Similarly we could have reduced out an outgoing particle to obtain

$$\boxed{\begin{aligned} S_{\alpha\beta} &= \langle \alpha_1 \cdots \alpha_{m-1}\, \text{out} |\, a^{in}_{\alpha_m} | \{\beta\}\, \text{in} \rangle \\ &+ i \int d^4x\, f^*_{\alpha_m}(x)\, Z^{-1/2} (\partial_x^2 + m^2) \langle \alpha_1 \cdots \alpha_{m-1}\, \text{out} | \phi(x) | \{\beta\}\, \text{in} \rangle \end{aligned}}$$

These are the <u>LSZ Reduction Formulae.</u>

We next continue reducing out the particles from the in- & out- states

for example we consider

$$\langle \{\alpha\}\text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1}\, \text{in} \rangle$$

$$= \langle \{\alpha\}\text{out} | \phi(x)\; a^{in\,\dagger}_{\beta_{n-1}} | \beta_1 \cdots \beta_{n-2}\, \text{in} \rangle$$

$$= \lim_{t \to -\infty} Z^{-1/2} \langle \{\alpha\}\text{out} | \phi(x)\, a^{\dagger}_{\beta_{n-1}}(t) | \beta_1 \cdots \beta_{n-2}\, \text{in} \rangle$$

$$= \lim_{t \to -\infty} -i Z^{-1/2} \int_{y^0 = t} d^3y\, f_{\beta_{n-1}}(y) \overleftrightarrow{\partial}_{y^0} \langle \{\alpha\}\text{out} | \phi(x) \phi(y) | \beta_1 \cdots \beta_{n-2}\, \text{in} \rangle$$

Now since $y^0 \to -\infty$ we have that $\phi(x)\phi(y)$ is in time order so we can replace $\phi(x)\phi(y) = T\phi(x)\phi(y)$ here. (Alternately we could replace $\phi(x)\phi(y) \to \theta(x^0-y^0)[\phi(x), \phi(y)]$ called the retarded product $R\phi(x)\phi(y)$ to obtain an alternate form of the reduction formulae based on R-products instead of T-products) Note that this operation is covariant as long as $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ since the sign of (x^0-y^0) is invariant for $(x-y)^2 \geq 0$. This commutivity at space-like separation of the fields is known as the principle of microcausality and we must make it one of our assumptions

Axiom 4: Principle of Microcausality

(or local commutivity)

The quantum field operators commute with each other at space-like separated points

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0.$$

This just demands that observables made from ϕ commute at $(x-y)^2 < 0$ meaning that measurements made at spacelike separations cannot interfer with each other since no signal with speed $\leq c$ can be sent between them; they are compatible.

Further this implies that

$[\dot{\phi}(x), \phi(y)]\big|_{x^0=y^0}$ (i.e. at space-like separated points) must vanish for all $\vec{x} \neq \vec{y}$. Hence $\vec{x}=\vec{y}$ is the only point the commutator can be non-zero. Thus it must be proportional to $\delta^3(\vec{x}-\vec{y})$ or a finite # of derivatives thereof. Whether the constants of proportionality are c-numbers or operators we don't know. The simplest choice is just a constant

$$\delta(x^0-y^0)\,[\dot{\phi}(x), \phi(y)] = -i\,\delta^4(x-y)$$

For now however we need only assume
$[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$.

Back to the reduction formula — So we have

$\langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$

$$= \lim_{t \to -\infty} -iZ^{-1/2} \int_{y^0 = t} d^3y \, f_{\beta_{n-1}}(y) \overleftrightarrow{\partial_{y^0}} \langle \{\alpha\} \text{out} | T \phi(x) \phi(y) | \beta_1 \cdots \beta_{n-2} \text{ in} \rangle.$$

Proceeding as previously using the surface integral

$$= +iZ^{-1/2} \int d^4y \, \partial_{y^0} \left[f_{\beta_{n-1}}(y) \overleftrightarrow{\partial_{y^0}} \langle \{\alpha\} \text{out} | T \phi(x) \phi(y) | \beta_1 \cdots \beta_{n-2} \text{ in} \rangle \right]$$

$$- iZ^{-1/2} \lim_{t \to +\infty} \int_{y^0 = t} d^3y \, f_{\beta_{n-1}}(y) \overleftrightarrow{\partial_{y^0}} \langle \{\alpha\} \text{out} | T \phi(x) \phi(y) | \beta_1 \cdots \beta_{n-2} \text{ in} \rangle$$

