but recall p.-94-

$$S = \frac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

So $$S^{-1} = U(0,+\infty)\,U^{-1}(0,-\infty)\,\langle 0|U(+\infty,-\infty)|0\rangle$$

and
$$\phi_{out}(x) = S^{-1}\,\phi_{in}(x)\,S$$

So far we have been concentrating on relating the S-operator to the in- and out- fields. We see that it is essentially just the time evolution operator in terms of the in- or out- fields. Again this is a perturbative expression for S and has all the drawbacks that S_{fi} does. It is nothing more than $U(0,\pm\infty)$, we would like to eliminate any explicit appearance of $U(0,\pm\infty)$ since it will involve approximations and our interpretive difficulties. Since it is $U(t,t_0)$ that contains the dynamical evolution information for the system we need another quantity that contains that at intermediate times. Certainly the full interacting Heisenberg picture fields $\phi(x)$ carry all the information at any time of our system, and up to now have been backstage. So towards our goal of eliminating the ambiguities of the explicit appearance of the time evolution operator in our S operator, let's begin to express it in terms of $\phi(x)$. Towards this we first relate the in- and out-fields to the full fields $\phi(x)$. The relation is known as the Yang-Feldman equation. It will demonstrate to us in what sense the interacting fields $\phi(x)$ interpolate for the asymptotic fields $\phi_{\substack{in \\ out}}(x)$.

For the sake of clarity we will derive the Yang-Feldman equation within the framework of a single, self-interacting Hermitian scalar field $\phi(x)$. We will assume that the equation of motion for $\phi(x)$ has the simple form

$$(\partial^2+m^2)\,\phi(x) = j(x)$$

where $j(x)$ is the "current" or "source" for ϕ and is itself a function of ϕ. The source is given by the derivative of the Lagrangian density

$$j(x) = -\frac{\partial \mathcal{L}_I}{\partial \phi(x)} = \frac{\partial \mathcal{H}_I}{\partial \phi(x)},$$

where we assume $\mathcal{H}_I = -\mathcal{L}_I$, that is non-derivative coupling.

Since we desire to relate the in-field $\phi_{in}(x)$ to the interpolating field $\phi(x)$ we start with the relation (1.2.114)

$$\phi_{in}(x) = U^{(t)}(t)\,\phi(x)\,U^{(t)-1}(t)$$

$$= e^{+iHt}\,U(0,-\infty)\,e^{-iHt}\,\phi(x)\,e^{+iHt}\,U^{\dagger}(0,-\infty)\,e^{-iHt}$$

Since these are Heisenberg picture fields

$$e^{-iHt}\,\phi(x)\,e^{+iHt} = \phi(\vec{x},0)$$

So

$$\phi_{in}(x) = e^{+iHt}\left(U(0,-\infty)\,\phi(\vec{x},0)\,U^{\dagger}(0,-\infty)\right)e^{-iHt}$$

$$= e^{+iHt}\left(\phi(\vec{x},0) - \phi(\vec{x},0) + U(0,-\infty)\phi(\vec{x},0)U^{-1}(0,-\infty)\right)$$

$$\times e^{-iHt}$$

$$\phi_{in}(x) = e^{+iHt}\phi(\vec{x},0)e^{-iHt}$$
$$+ e^{+iHt}\left([U(0,-\infty),\phi(\vec{x},0)]U^{-1}(0,-\infty)\right)e^{-iHt}$$

$$= \phi(x) + e^{+iHt}\left([U(0,-\infty),\phi^{ip}(\vec{x},0)]U^{-1}(0,-\infty)\right)e^{-iHt}$$

Since all pictures coincide at $t=0$, $\phi(\vec{x},0) = \phi^{ip}(\vec{x},0)$

and recalling that

$$U(0,-\infty) = T e^{-i\int_{-\infty}^{0} dt\, H_I^{ip}(t)}$$
$$= T\, e^{-i\int_{-\infty}^{0} dt\, d^3x\, \mathcal{H}_I^{ip}(t)}$$

with $H_I^{ip}(t) = \int d^3x\, \mathcal{H}_I^{ip}(x)$ where $\mathcal{H}_I^{ip}$ is the iP Hamiltonian density. Hence

$$[U(0,-\infty),\phi^{ip}(\vec{x},0)]$$
$$= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\int_{-\infty}^{0} d^4x_1 \cdots \int_{-\infty}^{0} d^4x_n \times$$

$$\times\left[T\mathcal{H}_I^{ip}(x_1)\cdots\mathcal{H}_I^{ip}(x_n),\, \phi^{ip}(\vec{x},0)\right].$$

