So we have differentiated form of Heisenberg's eq. of motion

$$-i \frac{\partial}{\partial t} B(x) = [H, B(x)] .$$

As we saw in the introduction we would like the system to reduce to freely moving particles as $t \to \pm\infty$ of a specific number with definite momentum and spin (α) and that these states would be a complete set of eigenstates of our observables.

Hence the second axiom states

Axiom 2: **Asymptotic Completeness**

$$\mathcal{H}_{in} = \mathcal{H} = \mathcal{H}_{out}$$

When $t \to -\infty$, the theory consists of freely moving particles of given # each with specified mass, spin, charge α etc. Each of which correspond to the single particle states of the theory. The totality of these in-states, $|0\rangle$ ($= |0_{in}\rangle$), $|\vec{k}_1, s_1, \alpha_1 in\rangle$, $|(\vec{k}_1, s_1, \alpha_1), (\vec{k}_2, s_2, \alpha_2) in\rangle$, ...

forms a basis for $\mathcal{H} = \mathcal{H}_{in}$.

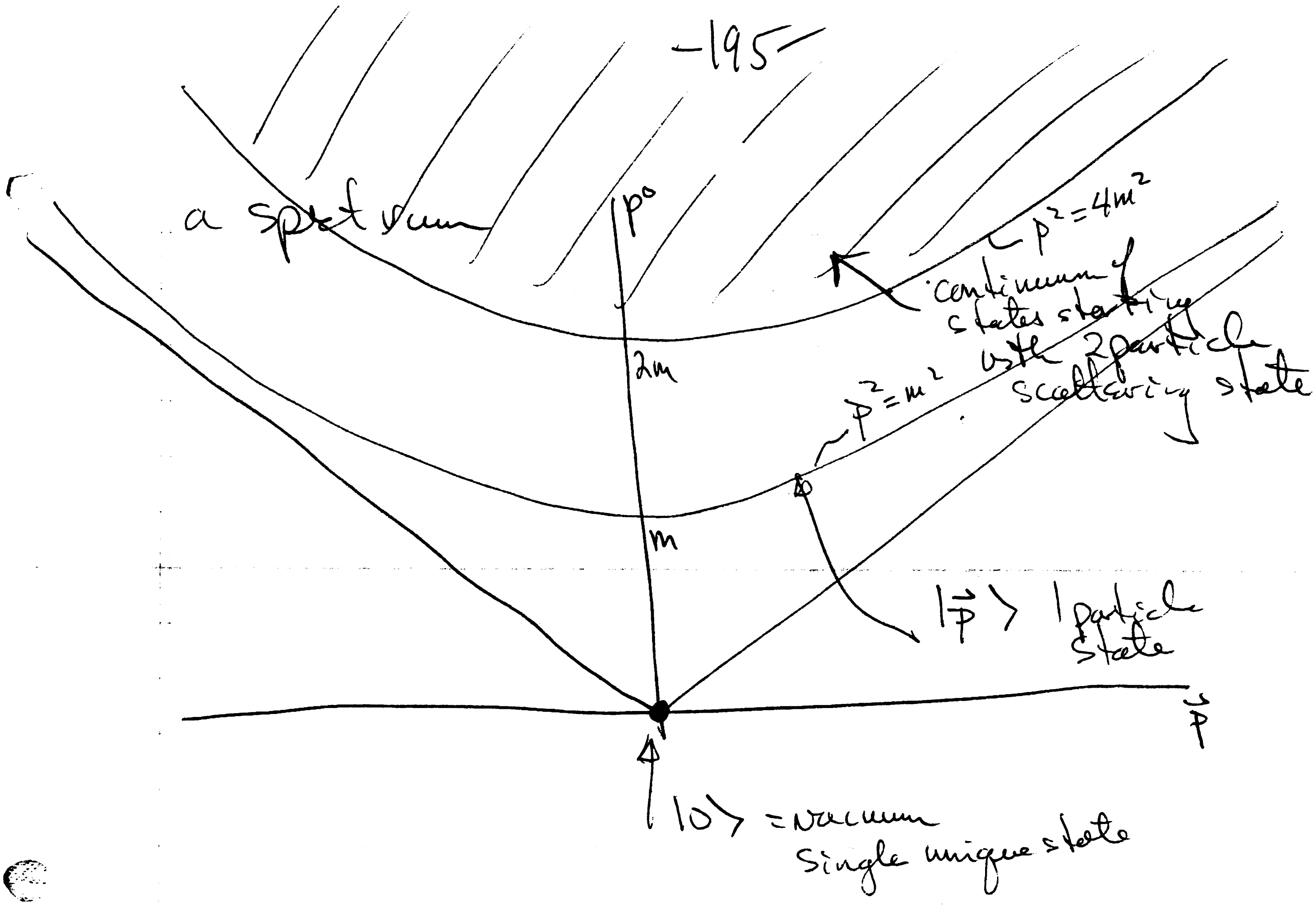
Similarly for $t \to +\infty$ the theory consists of freely moving particles of a given #, momentum, spin etc. The totality of these out-going states, $|0\, out\rangle = |0\rangle$, $|\vec{k}_1, s_1, \alpha_1, out\rangle$

