordered function is just the anti-time ordered function i.e.

$$\left(T\,\phi(x_1)\cdots\phi(x_n)\right)^\dagger = \overline{T}\,\phi(x_1)\cdots\phi(x_n)$$

So

$$\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle^* = \langle 0|\overline{T}\,\phi(x_1)\cdots\phi(x_n)|0\rangle$$

Since the Hermitian conjugate reverses the order of the fields in the product.

We can further use a combinatoric property of the step functions to obtain a non-linear relation amongst the time ordered & anti-time ordered products

$$0 = \sum_{l=0}^{n} \frac{(-1)^l}{l!\,(n-l)!} \sum_{(1\cdots n) \xrightarrow{P} (i_1\cdots i_l)(i_{l+1}\cdots i_n)} \overline{T}[\phi(x_{i_1})\cdots\phi(x_{i_l})] \times T[\phi(x_{i_{l+1}})\cdots\phi(x_{i_n})]$$

Where $(1\cdots n) \xrightarrow{P} (i_1\cdots i_l)(i_{l+1}\cdots i_n)$ is the unordered permutation of $1\cdots n$ into the 2 sets $(i_1\cdots i_l)$ $(i_{l+1}\cdots i_n)$.

If it was an ordered permutation with in each set P^0 we could write this as

$$0 = \sum_{(1\cdots n) \xrightarrow{P^0} (i_1\cdots i_l)(i_{l+1}\cdots i_n)} (-1)^l\, \overline{T}[\phi(x_{i_1})\cdots\phi(x_{i_l})]\, T[\phi(x_{i_{l+1}})\cdots\phi(x_{i_n})] .$$

Hence taking the VEV of this and inserting a complete set of in-states between T & T we have

$$0 = \sum_{l=0}^{n} \frac{(-1)^l}{l!(n-l)!} \sum_{P} \langle 0|\overline{T}\phi(x_{i_1})\cdots\phi(x_{i_l})\left(\sum_{\alpha_1\cdots\alpha_m} |\{\alpha\}in\rangle\langle\{\alpha\}in|\right)$$

$$T\phi(x_{i_{l+1}})\cdots\phi(x_{i_n})|0\rangle$$

Next we apply the LSZ reduction techniques to the in-states

$$0 = \sum_{l=0}^{n} \frac{(-1)^l}{l!(n-l)!} \sum_{P} \sum_{\alpha_1\cdots\alpha_m} i^{2m} Z^{-m} \int dy_1\cdots dy_m \int dz_1\cdots dz_m$$

$$f_{\alpha_1}(y_1)\cdots f_{\alpha_m}(y_m) f^*_{\alpha_1}(z_1)\cdots f^*_{\alpha_m}(z_m)$$

$$K_{y_1}\cdots K_{z_m} \langle 0|\overline{T}\phi(x_{i_1})\cdots\phi(x_{i_l})\phi(y_1)\cdots\phi(y_m)|0\rangle \times$$

$$\times \langle 0|T\phi(z_1)\cdots\phi(z_m)\phi(x_{i_{l+1}})\cdots\phi(x_{i_n})|0\rangle$$

But $\sum_{\alpha} f_\alpha(y_1) f^*_\alpha(z) = i\Delta^+(y-z)$

So we have that

$$0 = \sum_{l=0}^{n} \frac{(-1)^l}{l!\,(n-l)!} \sum_{P} \sum_{m=0}^{\infty} (-i)^m Z^{-m} \int dy_1 \cdots dy_m \int dz_1 \cdots dz_m$$

$$\Big(K_{y_1} \cdots K_{y_m} \langle 0|T\phi(x_{i_1}) \cdots \phi(x_{i_l})\phi(y_1) \cdots \phi(y_m)|0\rangle\Big) \Delta^+(y_1 - z_1) \cdots \Delta^+(y_m - z_m)$$
$$\Big(K_{z_1} \cdots K_{z_m} \langle 0|T\phi(z_1) \cdots \phi(z_m)\phi(x_{i_{l+1}}) \cdots \phi(x_{i_n})|0\rangle\Big)$$