Now using the identity

$$[AB, C] = A[B, C] + [A, C]B$$

and a re-labelling of the dummy integration variables along with the symmetry of the T-operator $T(AB) = T(BA)$ we have

$$[U(0,-\infty), \phi^{ip}(\vec{x},0)]$$
$$= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} d^4x_1 \cdots \int_{-\infty}^{0} d^4x_n \, n\, T\Big([\mathcal{H}_I^{ip}(x_1), \phi^{ip}(\vec{x},0)] \times$$
$$\times \mathcal{H}_I^{ip}(x_2) \cdots \mathcal{H}_I^{ip}(x_n) \Big)$$

Now $$[\mathcal{H}_I^{ip}(x_1), \phi^{ip}(\vec{x},0)] = \frac{\partial \mathcal{H}_I^{ip}(x_1)}{\partial \phi^{ip}(x_1)} [\phi^{ip}(x_1), \phi^{ip}(\vec{x},0)]$$

$$= j^{ip}(x_1) [\phi^{ip}(x_1), \phi^{ip}(\vec{x},0)]$$

But recall that $\phi^{ip}(x)$ is a free field

$$[\phi^{ip}(x), \phi^{ip}(y)] = i\Delta(x-y)$$
$$= -i\Delta(y-x)$$

with Δ a c-number.

Thus

$$[U(0,-\infty), \phi^{i\rho}(\vec{x},0)] = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} n(-i) \int_{-\infty}^{0} d^4x_1 \cdots d^4x_n$$

$$\Delta(\vec{x}-\vec{x}_1, -t_1)\, T\left(j^{i\rho}(x_1)\, \mathcal{H}_I^{i\rho}(x_2) \cdots \mathcal{H}_I^{i\rho}(x_n)\right)$$

$$= -\int_{-\infty}^{0} d^4x_1\, \Delta(\vec{x}-\vec{x}_1, -t_1) \times$$

$$\times \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} \int_{-\infty}^{0} d^4x_2 \cdots \int_{-\infty}^{0} d^4x_n\, T\left[j^{i\rho}(x_1)\, \mathcal{H}_I^{i\rho}(x_2) \cdots \mathcal{H}_I^{i\rho}(x_n)\right]$$

Thus re-labelling the dummy indices again,

$$x_1 \rightarrow y\,;\; x_2 \rightarrow x_1,\, x_3 \rightarrow x_2, \ldots, x_n \rightarrow x_{n-1}\,,$$

we obtain

$$\boxed{\begin{aligned} [U(0,-\infty), \phi^{i\rho}(\vec{x},0)] &= -\int_{-\infty}^{0} d^4y\, \Delta(\vec{x}-\vec{y}, -y^0) \times \\ &\times \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} d^4x_1 \cdots \int_{-\infty}^{0} d^4x_n\, T\left[j^{i\rho}(y)\, \mathcal{H}_I^{i\rho}(x_1) \cdots \mathcal{H}_I^{i\rho}(x_n)\right] \\ &\equiv -\int_{-\infty}^{0} d^4y\, \Delta(\vec{x}-\vec{y}, -y^0)\, T\left(j^{i\rho}(y)\, U(0,-\infty)\right) \end{aligned}}$$