$|(\vec{k}_1, s_1, \alpha_1)(\vec{k}_2, s_2, \alpha_2)\; out\rangle$, ...

form a complete basis for $\mathcal{H} = \mathcal{H}_{out}$.

Thus any physical state of $\mathcal{H}$ can be made from a superposition of either in- or out-states. (bound states must be included explicitly as $t \to \pm\infty$, or excluded, by assumption)

Hence we have in the case of a single scalar spin 0 particle of mass m



The above states are labelled by the momenta of the individual particles making up the state

$|k_1, k_2, \ldots, k_n \text{ in}\rangle$ for the n-particle in-state.

These are normalized to invariant continuum normalization condition

$\langle \text{in } k_1', \ldots, k_m' | k_1 \ldots, k_n \text{ in}\rangle$

$= \delta_{mn} \sum_P (2\pi)^3 2\omega_{k_1} \delta^3(\vec{k}_1 - \vec{k}_{i_1}') \cdots (2\pi)^3 2\omega_{k_n} \delta^3(\vec{k}_n - \vec{k}_{i_n}')$

and the totality add up to one; they are complete in $\mathcal{H}$

$$\mathbb{1} = \sum_n{}' |n\rangle\langle n|$$

$$= |0\rangle\langle 0| + \int \frac{d^3k}{(2\pi)^3 2\omega_k} |k\ in\rangle\langle k\ in|$$

$$+ \cdots + \frac{1}{n!}\int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2\omega_{k_n}} |k_1 \ldots k_n\ in\rangle\langle k_1 \ldots k_n\ in|$$

$$+ \cdots$$

Similarly for out-states.

As we well know this system of free in- and out-states can be simply described by free in- or out-field theory.

$$\mathcal{L}_{in} = \frac{1}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \frac{1}{2}m^2\phi_{in}^2$$

$$\Rightarrow \quad (\partial^2 + m^2)\phi_{in} = 0$$

and $\displaystyle \Pi_{in} \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi_{in}} = \dot\phi_{in}$

Commutators with $\dot\phi_{in}$

$$\delta(x^0-y^0)\,[\dot\phi_{in}(x),\phi_{in}(y)] = -i\,\delta^4(x-y)$$

$$\Rightarrow \quad [\phi_{in}(x),\phi_{in}(y)] = i\,\Delta(x-y)$$ as reviewed

in the introduction. The Poincaré operators can be formed in terms of the in- out- fields since we know the action of P^μ, $M^{\mu\nu}$ etc. on the in-, out- states by definition