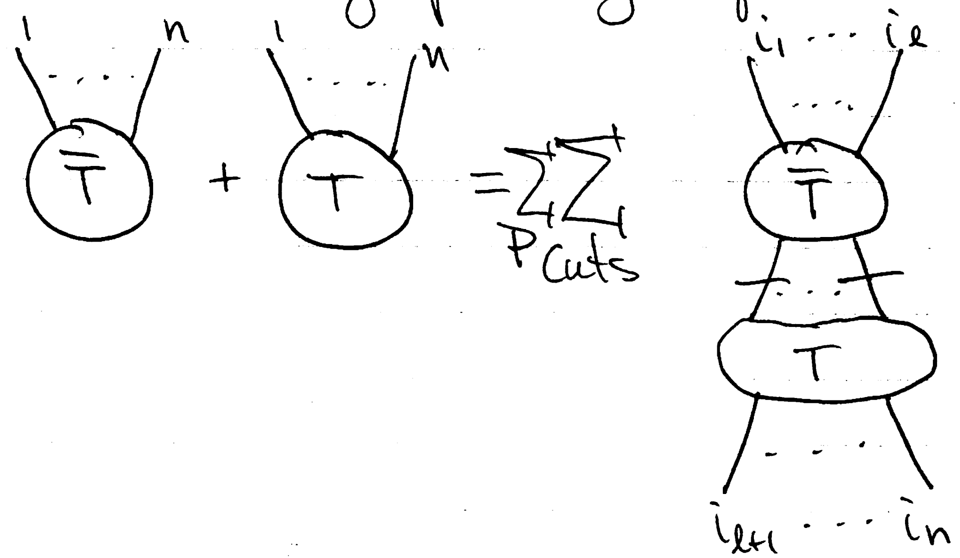
Or factoring out the $|0\rangle\langle 0|$ contribution $(m=0)$ we have

$$\boxed{\begin{aligned} &(-1)^n \langle 0|T\phi(x_1) \cdots \phi(x_n)|0\rangle + \langle 0|T\phi(x_1) \cdots \phi(x_n)|0\rangle \\ &= -\sum_{l=1}^{n-1} \frac{(-1)^l}{l!\,(n-l)!} \sum_{\substack{(1 \cdots n) \\ P (i_1 \cdots i_l)(i_{l+1} \cdots i_n)}} \sum_{m=0}^{\infty} \frac{(-i)^m}{Z^m} \int dy_1 \cdots dy_m \int dz_1 \cdots dz_m \\ &\Big(K_{y_1} \cdots K_{y_m} \langle 0|T\phi(x_{i_1}) \cdots \phi(x_{i_l})\phi(y_1) \cdots \phi(y_m)|0\rangle\Big) \Delta^+(y_1 - z_1) \cdots \Delta^+(y_m - z_m) \\ &\Big(K_{z_1} \cdots K_{z_m} \langle 0|T\phi(z_1) \cdots \phi(z_m)\phi(x_{i_{l+1}}) \cdots \phi(x_{i_n})|0\rangle\Big) \end{aligned}}$$

These are called the <u>generalized unitarity relations</u>

(If we recall our Gell-Mann-Low perturbative expansion for $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ and similarly the expansion for $\langle 0|\bar{T}\phi(x_1)\cdots\phi(x_n)|0\rangle$ $\left(\frac{i}{p^2-m^2+i\epsilon} \to \frac{-i}{p^2-m^2-i\epsilon},\ -i\lambda \to +i\lambda\right)$

this can be graphically represented as



where $+$ is called a cut line

and means we have replaced Δ_F with Δ^+ for these lines. Then the generalized unitarity relations are referred to as the (Veltman) <u>cutting formula</u>)

Now suppose we get back to our original point if the Green functions obey the generalized unitarity relation then $S^\dagger S = 1$ follows; i.e.

$$\langle\{\gamma\}\text{in}|S^\dagger S|\{\beta\}\text{in}\rangle$$
$$= \sum_{\alpha_1 \cdots \alpha_m} \langle\{\gamma\}\text{in}|S^\dagger|\{\alpha\}\text{in}\rangle\langle\{\alpha\}\text{in}|S|\{\beta\}\text{in}\rangle$$

Applying the LSZ reduction formula to these in-states we'd obtain the above formula so that

$$= \langle\{\gamma\}\text{in}|\{\beta\}\text{in}\rangle$$

Hence $S^\dagger S = 1$ or $S^\dagger = S^{-1}$.