In the second term on the RHS $y^0 \to +\infty$ now so $T\,\phi(x)\phi(y) = \phi(y)\phi(x)$ and

we can apply the outgoing asymptotic condition

to obtain $\langle\{\alpha\}\text{out}|\phi(x)|\beta_1\cdots\beta_{n-1}\text{ in}\rangle$

$$= \langle\{\alpha\}\text{out}|a^{\text{out}\,\dagger}_{\beta_{n-1}}\,\phi(x)|\beta_1\cdots\beta_{n-2}\text{ in}\rangle$$

$$+ iZ^{-1/2}\int d^4y\,\left(f_{\beta_{n-1}}(y)\,\partial^2_{y^0} - \ddot{f}_{\beta_{n-1}}(y)\right)\langle\{\alpha\}\text{out}|$$

$$T\,\phi(x)\phi(y)\,|\beta_1\cdots\beta_{n-2}\text{ in}\rangle$$

Again applying the K-G equation to $\ddot{f}_{\beta_{n-1}} = -\nabla^2 f_{\beta_{n-1}} + m^2 f_{\beta_{n-1}}$

and integrating by parts with surface terms vanishing yields

$$\boxed{\begin{aligned}&\langle\{\alpha\}\text{out}|\phi(x)|\beta_1\cdots\beta_{n-1}\text{ in}\rangle\\ &= \langle\{\alpha\}\text{out}|a^{\text{out}\,\dagger}_{\beta_{n-1}}\,\phi(x)|\beta_1\cdots\beta_{n-2}\text{ in}\rangle\\ &+ iZ^{-1/2}\int d^4y\,f_{\beta_{n-1}}(y)\,(\partial_y^2+m^2)\langle\{\alpha\}\text{out}|T\,\phi(x)\phi(y)\\ &\qquad|\beta_1\cdots\beta_{n-2}\text{ in}\rangle.\end{aligned}}$$

Substituting this into our first LSZ formula we find

$$\boxed{\begin{aligned} S_{\alpha\beta} &= \langle\{\alpha\}\text{out}|\,a^{\text{out}\dagger}_{\beta_n}\,|\beta_1\cdots\beta_{n-1}\,\text{in}\rangle \\ &+ iZ^{-1/2}\int d^4x\, f_{\beta_n}(x)\,(\partial_x^2+m^2)\langle\{\alpha\}\text{out}|\,a^{\text{out}\dagger}_{\beta_{n-1}}\,\phi(x)\,|\beta_1\cdots\beta_{n-2}\,\text{in}\rangle \\ &+ i^2 Z^{-1/2}Z^{-1/2}\int d^4x\int d^4y\; f_{\beta_n}(x)\,f_{\beta_{n-1}}(y)\,(\partial_x^2+m^2)(\partial_y^2+m^2) \\ &\qquad\qquad \langle\{\alpha\}\text{out}|\,T\,\phi(x)\,\phi(y)\,|\beta_1\cdots\beta_{n-2}\,\text{in}\rangle \end{aligned}}$$

The first 2 terms on the RHS correspond to scattering processes in which the f_{β_n} or the $f_{\beta_{n-1}}$ particle pass through the interaction without scattering. The third term has the particles entering the interaction.

In order to simplify the algebra a log let's choose $\alpha_i \neq \beta_j$ for all i & j

Hence $\langle\{\alpha\}\text{out}|\,a^{\text{out}\dagger}_{\beta_n} = 0$

$\langle\{\alpha\}\text{out}|\,a^{\text{out}\dagger}_{\beta_{n-1}} = 0$ etc.

Since $\langle f_\alpha | f_\beta \rangle = \delta_{\alpha\beta}$.