For example $\phi_{in}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a_{in}(\vec k)e^{-ikx} + a^\dagger_{in}(\vec k)e^{+ikx}\right]$

$$a_{in}(\vec k)|0\rangle = 0 \quad \text{as usual}$$

with $\quad [a_{in}(\vec k), a^\dagger_{in}(\vec l)] = (2\pi)^3 2\omega_k\,\delta^3(\vec k-\vec l)$

and

$$|k_1,\dots k_n\ in\rangle = a^\dagger_{in}(\vec k_1)\dots a^\dagger_{in}(\vec k_n)|0\rangle$$

Since $\quad [P^\mu, \phi_{in}(x)] = -i\partial^\mu \phi_{in}(x)$

or $\quad [P^\mu, a^\dagger_{in}(\vec k)] = +k^\mu a^\dagger_{in}(\vec k)$

we can construct P^μ to find

$$P^\mu = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, k^\mu\, a^\dagger_{in}(\vec k)a_{in}(\vec k) \quad \text{etc.}$$

Finally instead of dealing with improper states $|k_1 \ldots k_n\ in\rangle$ as basis vectors it is useful to introduce normalizable states by making wave packets out of superpositions of $|\{k\}\ in\rangle$.

Consider the complete set $f_\alpha(x)$ of normalizable, single particle positive energy solutions to the K-G eq., $\alpha = 1, 2, \ldots$

$$f_\alpha(x) \equiv \int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta(k^2 - m^2)\,\theta(k^0)\,\tilde{f}_\alpha(\vec{k})\,e^{-ikx}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\tilde{f}_\alpha(\vec{k})\,e^{-ikx}$$

$f_\alpha(x)$ obeys the K-G eq. $(\partial^2 + m^2) f_\alpha(x) = 0$

further we choose $\tilde{f}_\alpha(k)$ to be normalizable

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\tilde{f}^*_\alpha(\vec{k})\,\tilde{f}_\beta(\vec{k}) = \delta_{\alpha\beta}$$

and complete $\sum_\alpha \tilde{f}_\alpha(\vec{k})\,\tilde{f}^*_\alpha(\vec{p}) = (2\pi)^3\, 2\omega_k\, \delta^3(\vec{p} - \vec{k})$

in Dirac notation we are introducing single particle wavepacket states $|f_\alpha\rangle$, Any state can be expanded in terms of $|k\rangle$ which have normalization $\langle p|k\rangle = (2\pi)^3 2\omega_k \delta^3(\vec{p}-\vec{k})$

(margin: [warm])

$$|f_\alpha\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \langle k|f_\alpha\rangle |k\rangle$$

The inner product is given by

$$\langle f_\alpha|f_\beta\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \langle f_\alpha|k\rangle\langle k|f_\beta\rangle$$

and we choose normalization $\langle f_\alpha|f_\beta\rangle = \delta_{\alpha\beta}$

and completeness for the wavepacket states

$$\sum_\alpha |f_\alpha\rangle\langle f_\alpha| = 1$$

$$\left(\underline{\text{i.e.}} \quad |k\rangle = \sum_\alpha \langle f_\alpha|k\rangle |f_\alpha\rangle\right)$$

by sandwiching 1 between $\langle k|1|k'\rangle$ we find

$$\langle k|1|k'\rangle = \sum_{\alpha} \langle k|f_\alpha\rangle\langle f_\alpha|k'\rangle$$
$$= (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}')$$

for our completeness relation.

Evidently $\langle k|f_\alpha\rangle = \tilde{f}_\alpha(\vec{k})$ in the above notation.