Now we can re-write the time ordered product by using the fact that $y^0 < 0$ (use $T[j^{iP}(y) \mathcal{H}_I^{iP}(x_1) \cdots \mathcal{H}_I^{iP}(x_n)]$ to be more rigorous explicitly)

$$T\, j^{iP}(y)\, U(0,-\infty) = T\, j^{iP}(y)\, U(0,y^0)\, U(y^0,-\infty)$$

$$= U(0,y^0)\, j^{iP}(y)\, U(y^0,-\infty)$$

$$= U^{-1}(y^0,0)\, j^{iP}(y) \big(U(y^0,0)\, U(0,y^0) \big) U(y^0,-\infty)$$

$$= \big(U^{-1}(y^0,0)\, j^{iP}(y)\, U(y^0,0) \big)\, U(0,-\infty)$$

Now recall

$$j(y) = U^{-1}(y^0,0)\, j^{iP}(y)\, U(y^0,0)$$

So

$$\boxed{T\big(j^{iP}(y)\, U(0,-\infty)\big) = j(y)\, U(0,-\infty)}\,.$$

Hence we obtain

$$[U(0,-\infty), \phi^{iP}(\vec{x},0)] = -\int_{-\infty}^{0} d^4y\, \Delta(\vec{x}-\vec{y}, -y^0)\, j(y)\, U(0,-$$

Thus we secure the formula ($t = x^0$)

$$\phi_{in}(x) = \phi(x) + e^{+iHt}\left(-\int_{-\infty}^{0} d^4y\, \Delta(\vec{x}-\vec{y}, -y^0)\, j(y)\right) e^{-iHt}$$

$$= \phi(x) - \int_{-\infty}^{0} d^4y\, \Delta(\vec{x}-\vec{y}, -y^0)\, j(\vec{y}, y^0 + x^0)$$

Changing integration variables

$y^0 \rightarrow y^0 + x^0$ we find

$$\phi_{in}(x) = \phi(x) - \int_{-\infty}^{x^0} d^4y\, \Delta(x-y)\, j(y) .$$

Thus we obtain the first Yang-Feldman equation

$$\boxed{\phi(x) = \phi_{in}(x) - \int_{-\infty}^{+\infty} d^4y\, \Delta_R(x-y)\, j(y)}$$

where $\Delta_R(x-y) \equiv -\theta(x^0 - y^0)\, \Delta(x-y)$ is called the retarded function.

Similarly one can relate the out fields to ϕ to obtain the other Yang-Feldman equation

$$\boxed{\phi(x) = \phi_{out}(x) - \int_{-\infty}^{+\infty} d^4y\, \Delta_A(x-y)\, j(y)}$$

with the advanced function defined by

$$\Delta_A(x-y) \equiv \theta(y^0-x^0)\,\Delta(x-y) .$$

Taking the difference of these two equations we obtain a direct relation between ϕ_{in} and ϕ_{out}

$$\phi_{out}(x) = \phi_{in}(x) + \int_{-\infty}^{+\infty} d^4y \left[\Delta_A(x-y) - \Delta_R(x-y)\right] j(y)$$

$$= \phi_{in}(x) + \int_{-\infty}^{+\infty} d^4y \left(\theta(y^0-x^0) + \theta(x^0-y^0)\right)\Delta(x-y)\, j(y)$$

Thus

$$\boxed{\phi_{out}(x) = \phi_{in}(x) + \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)\, j(y)}$$

Recalling the properties of the singular function

$$(\partial^2+m^2)\Delta_{\substack{A\\R}}(x-y) = -\delta^4(x-y)$$

$$(\partial^2+m^2)\Delta(x-y) = 0$$

and since $(\partial^2+m^2)\phi_{\substack{in\\out}} = 0$ we have

$$(\partial^2+m^2)\phi(x) = -\int_{-\infty}^{+\infty} d^4y\,(\partial_x^2+m^2)\Delta_{\substack{A\\R}}(x-y)\,j(y)$$

$$= \int_{-\infty}^{+\infty} d^4y\,\delta^4(x-y)\,j(y) = j(y)$$

as required.