So for all different particles we find

$$\boxed{S_{\alpha\beta} = i^2 Z^{-1/2} Z^{-1/2} \int d^4x\, d^4y\, f_{\beta_n}(x) f_{\beta_{n-1}}(y) (\partial_x^2 + m^2)(\partial_y^2 + m^2) \langle \{\alpha\}\, out | T\phi(x)\phi(y) | \beta_1 \cdots \beta_{n-2}\, in \rangle}$$

Similarly if we reduced out 2 outgoing particles we would find

$$S_{\alpha\beta} = i^2 Z^{-1/2} Z^{-1/2} \int d^4x\, d^4y\, f^*_{\alpha_m}(x) f^*_{\alpha_{m-1}}(y) (\partial_x^2 + m^2)(\partial_y^2 + m^2) \langle \alpha_1 \cdots \alpha_{m-2}\, out | T\phi(x)\phi(y) | \{\beta\}\, in \rangle.$$

Continuing the reduction procedure until no in- or out- particles are left in the states we obtain the final form of the LSZ reduction formulae

$$\boxed{\begin{aligned} S_{\alpha\beta} = {} & i^{m+n} Z^{-\left(\frac{m+n}{2}\right)} \int d^4x_1 \cdots d^4x_m\, d^4y_1 \cdots d^4y_n \\ & f^*_{\alpha_1}(x_1) \cdots f^*_{\alpha_m}(x_m)\, f_{\beta_1}(y_1) \cdots f_{\beta_n}(y_n) \\ & (\partial^2_{x_1} + m^2) \cdots (\partial^2_{x_m} + m^2)(\partial^2_{y_1} + m^2) \cdots (\partial^2_{y_n} + m^2) \times \\ & \times \langle 0 | T \phi(x_1) \cdots \phi(x_m) \phi(y_1) \cdots \phi(y_n) | 0 \rangle \end{aligned}}$$

As we obtained in Perturbation theory.

Now that integrations by parts have been carefully performed we can go to the plane wave limit. The wave packets then become

$$f_\alpha(x) \to f_k(x) = e^{-ikx}$$

So

$$S_{\alpha\beta} = (-i)^{m+n} Z^{-1/2} (q_1^2 - m^2) \cdots Z^{-1/2} (p_n^2 - m^2)$$

$$\langle 0 | T \tilde{\phi}(q_1) \cdots \tilde{\phi}(q_m) \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) | 0 \rangle \Big|_{\text{mass shell}}$$

From our perturbative analysis of the Feynman-Dyson expansion of the S-matrix elements and that of the Gell-Mann-Low expansion for the time ordered functions we also arrived at this LSZ formula. In fact if we recall that the bare field renormalization theory of S_{fi} led to a finite expression since the external lines with careful use of the adiabatic hypothesis had factors of $Z^{1/2}$ associated with them, ex:

QED: $\chi_R = Z^{1/2}\chi$. On the other hand the time ordered functions had exactly the same Feynman rules except the external lines were full propagators that close to mass shell went like $\Delta_F' = \frac{iZ}{p^2-m^2+i\epsilon}$.

There is an extra factor of $Z^{1/2}$ for each ext. line. Hence amputating the lines i.e. multiplying the time ordered function by $(-i)(p^2-m^2)$ and removing the extra $Z^{1/2}$ by multiplying by $Z^{-1/2}$ for each ext. line yields the finite (renormalized) S-matrix elements

$$S_{\alpha\beta} = (-i)^{m+n} Z^{-1/2}(q_1^2-m^2)\cdots Z^{-1/2}(p_n^2-m^2)\times$$

$$\times\langle 0|T\,\tilde\Phi(-q_1)\cdots\tilde\Phi(-q_m)\,\tilde\Phi(p_1)\cdots\tilde\Phi(p_n)|0\rangle\Big|_{\forall\, q_i^2=m^2,\ p_j^2=m^2}$$

Of course we can apply the reduction procedure to any matrix elements of T-products of operators to obtain

$$\langle\{\alpha\}\text{out}|T\,B_1(x_1)\cdots B_\ell(x_\ell)|\{\beta\}\text{ in}\rangle$$
$$= i^{m+n}\int d^4y_1\cdots d^4y_m\,d^4z_1\cdots d^4z_n$$
$$f^*_{\alpha_1}(y_1)\cdots f^*_{\alpha_m}(y_m)\,f_{\beta_1}(z_1)\cdots f_{\beta_n}(z_n)$$
$$Z^{-1/2}K_{y_1}\cdots Z^{-1/2}K_{y_m}\,Z^{-1/2}K_{z_1}\cdots Z^{-1/2}K_{z_n}\times$$
$$\times\langle 0|T\,B_1(x_1)\cdots B_\ell(x_\ell)\,\phi(y_1)\cdots\phi(y_m)\,\phi(z_1)\cdots\phi(z_n)|0\rangle$$

Hence we can define an operator by means of its matrix elements as usual. The LSZ Reduction formulae then relates these to the operators time ordered functions.