2) So we have gone from a given field theory that obeys ~~the~~ 4 axioms to relate the matrix elements of operators, in particular the S-matrix, to the VEV of the time ordered products of field operators. How about the other way around? Suppose we ask for the properties a given set of functions (candidates for the Green functions) should have so that we are able to reconstruct the quantum field operators $\phi(x)$ from the functions and further be assured that the VEV of the time ordered products of these reconstructed fields are indeed these same candidate functions and that all the axioms are obeyed. The conditions that the functions must obey were given by Glaser, Lehmann and Zimmermann in the GLZ Reconstruction Theorem.

We will not cover the theorem here but simply state that beyond the obvious time orderedness, Poincaré invariance properties

The candidate time ordered functions should obey the generalized unitarity conditions.

(Actually GLZ stated their theorem in the framework of retarded functions since locality of the field theory is easier to demonstrate. They stated that the necessary & sufficient conditions that the retarded functions $r(x\,x_1 \ldots x_n)$ must obey in order that they define a local quantum field theory that satisfies the asymptotic conditions. The $r(x \ldots x_n)$ are

1) real symmetric and invariant functions of the difference variables $\xi_i = x - x_i \quad i = 1, \ldots, n$.

2) they are retarded $r(\xi_1 \cdots \xi_n)$ vanishes unless all ξ_i are in the forward light cone V^+

3) they obey the generalized unitarity equations for retarded functions — analogous to those for the time ordered functions.

4) if $\tilde{r}(k_1 \ldots k_n)$ is the F.T of $K_x K_{x_1} \ldots K_{x_n} r(x - x_1, \ldots x - x_n)$ then

$\tilde{r}$ is finite on the mass shell

$k_i^2 = m^2$, $\left(\sum_{i=1}^{n} k_i\right)^2 = m^2$ and does not depend on the order in which the limits $k_i^2 \to m^2$, $\left(\sum_i k_i\right)^2 \to m^2$ are taken.

Then $\phi(x)$ defined by the Yang-Feldman eq.

$$\phi(x) = Z^{1/2}\phi_{in}(x) + \sum_{n=2}^{\infty} \frac{1}{n!}\int dx_1 \cdots dx_n$$

$$Z^{-1/2} K_{x_1} \cdots Z^{-1/2} K_{x_n}\, r(x, x_1 \cdots x_n) :\phi_{in}(x_1) \cdots \phi_{in}(x_n):$$

and the $r(x, x_1 \cdots x_n)$ are its retarded functions and $\phi(x) \xrightarrow[x^0 \to \pm\infty]{LSZ} Z^{1/2}\phi_{\substack{out \\ in}}(x)$, it is local and Lorentz covariant.

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3) We can more conveniently express the operator

$STB_1(x_1)\cdots B_\ell(x_\ell)$ by introducing the generating functional for Green functions. Suppose we introduce the formal sum (it does not have to converge)

$$Z_B[J]$$
$$\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4y_1 \cdots d^4y_n\, J(y_1)\cdots J(y_n)\times$$
$$\times \langle 0|T B_1(x_1)\cdots B_\ell(x_\ell)\, \phi(y_1)\cdots\phi(y_n)|0\rangle$$
$$= \langle 0|T B_1(x_1)\cdots B_\ell(x_\ell)\, e^{i\int d^4y\, J(y)\phi(y)}|0\rangle$$

more succinctly written.

Note that we can recover a particular Green function by differentiating the generating functional wrt $J(y)$ and then setting $J=0$ after all the derivatives are taken

$$\langle 0 | T B_1(x_1) \cdots B_\ell(x_\ell) \phi(y_1) \cdots \phi(y_n) | 0 \rangle$$

$$= \frac{\delta}{i\delta J(y_1)} \cdots \frac{\delta}{i\delta J(y_n)} Z_B[J] \Big|_{J=0}$$

where the functional derivative

$$\frac{\delta}{\delta J(y)} J(x) \equiv \delta^4(x-y) .$$

[That is a functional is a mapping of a set of point sets onto a point set;

$J(x)$ is a function of x

if we consider J to be an independent function at each x_i; we have a set of functions $J(x_1) \cdots J(x_k)$ for x^μ continuous we have an ∞ set of functions

Then $Z[J]$ is a function of ∞ variables or a functional. The variation in J at one point y is $J'(x) = J(x) + \epsilon\delta(x-y)$

then $$\frac{\delta J(x)}{\delta J(y)} = \lim_{\epsilon \to 0} \frac{J'(x) - J(x)}{\epsilon} = \delta(x-y)$$

Equivalently we can define the variation of a functional to be the Functional Taylor expansion $\delta Z[J] = Z[J+\delta J] - Z[J]$

$$\delta Z[J] = \int \frac{\delta Z[J]}{\delta J(y)} \delta J(y)\, dy$$

let $Z[J] = J(x)$ then

$$\delta J(x) = \int \frac{\delta J(x)}{\delta J(y)} \delta J(y)\, dy$$

$$\Rightarrow \quad \frac{\delta J(x)}{\delta J(y)} = \delta(x-y)$$

Note: $\dfrac{\delta}{\delta J(y)} \partial_\mu^x J(x) = \partial_\mu^x \delta^4(x-y)$ etc.