Also we find that $\phi(x)$ for early and late times goes over to $\phi_{\substack{in\\out}}(x)$

$$\phi(x) = \phi_{in}(x) + \int_{-\infty}^{x^0} d^4y\,\Delta(x-y)\,j(y)$$

$$\phi(x) = \phi_{out}(x) - \int_{x^0}^{+\infty} d^4y\,\Delta(x-y)\,j(y)$$

Thus as $x^0 \to \mp\infty$

$$\phi(x) \xrightarrow{x^0\to\mp\infty} \phi_{\substack{in\\out}}(x)$$

LSZ showed that this asymptotic strong operator convergence of $\phi \xrightarrow{t \to \mp\infty} \phi_{\substack{in \\ out}}$ leads to contradictions. For instance, we found in perturbation theory that the adiabatic hypothesis implied the full propagator, after mass renormalization, goes like $\frac{iZ}{p^2 - m^2 + i\epsilon}$ with residue Z while the external lines, that is wave functions, go like $Z^{-1/2}$, that is the fields must be rescaled $\phi_R' = Z^{-1/2}\phi$. Hence if we include the adiabatic hypothesis factors $e^{-\epsilon|t|}$ the correct form of the Yang-Feldman equation would be

$$Z^{-1/2}\phi(x) = \phi_{in}(x) - \int_{-\infty}^{+\infty} d^4y\, \Delta_R(x-y)(\partial_y^2 + m^2) Z^{-1/2}\phi(y)$$

$$Z^{-1/2}\phi(x) = \phi_{out}(x) - \int_{-\infty}^{+\infty} d^4y\, \Delta_A(x-y)(\partial_y^2 + m^2) Z^{-1/2}\phi(y)$$

So

$$Z^{1/2}\phi_{out}(x) = Z^{1/2}\phi_{in}(x) + \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)(\partial_y^2 + m^2)\phi(y)$$

Also LSZ showed the necessity for smearing the fields and states with normalizable wave packets. And the asymptotic condition and Yang-Feldman equation is a matrix element identity.

For now we will ignore the factors of Z but will recall them later.

Recall $\phi_{out}(x) = S^{-1}\phi_{in}(x) S$, thus we find

$$S^{-1}\phi_{in}(x) S - \phi_{in}(x) = \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y) j(y)$$ -110-

Or

$$[\phi_{in}(x), S] = \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)\, S j(y)$$

$$= \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)(\partial_y^2 + m^2) S\phi(y).$$

We could continue on in this manner to find the generalized formula

$$[\phi_{in}(x_n), [\phi_{in}(x_{n-1}), [\cdots, [\phi_{in}(x_1), ST\phi(y_1)\cdots\phi(y_m)]\cdots]]]$$

$$= \int_{-\infty}^{+\infty} d^4z_1 \cdots d^4z_n\, \Delta(x_1 - z_1)\cdots\Delta(x_n - z_n) \times$$

$$\times (\partial_{z_1}^2 + m^2)\cdots(\partial_{z_n}^2 + m^2)\, ST\phi(y_1)\cdots\phi(y_m)\phi(z_1)\cdots\phi(z_n)$$

where $T\phi(y_1)\cdots\phi(z_n)$ is the time ordered product of fields ϕ.

Since $[\phi_{in}(x), \phi_{in}(y)] = i\Delta(x-y)$

one can show that the above formula

yields

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \cdots d^4x_n :\phi_{in}(x_1) \cdots \phi_{in}(x_n): \times$$

$$\times (\partial^2_{x_1} + m^2) \cdots (\partial^2_{x_n} + m^2) \langle 0_{out} | T\phi(x_1) \cdots \phi(x_n) | 0_{in} \rangle$$

as the solution for the S-operator.

This is one form of the LSZ-reduction formulae for the S-matrix. In particular taking the $\langle 0_{in} |$ $| 0_{in} \rangle$ matrix element of the above commutator when $n=0$ we find the various in-state matrix els of S while on the RHS we find $\langle 0_{out} | T\phi(z_1) \cdots \phi(z_n) | 0_{in} \rangle$. Thus the above solution.

Since this result is rather abstract, we will approach the reduction formula from the perturbative expansion of S_{fi} and compare this to the perturbative formula for $\langle 0_{in} | T\phi(x_1) \cdots \phi(x_n) | 0_{in} \rangle$. Before this direct comparison however let's consider the consequences of the Yang-Feldman equation further.