By Axiom 2 Asymptotic completeness we have that ϕ_{in} or ϕ_{out} form an irreducible operator ring. That is we can expand any operator in terms of polynomials of $\phi_{\substack{in\\out}}$. The S-operator is no exception (or $\phi(x)$ for that matter)

So the S operator, defined by

$$S_{\alpha\beta} = \langle \{\alpha\} in | S | \{\beta\} in \rangle = \langle \{\alpha\} out | \quad \{\beta\} in \rangle ,$$

has an expansion

$$S = \sum_{k=0}^{\infty} \frac{1}{k!} \int d^4x_1 \cdots d^4x_k \, \sigma(x_1, \dots, x_k) \, :\phi_{in}(x_1) \cdots \phi_{in}(x_k):$$

we can determine the $\sigma(x_1, \dots, x_k)$ by the LSZ reduction formula for $S_{\alpha\beta}$.

$$S_{\alpha\beta} = \sum_{k=0}^{\infty} \frac{1}{k!} \int dx_1 \cdots dx_k \, \sigma(x_1 \cdots x_k) \times$$

$$\times \langle \{\alpha\} in | :\phi_{in}(x_1) \cdots \phi_{in}(x_k): | \{\beta\} in \rangle$$

$$= i^{m+n} \int dy_1 \cdots dy_m \int dz_1 \cdots dz_n \, f^*_{\alpha_1}(y_1) \cdots f^*_{\alpha_m}(y_m)$$
$$f_{\beta_1}(z_1) \cdots f_{\beta_n}(z_n) \, Z^{-1/2} K_{y_1} \cdots Z^{-1/2} K_{z_n} \times$$
$$\times \langle 0 | T \, \phi(y_1) \cdots \phi(z_n) | 0 \rangle$$

Again we have taken $\{\alpha\} \cap \{\beta\} = \emptyset$ hence the only way ~~the~~ matrix element of the Wick product is non-zero is if $:\phi_{in}(x_1) \cdots \phi_{in}(x_k):$ contains m-creation operators and n-annihilation operators that is for $k = m+n$. Since $:\phi_{in}(x_1) \cdots \phi_{in}(x_k):$ is symmetric, so is $\sigma(x_1, \ldots, x_k)$

Hence we have that there are $\frac{k!}{m!\,n!}$ terms in the Wick product with m-creation operators and n-creation and we can relabel the integration variables so that we find

$$S_{\alpha\beta} = \int dy_1 \cdots dy_m\, dz_1 \cdots dz_n \frac{1}{m!\,n!} \sigma(y_1 \cdots z_n)$$

$$\langle 0| a^{in}_{\alpha_1} \cdots a^{in}_{\alpha_m} \phi^{-}_{in}(y_1) \cdots \phi^{-}_{in}(y_m) \phi^{+}_{in}(z_1) \cdots \phi^{+}_{in}(z_n) a^{int}_{\beta_1} \cdots a^{int}_{\beta_n} |0\rangle$$

but $$[\phi^{+}_{in}(z), a^{int}_{\beta}] = \sum_{\alpha}{}' f_{\alpha}(z) [a^{in}_{\alpha}, a^{int}_{\beta}]$$

$$= f_{\beta}(z)$$

and $$[a^{in}_{\alpha}, \phi^{-}_{in}(y)] = \sum_{\beta}{}' f^{*}_{\beta}(y) [a^{in}_{\alpha}, a^{int}_{\beta}]$$

$$= f^{*}_{\alpha}(y)$$

Hence we find; since there are $m!\,n!$ ways to calculate the commutators and re-label the dummy indices

$$S_{\alpha\beta} = \int dy_1 \cdots dy_m\, dz_1 \cdots dz_n\, f^*_{\alpha_1}(y_1) \cdots f^*_{\alpha_m}(y_m)$$
$$f_{\beta_1}(z_1) \cdots f_{\beta_n}(z_n)\, \sigma(y_1 \cdots z_n).$$

$$= i^{m+n} \int dy_1 \cdots dz_n\, f^*_{\alpha_1}(y_1) \cdots f_{\beta_n}(z_n)$$
$$Z^{-1/2} K_{y_1} \cdots Z^{-1/2} K_{z_n} \langle 0 | T\, \phi(y_1) \cdots \phi(z_n) | 0 \rangle$$