Then we can write the S-operator as

$$\boxed{\begin{aligned} & S\,T\,B_1(x_1)\cdots B_\ell(x_\ell) \\ & = N\left[e^{\int d^4y\,\phi_{in}(y)\,Z^{-1/2}K_y\frac{\delta}{\delta J(y)}}\right] Z_B[J]\Big|_{J=0} \end{aligned}}$$

And in particular

$$S = N\left[e^{\int d^4y\,\phi_{in}(y)\,Z^{-1/2}K_y\frac{\delta}{\delta J(y)}}\right] Z[J]\Big|_{J=0}.$$

In particular we can return to our eq. for STB and let $B_i = \phi$ and mult. by J & sum up

$$S T e^{i\int J(x)\phi(x)\,dx}$$

$$\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n\, J(x_1)\cdots J(x_n)\, S T \phi(x_1)\cdots\phi(x_n)$$

$$= :e^{\int d^4y\, \phi_{in}(y)\, Z^{-1/2} K_y \frac{\delta}{\delta J(y)}}: Z[J]$$

note: $J \neq 0$ here

$$2)\ Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n\, J(x_1)\cdots J(x_n)$$
$$\langle 0| T\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$= \langle 0| T e^{i\int dx\, J(x)\phi(x)} |0\rangle$$

So expanding the RHS we have

$$= \sum_{n,m=0}^{\infty} \frac{1}{n!}\frac{i^m}{m!} \int dy_1 \cdots dy_n :\phi_{in}(y_1)\cdots\phi_{in}(y_n): Z^{-1/2}K_{y_1}\cdots Z^{-1/2}K_{y_n} \times$$
$$\times \frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)} \int dz_1\cdots dz_m\, J(z_1)\cdots J(z_m)\langle 0|T\phi(z_1)\cdots\phi(z_m)$$

Now we take the n derivatives to obtain

$\frac{m!}{(m-n)!}$ identical terms due to symmetry of $\langle T \quad \rangle$

So

$$= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{1}{n!} \frac{i^m}{(m-n)!} \int dz_1 \cdots dz_{m-n} J(z_1) \cdots J(z_{m-n})$$

$$\int dy_1 \cdots dy_n :\phi_{in}(y_1) \cdots \phi_{in}(y_n): Z^{-1/2} K_{y_1} \cdots Z^{-1/2} K_{y_n} \langle T\phi(y_1) \cdots \phi(y_n) \cdots \phi(z_1) \cdots \phi(z_{m-n}) \rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{l=0}^{\infty} \frac{i^l}{l!} \int dz_1 \cdots dz_l \, J(z_1) \cdots J(z_l) \times$$

$$\int dy_1 \cdots dy_n :\phi_{in}(y_1) \cdots \phi_{in}(y_n): \cdots Z^{-1/2} K_{y_n} \langle T\phi(y_1) \cdots \phi(z_l) \rangle$$

which is just $S T e^{i\int J(x)\phi(x)\,dx}$.

Thus we have defined the operator

$S T e^{i\int J(x)\phi(x)\,dx}$ in terms of its VEV $\langle 0 | T e^{i\int J\phi\,dx} | 0 \rangle$

($\langle 0 | S = \langle 0 |$)

in our functional approach to field theory. We will need some more dynamics in order to find the Green functions.