In fact we can derive the necessary relations between in- and out-fields and interpolating fields by considering

$$T\left(\phi_{in}(x_1) \cdots \phi_{in}(x_n) S\right)$$

$$= T\left[\phi_{in}(x_1) \cdots \phi_{in}(x_n)\, U^{in}(+\infty,-\infty)\right] \frac{1}{\langle 0_{in}| U^{in}(+\infty,-\infty)|0_{in}\rangle}$$

$$= \left[U^{in}(+\infty,t_1)\phi_{in}(x_1)U^{in}(t_1,t_2)\phi_{in}(x_2)U^{in}(t_2,t_3)\right.$$

$$\left.\cdots U^{in}(t_{n-1},t_n)\phi_{in}(x_n)U^{in}(t_n,-\infty)\right]\frac{1}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

for $x_1^0 > x_2^0 > \cdots > x_n^0$

$$= \left[U^{in}(+\infty,0)U^{in}(0,t_1)\phi_{in}(x_1)U^{in}(t_1,0)U^{in}(0,t_2)\phi_{in}(x_2)\right.$$

$$\left.\cdots U^{in}(t_{n-1},0)U^{in}(0,t_n)\phi_{in}(x_n)U^{in}(t_n,0)U^{in}(0,-\infty)\right]$$

$$\times \frac{1}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

But recall that

$$\phi(x) = U^{-1}(t,0)\,\phi^{IP}(x)\,U(t,0)$$

$$= U^{-1}(t,0)\,U^{-1}(0,-\infty)\,\phi_{in}(x)\,U(0,-\infty)\,U(t,0)$$

however $\quad U(0,-\infty)\,U(t,0)\,U^{-1}(0,-\infty) = U_{in}(t,0)$

So

$$\phi(x) = U^{-1}(0,-\infty)\,U_{in}^{-1}(t,0)\,\phi_{in}(x)\,U_{in}(t,0)\,U(0,-\infty)$$

So $\quad \boxed{U^{in}(0,t)\,\phi_{in}(x)\,U^{in}(t,0) = U(0,-\infty)\,\phi(x)\,U^{-1}(0,-\infty)}$

Thus

$$T\left(\phi_{in}(x_1)\cdots\phi_{in}(x_n)\,S\right)$$

$$= U^{in}(+\infty,0)\,U(0,-\infty)\,\phi(x_1)\,U^{-1}(0,-\infty)\,U(0,-\infty)\,\phi(x_2)\,U^{-1}(0,-\infty)$$

$$\cdots U(0,-\infty)\,\phi(x_n)\,U^{-1}(0,-\infty)\,U^{in}(0,-\infty)\,\frac{1}{(0|U(+\infty,-\infty)|0)}$$

$$= \frac{U(0,-\infty)\,U(+\infty,0)}{(0|U(+\infty,-\infty)|0)}\,\phi(x_1)\cdots\phi(x_n)\,U(0,-\infty)\,U^{-1}(0,-\infty)$$

$$= \frac{U(0,-\infty)\,U(+\infty,0)}{(0|U(+\infty,-\infty)|0)}\left(T\,\phi(x_1)\cdots\phi(x_n)\right)$$

where we obtain the last line for any chronological ordering of $x_1^0,\dots,x_n^0$

Now recall that $S = \dfrac{U(0,-\infty)\, U^{+}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}$

So

$$\boxed{T\left[\phi_{in}(x_1)\cdots\phi_{in}(x_n)\, S\right] = S\, T\,\phi(x_1)\cdots\phi(x_n)}$$

We can exploit this formula for several results. First the LSZ reduction formula for the S-operator

Recall the Yang-Feldman equation

$$[\phi_{in}(x), S] = \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)\, K_y\, S\phi(y)$$

$$= \int_{-\infty}^{+\infty} d^4y\, \Delta(x-y)\, K_y\, T\left[\phi_{in}(y)\, S\right]$$

We first consider $[\phi_{in}(x_2), [\phi_{in}(x_1), S]]$.