Thus we obtain

$$\boxed{\begin{aligned} \sigma(x_1 \cdots x_n) = i^n Z^{-1/2} K_{x_1} \cdots Z^{-1/2} K_{x_n} \times \\ \times \langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle \end{aligned}}$$

So the S-operator can be written as

$$\boxed{S=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots d^4x_n\left[Z^{-1/2}K_{x_1}\cdots Z^{-1/2}K_{x_n}\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle\right]:\phi_{in}(x_1)\cdots\phi_{in}(x_n):}$$

In a similar manner we could expand

$$\phi(x)=Z^{1/2}\phi_{in}(x)+\sum_{n=2}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n$$

$$\times\left[Z^{-1/2}K_{x_1}\cdots Z^{-1/2}K_{x_n}\langle 0|R\,\phi(x)\phi(x_1)\cdots\phi(x_n)|0\rangle\right]$$

$$:\phi_{in}(x_1)\cdots\phi_{in}(x_n):$$

where $R\,\phi(x)\phi(x_1)\cdots\phi(x_n)$ is the retarded product of fields

$$R\phi(x)\phi(x_1)\cdots\phi(x_n) \equiv (-i)^n \sum_P \theta(x^0-x^0_{i_1})\theta(x^0_{i_1}-x^0_{i_2})\cdots\theta(x^0_{i_{n-1}}-x^0_{i_n})$$
$$[\cdots[[\phi(x),\phi(x_{i_1})],\phi(x_{i_2})],\ldots],\phi(x_{i_n})]$$

i.e. $\langle 0|R\phi(x)\phi(y)|0\rangle = \theta(x-y)\Delta'(x-y)$ etc.

We have now reached our goal of expressing matrix elements of operators in terms of their time ordered functions by means of the LSZ reduction formulae. This focuses our attention upon Green functions. We must now try to evaluate these, in order to do so we must assume further information about the dynamics of the

And of course we are able to determine the S operator times any time ordered product of operators $T B_1(x_1) \cdots B_\ell(x_\ell)$ in a similar manner

$$\boxed{\begin{aligned} & S\,T B_1(x_1) \cdots B_\ell(x_\ell) \\ & = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4y_1 \cdots d^4y_n \times \\ & \times \left[Z^{-1/2} K_{y_1} \cdots Z^{-1/2} K_{y_n} \langle 0 | T B_1(x_1) \cdots B_\ell(x_\ell) \phi(y_1) \cdots \phi(y_n) | 0 \rangle \right] \\ & \qquad \times N\left[\phi_{in}(y_1) \cdots \phi_{in}(y_n) \right] . \end{aligned}}$$

Note: $\langle 0 | S\,T B_1(x_1) \cdots B_\ell(x_\ell) | 0 \rangle = \langle 0 | T B_1(x_1) \cdots B_\ell(x_\ell) | 0 \rangle$

only the $n=0$ term contributes due to Wick ordering and of course $\langle 0 | S = \langle 0 |$ so that the LHS and RHS are in agreement.

field theory under consideration. However before embarking on this let's make some final comments about our LSZ formulae and time ordered products.

Remarks:

1) $S^{-1} = S^{\dagger}$, S is a unitary operator.
since $\langle {}_{out}^{in} | {}_{out}^{in} \rangle = 1 \Rightarrow \sum_f |S_{fi}|^2 = 1 \Rightarrow S^{-1} = S^{\dagger}$!

In order to show this we must exploit the generalized unitarity formula for the time ordered functions.

Besides the time ordered product of fields

$$T\,\phi(x_1)\cdots\phi(x_n) \equiv \sum_P \theta(x^0_{i_1}-x^0_{i_2})\,\theta(x^0_{i_2}-x^0_{i_3})\cdots\theta(x^0_{i_{n-1}}-x^0_{i_n})\,\phi(x_{i_1})\cdots\phi(x_{i_n})$$

we can define the anti-time ordered product denoted by $\bar{T}$