Again we can invert the FT

$$i\int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} f_\alpha(x)$$

$$= i\int d^3x\, e^{ikx}\left(-i\omega_k f_\alpha(x) + \partial_0 f_\alpha(x)\right)$$

$$= i\int d^3x\, e^{ikx}\Big[-i\omega_k \int \frac{d^3p}{(2\pi)^3 2\omega_p} \tilde{f}_\alpha(\vec{p}) e^{-ipx}$$

$$- i\int \frac{d^3p}{(2\pi)^3 2\omega_p} \omega_p \tilde{f}_\alpha(\vec{p}) e^{-ipx}\Big]$$

$$= \tfrac{1}{2}\tilde{f}_\alpha(\vec{k}) + \tfrac{1}{2}\tilde{f}_\alpha(\vec{k})$$

So

$$\boxed{\tilde{f}_\alpha(\vec{k}) = i\int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} f_\alpha(x)}$$

The inner product in coordinate space is given by

$$(f_\alpha, f_\beta) \equiv i\int d^3x\, f_\alpha^*(x) \overleftrightarrow{\partial_0} f_\beta(x) \quad \left(\text{indep of time}\right)$$

$$= i\int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{f}_\alpha^*(\vec{k})\, e^{+ikx} \overleftrightarrow{\partial_0} f_\beta(x)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{f}_\alpha^*(\vec{k}) \tilde{f}_\beta(\vec{k}) = \delta_{\alpha\beta}.$$

Since $\int d^3x\, e^{-ikx} \overleftrightarrow{\partial_0} f_\alpha(x) = 0$

we also have $(f^*_\alpha, f_\beta) = 0$

$$= i\int d^3x\, f_\alpha(x) \overleftrightarrow{\partial_0} f_\beta(x).$$

and

$$(f^*_\alpha, f^*_\beta) = (f_\beta, f_\alpha) = \delta_{\alpha\beta}.$$

Finally the unit operator in the space of positive freq. solutions to the K-G eq. is Δ^+ and neg. freq. sol. Δ^-.

i.e. let $F(x)$ be arb. pos freq sol.

$$F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (2\pi)\delta(k^2-m^2)\theta(k_0)\tilde{F}(k)$$

Then $F(x) = i\int d^3y\, \Delta^+(x-y) \overleftrightarrow{\partial_0} F(y)$

$$= i\int d^3y \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \overleftrightarrow{\partial^y_0} \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ipy}\tilde{F}(p)$$

$$= \int d^3y \frac{d^3k\, d^3p}{(2\pi)^6 2\omega_k 2\omega_p} \tilde{F}(p) [\omega_p + \omega_k] e^{-ikx} e^{-i(p-k)y}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{F}(k) e^{-ikx} = F(x) \checkmark$$

Similarly for neg. freq. sol. $G(x)$

$$G(x) = -\int d^3y\, \Delta^-(x-y) \overleftrightarrow{\partial_0} G(y).$$

That is the momentum space completeness relation becomes in coordinate space

$$\sum_\alpha f_\alpha(x) f_\alpha^*(y) = i\Delta^+(x-y)$$

and $$\sum_\alpha f_\alpha^*(x) f_\alpha(y) = i\Delta^-(x-y).$$

This can be obtained by FT the momentum space results

$$\sum_\alpha \tilde{f}_\alpha(\vec{k}) \tilde{f}^*_\alpha(\vec{p}) = (2\pi)^3 2\omega_k \delta^3(\vec{p}-\vec{k})$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ikx} e^{+ipy} (2\pi)^3 2\omega_p \delta^3(\vec{p}-\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} = i\Delta^+(x-y)$$

$$= \sum_\alpha \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ikx} \tilde{f}_\alpha(\vec{k}) \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{+ipy} \tilde{f}^*_\alpha(\vec{p})$$

$$= \sum_{\alpha}{}' f_\alpha(x) f_\alpha^*(y) = i\Delta^+(x-y) .$$

Thus any soln. of K-G eq. can be expanded in terms of f_α, f_α^* instead of plane waves

Thus we can consider

$$\phi_{in}(x) = \sum_\alpha' [a_\alpha^{in} f_\alpha(x) + a_\alpha^{in\dagger} f_\alpha^*(x)]$$

$$\left(= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ikx} \sum_\alpha' a_\alpha^{in} \tilde{f}_\alpha(\vec{k}) + e^{+ikx} \sum_\alpha' a_\alpha^{in\dagger} \tilde{f}_\alpha^*(\vec{k}) \right] \right.$$