4.) Finally our last remark has to do with the Lorentz invariance of our timeordered product. Recall the S matrix is to be Lorentz invariant, if this is so $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \langle 0|T\phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)|0\rangle$ should also be Lorentz inv:

$$\langle in|S|in\rangle = \langle in'|S|in'\rangle = \langle in|U^{-1}SU|in\rangle$$

i.e.

$$S = \sum_n \frac{i^n}{n!}\int dx_1\cdots dx_n\, Z^{-1/2}K_{x_1}\cdots Z^{-1/2}K_{x_n}$$

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle :\phi_{in}(x_1)\cdots\phi_{in}(x_n):$$

$$U(\Lambda,a)SU^{-1}(\Lambda,a) = \sum_n \frac{i^n}{n!}\int dx_1\cdots dx_n\, Z^{-1/2}K_{x_1}\cdots Z^{-1/2}K_{x_n}$$

after change of variables $\langle 0|T\phi(\Lambda^{-1}x_1-a)\cdots\phi(\Lambda^{-1}x_n-a)|0\rangle :\phi_{in}(x_1)\cdots\phi_{in}(x_n):$

Thus if $USU^{-1} = S$ then $UT\phi\cdots\phi U^{-1} = T\phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)$

So consider the 2-point function $TA(x)B(y)$

$T A(\Lambda x+a) B(\Lambda y+a)$

$= \Theta(\Lambda x^0 - \Lambda y^0)\, A(\Lambda x+a) B(\Lambda y+a) + \Theta(\Lambda y^0 - \Lambda x^0) B(\Lambda y+a) A(\Lambda x+a)$

$= U(\Lambda, a)\left\{ \Theta(\Lambda(x-y)^0)\, A(x) B(y) + \Theta(\Lambda(y-x)^0)\, B(y) A(x) \right\}$
$U^{-1}(\Lambda, a)$

Now 1) if $(x-y)^2 > 0$ and $x^0 > y^0$ then

$x'^0 > y'^0$ i.e. $\Theta(x^0 - y^0) = \Theta(\Lambda(x-y)^0)$

Similarly if $x^0 < y^0 \Rightarrow x'^0 < y'^0$ and

So for timelike separation $\Theta(\Lambda(x^0-y^0)) = \Theta(x^0-y^0)$

2) if $(x-y)^2 < 0$ then $(x-y)^0$ can

change sign so Θ's
change order of A B
but for spacelike separation $[A(x), B(y)] = 0$
by microcausality principle

So the timeorder product for space-like separators is AB or BA.

3) for $(x-y)^2=0$ if $(x-y)^0>0$; $(x-y)^0=|\vec{x}-\vec{y}|$

then

$z=x-y$ $\quad z'^0=\gamma\left(z^0-\frac{\vec{v}\cdot\vec{z}}{c^2}\right)$

$$=\gamma|\vec{z}|\left(1-\frac{\vec{v}\cdot\vec{z}}{|\vec{z}|}\right)$$

$$>\gamma|\vec{z}|(1-|\vec{v}|)$$

So for $\vec{z}\neq 0$ the ordering is preserved

So we find that

$$T\,A(\Lambda x+a)\,B(\Lambda y+a)$$

$$=U(\Lambda,a)\,T\,A(x)B(y)\,U^{-1}(\Lambda,a)$$

except possibly at $x^\mu=y^\mu$ (unless $[A,B]=0$ their too)

However this is just a single point and it will not contribute to S. That is we can always define a new Time ordering operator

$$T^* A(x) B(y) \doteq T A(x) B(y) + C^{n}(x)\, \delta^{4(n)}(x-y)$$

That is, we know the Lorentz non-invariance is contained at one point and a distribution with support at one point is a sum of δ's & their derivatives. This will not change the S-operator.

First an example which we will make use of later: asymptotic fields

$$T\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y)$$

$$= \theta(x^0-y^0)\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y) + \theta(y^0-x^0)\, \partial^y_\nu \phi_{in}(y)\, \partial^x_\mu \phi_{in}(x)$$

But

$$\partial^x_\mu \partial^y_\nu\, T \phi_{in}(x) \phi_{in}(y) = \partial^x_\mu \partial^y_\nu \left[\theta(x^0-y^0)\, \phi_{in}(x) \phi_{in}(y) + \theta(y^0-x^0)\, \phi_{in}(y) \phi_{in}(x) \right]$$

$$= \partial^x_\mu \Big[g_{\nu 0} - \delta(x^0-y^0)\, \phi_{in}(x) \phi_{in}(y) + g_{\nu 0}\, \delta(y^0-x^0)\, \phi_{in}(y) \phi_{in}(x) + \theta(x^0-y^0)\, \phi_{in}(x) \partial^y_\nu \phi_{in}(y) + \theta(y^0-x^0)\, \partial^y_\nu \phi_{in}(y)\, \phi_{in}(x) \Big)$$

but $\delta(x^0-y^0)[\phi_{in}(x), \phi_{in}(y)] = 0$
So

$\partial^x_\mu \partial^y_\nu T\, \phi_{in}(x)\, \phi_{in}(y)$

$= \partial^x_\mu [\theta(x^0-y^0)\, \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y) + \theta(y^0-x^0)\, \partial^y_\nu \phi_{in}(y)\, \phi_{in}(x)]$