$\Rightarrow$ Using $\int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{f}_\alpha^*(\vec{k}) \tilde{f}_\beta(\vec{k}) = \delta_{\alpha\beta}$ we find

$$a_{in}(\vec{k}) = \sum_\alpha' a_\alpha^{in} \tilde{f}_\alpha(\vec{k})$$

$$a_{in}^\dagger(\vec{k}) = \sum_\alpha' a_\alpha^{in\dagger} \tilde{f}_\alpha^*(\vec{k})$$

So

$$\left. \boxed{\int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{f}_\beta^*(\vec{k})\, a_{in}(\vec{k}) = a_\beta^{in}} \right)$$

Note we can recover our plane wave results by choosing

$$f_\alpha(x) \Rightarrow f_k(x) = e^{-ikx}$$

$$\sum_\alpha \rightarrow \int \frac{d^3k}{(2\pi)^3 2\omega_k}$$

(that is $\tilde{f}_\alpha(k) \rightarrow \tilde{f}_p(k) = (2\pi)^3 2\omega_k \delta^3(\vec{p}-\vec{k})$)

Then mult. by $\int i f_\beta^* \overleftrightarrow{\partial_0}\, d^3x$ to invert:

$$a_\alpha^{in} = i\int d^3x\, f_\alpha^*(x) \overleftrightarrow{\partial_0}\, \phi_{in}(x)$$

$$a_\alpha^{in\dagger} = -i\int d^3x\, f_\alpha(x) \overleftrightarrow{\partial_0}\, \phi_{in}(x) .$$

a_α^{in} annihilates a wave packet f_α

$a_\alpha^{in\dagger}$ creates a wave packet f_α

since algebra is

$$[a_\alpha^{in}, a_\alpha^{in}] = [a_\alpha^{in\dagger}, a_\alpha^{in\dagger}] = 0$$

$$[a_\alpha^{in}, a_\beta^{in\dagger}] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3p}{(2\pi)^3 2\omega_p}$$

$$\tilde{f}_\alpha^*(\vec{k})\, \tilde{f}_\beta(\vec{p})\, [a_{in}(\vec{k}), a_{in}^\dagger(\vec{p})]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{f}_\alpha^*(\vec{k}) \tilde{f}_\beta(\vec{k}) = \delta_{\alpha\beta} .$$

We can define number operators to count the # of particles with wave packet f_α

$$N_\alpha^{in} = a_\alpha^{in\dagger} a_\alpha^{in}$$

The total Number operator is then

$$N = \sum_\alpha N_\alpha^{in}$$

A set of normalizable basis states can now be constructed for $\mathcal{H}_{in} = \mathcal{H}$. as before we define the vacuum state by $a_\alpha^{in}|0\rangle = 0$ then we construct the multi-particle states by the action of $a_\alpha^{in\dagger}$ on the vacuum

$$|0\rangle$$

$$|\alpha\ in\rangle = a_\alpha^{in\dagger}|0\rangle$$

$$|\alpha_1,\alpha_2\ in\rangle = \frac{1}{\sqrt{p_{\alpha_1\alpha_2}}}\, a_{\alpha_1}^{in\dagger} a_{\alpha_2}^{in\dagger}|0\rangle$$

$$\vdots$$

$$|\alpha_1,\ldots,\alpha_n\ in\rangle = \frac{1}{\sqrt{p_{\alpha_1\cdots\alpha_n}}}\, a_{\alpha_1}^{in\dagger}\cdots a_{\alpha_n}^{in\dagger}|0\rangle$$

$$\vdots$$

where $p_{\alpha_1\ldots\alpha_n} = n_1!\cdots n_k!$; $k \leq n$.