$= T\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y) + g_{\mu 0}\, \delta(x^0-y^0)[\phi_{in}(x), \partial^y_\nu \phi_{in}(y)]$

$= T\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y) + g_{\mu 0} g_{\nu 0}\, i\delta^4(x-y)$

We see that $T\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y)$
have a non-covariant term
$g_{\mu 0} g_{\nu 0}\, i\delta^4(x-y)$ at the $x^\mu = y^\mu$ point.
we can define a new product

$T^*\, \partial^x_\mu \phi_{in}(x)\, \partial^y_\nu \phi_{in}(y) \equiv \partial^x_\mu \partial^y_\nu T\, \phi_{in}(x)\, \phi_{in}(y)$

which is covariant by subtracting off this non-covariant piece

So

$$T^* \partial_\mu^x \phi_i(x) \partial_\nu^y \phi_i(y)$$
$$= T \partial_\mu^x \phi_i(x) \partial_\nu^y \phi_i(y) + g_{\mu 0} g_{\nu 0}\, i \delta^4(x-y)$$

Now suppose we add such a term to the timeordered functions how will it contribute to S

So let's consider

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \to \langle\ \rangle + \delta(x_1 - x_2)\, d(x_2 \cdots x_n)$$

So this new term contributes to the FT

$\langle 0|T\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)|0\rangle$

$$= \int dx_1 dx_2\, e^{+ip_1x_1} e^{ip_2x_2} \langle 0|T\phi(x_1)\phi(x_2)\tilde{\phi}(p_3)\cdots|0\rangle$$

$$= \int dx_1 dx_2\, e^{+ip_1x_1} e^{+ip_2x_2}\, \delta(x_1-x_2)\, d(x_1,\tilde{p}_3\cdots)$$

$$= \int dx_1\, e^{i(p_1+p_2)x_1}\, d(x_1,\tilde{p}_3\cdots)$$

$$= \tilde{d}(p_1+p_2,p_3\cdots)$$

Now

$\langle \text{out}\{p\}|\{q\}\text{in}\rangle$

$$\sim (p_1^2-m^2)(p_2^2-m^2)\cdots\langle 0|T\tilde{\phi}(p_1)\tilde{\phi}(p_2)\cdots|0\rangle\Big|_{p_i^2=m^2}$$

$$\sim (p_1^2-m^2)(p_2^2-m^2)\,\tilde{d}(p_1+p_2,p_3\cdots)\Big|_{p_i^2=m^2}$$

$\tilde{d}$ no longer contains poles at $p_i^2 = m^2$

So this vanishes.

i.e. Pick PCAC the non-covariant terms in $T\partial_\mu j^\mu \partial_\nu j^\nu$ vanish on shell.

$$\langle \text{out}\, p_1p_2|q_1q_2\,\text{in}\rangle \sim (p_1^2-m_\pi^2)(p_2^2-m_\pi^2)\langle 0|T\,\widetilde{\partial_\mu j^\mu}(p_1)\,\widetilde{\partial_\nu j^\nu}(p_2)\cdots$$

So these quasi-local terms do not contribute to S. Thus we could write S in terms of T or T^*; it will still be covariant. Since we will deal with $\langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle$ directly, it is often convenient to use the manifestly covariant definition. Hence in what follows, unless otherwise stated, we will use the covariant time ordered product T^*, but of course we will drop the superscript $*$ from it.

Of course $T\, B_1(x_1) \cdots B_\ell(x_\ell)$ will have matrix els. that are not invariant since the x_i are not reduced as are the in-out-states and so we should consider $T^* B_1(x_1) \cdots B_\ell(x_\ell)$ if we need manifest Lorentz covariance.

Thus we have reduced our field theoretic problem to one of calculating VEV's of time ordered products of fields. As we have seen the generating functional for Green functions $Z_B[J]$ plays a central role in the theoretical formulation of the theory. We now must develop techniques to calculate $Z_B[J]$.