with n_i the number of wavepackets equal to f_{α_i} for example

$$|\alpha_1,\alpha_1,\alpha_2\ in\rangle = \frac{1}{\sqrt{2}}\left(a_{\alpha_1}^{in\dagger}\right)^2 a_{\alpha_2}^{in\dagger}|0\rangle .$$

These are normalized to

$$\langle in\ \alpha_1\cdots\alpha_m|\beta_1\cdots\beta_n\ in\rangle$$

$$= \frac{\delta_{mn}}{p_{\alpha_1\cdots\alpha_n}} \sum_P \delta_{\alpha_1\beta_{i_1}}\cdots\delta_{\alpha_n\beta_{i_n}}$$

and completeness in $\mathcal{H}$ takes the form

$$\underline{1} = \sum_{(\alpha)}{}^{f} |\alpha \, in\rangle\langle\alpha \, in|$$

$$= |0\rangle\langle 0| + \sum_{\alpha}{}^{f} |\alpha \, in\rangle\langle\alpha \, in|$$

$$+ \dots + \sum_{\alpha_1 \dots \alpha_n}{}^{f} |\alpha_1 \dots \alpha_n \, in\rangle\langle\alpha_1 \dots \alpha_n \, in| + \dots$$

Similarly we can proceed for the out-states.

All observables can now be written in term of the wavepacket creation and annihilation operators, in particular we are interested in the transition amplitude to go from an initial incomming state of wavepackets to a final outgoing state of wavepackets

$$S_{\alpha\beta} = \langle\alpha_1 \dots \alpha_m \, out | \beta_1 \dots \beta_n \, in\rangle$$

As before, since ϕ_{in} and ϕ_{out} obey the same ETCR and since we assumed

The vacuum is unique $|0\rangle = |0\,in\rangle = |0\,out\rangle$

Wightman has shown that they belong to equivalent irreducible representations of the ETCR, Thus there exists a unitary operator S, $S^{-1} = S^{\dagger}$ so that

$$\phi_{in}(x) = S\,\phi_{out}(x)\,S^{-1}.$$

that is $a_\alpha^{in} = S a_\alpha^{out} S^{-1}$

so $a_\alpha^{in\dagger} = S a_\alpha^{out\dagger} S^{-1}$.

Since $|\{\alpha\}\,out\rangle = a_{\alpha_1}^{out\dagger} \cdots a_{\alpha_n}^{out\dagger} |0\rangle$

we have

$$S|\{\alpha\}\,out\rangle = S a_{\alpha_1}^{out\dagger} S^{-1} S a_{\alpha_2}^{out\dagger} S^{-1} S \cdots$$
$$\cdots a_{\alpha_n}^{out\dagger} S^{-1} S |0\rangle$$

$$= a_{\alpha_1}^{in\dagger} \cdots a_{\alpha_n}^{in\dagger} |0\rangle \quad \text{since } S|0\rangle = |0\rangle$$

because $|0\rangle$ is unique

that is $|0\rangle$ is defined by

$$a_\alpha^{out}|0_{out}\rangle = 0 \Rightarrow S a_\alpha^{out} S^{-1} S|0_{out}\rangle = 0$$

$$\Rightarrow a_\alpha^{in}(S|0_{out}\rangle) = 0$$

but $|0\, in\rangle = S|0\, out\rangle = |0\rangle$ ~~by~~ by uniqueness assumption.

Hence $\boxed{S|\{\alpha\}out\rangle = |\{\alpha\}in\rangle}$.

So $S_{\alpha\beta} = \langle\{\alpha\}out|\{\beta\}in\rangle$

$$= \langle\{\alpha\}out|S|\{\beta\}out\rangle$$

$$= \langle\{\alpha\}in|S|\{\beta\}in\rangle$$

Since $\langle out| = \langle in|S$.

Finally due to the completeness of $|\{\alpha\}{}^{in}_{out}\rangle$ we have the Scattering operator

$$S = \sum_\alpha |\{\alpha\}in\rangle\langle\{\alpha\}out|.$$