Now

$$[\phi_{in}(x_2), T(\phi_{in}(y_1), S)]$$

$$= [\phi_{in}(x_2),\ U^{in}(+\infty, y_1)\,\phi_{in}(y_1)\, U^{in}(y_1, -\infty)]\, e^{-i\theta}$$

$$= i\Delta(x_2 - y_1)\, S + [\phi_{in}(x_2), U^{in}(+\infty, y_1)]\,\phi_{in}(y_1)\, \times$$

$$\times\, U^{in}(y_1, -\infty)\, e^{-i\theta} + U^{in}(+\infty, y_1)\,\phi_{in}(y_1)\, \times$$

$$\times\, [\phi_{in}(x_2), U^{in}(y_1, -\infty)]\, e^{-i\theta}$$

Using that $U^{in}(+\infty, y_1) = T e^{-i\int_{y_1}^{+\infty} dt\, H_I^{in}(\phi_{in}, \pi_{in}; t)}$
we have

$$[\phi_{in}(x_2), U^{in}(+\infty, y_1)]$$

$$= +\int_{y_1}^{+\infty} d^4y_2\, \Delta(x_2 - y_2)\, T j_{in}(y_2)\, U^{in}(+\infty, y_1)$$

and analogously for the other commutator

So

$$[\phi_{in}(x_2), T(\phi_{in}(y_1) S)]$$

$$= i\Delta(x_2 - y_1) S + \int_{y_1}^{+\infty} d^4y_2\, \Delta(x_2 - y_2) \left(T j_{in}(y_2) U^{in}(+\infty, y_1)\right) \times$$

$$\times \phi_{in}(y_1) U^{in}(y_1, -\infty) e^{i\theta}$$

$$+ \int_{-\infty}^{y_1} d^4y_2\, \Delta(x_2 - y_2)\, U^{in}(+\infty, y_1) \phi_{in}(y_1) \times$$

$$\times T j_{in}(y_2) U^{in}(y_1, -\infty)$$

But $y_2^0 > y_1^0$ in the first term and $y_2^0 < y_1^0$ in the second term, hence they are each totally time ordered

$$[\phi_{in}(x_2), T(\phi_{in}(y_1) S)] = i\Delta(x_2 - y_1) S$$

$$+ \int_{y_1}^{+\infty} d^4y_2\, \Delta(x_2 - y_2)\, T\left(j_{in}(y_2)\, U^{in}(+\infty, y_1)\, \phi_{in}(y_1)\, U^{in}(y_1, -\infty\right) \times e^{-i\theta}$$

$$+ \int_{-\infty}^{y_1} d^4y_2\, \Delta(x_2 - y_2)\, T\left(j_{in}(y_2)\, U^{in}(+\infty, y_1)\, \phi_{in}(y_1)\, U^{in}(y_1, -\infty)\right) \times e^{-i\theta}$$

$$= i\Delta(x_2 - y_1) S$$

$$+ \int_{y_1}^{+\infty} d^4y_2\, \Delta(x_2 - y_2)\, T\left(j_{in}(y_2)\, \phi_{in}(y_1)\, S\right)$$

$$+ \int_{-\infty}^{y_1} d^4y_2\, \Delta(x_2 - y_2)\, T\left(j_{in}(y_2)\, \phi_{in}(y_1)\, S\right)$$

$$= i\Delta(x_2 - y_1) S$$

$$+ \int_{-\infty}^{+\infty} d^4y_2\, \Delta(x_2 - y_2)\, T\left(j_{in}(y_2)\, \phi_{in}(y_1)\, S\right)$$

So using $T(j_{in}(y_2)\phi_{in}(y_1)S)$

$$= S\,T\,\overset{\circ}{j}(y_2)\phi(y_1)$$

$$= K_{y_2} S\,T\,\phi(y_1)\phi(y_2) - \underbrace{i\delta^4(y_1-y_2)S}_{= -\delta(y_1^0-y_2^0)[\phi(y_1),\phi(y_2)] \times S}$$

So we obtain

$$[\phi_{in}(x_2),[\phi_{in}(x_1),S]]$$

$$= \int_{-\infty}^{+\infty} d^4y_1\, \Delta(x_1-y_1) K_{y_1} [\phi_{in}(x_2), T(\phi_{in}(y_1)S)]$$

$$= \int_{-\infty}^{+\infty} d^4y_1\, \Delta(x_1-y_1) K_{y_1} \Big\{ \cancel{i\Delta(x_2-y_1)S}$$

$$+ \int_{-\infty}^{+\infty} d^4y_2\, \Delta(x_2-y_2) K_{y_2} S\,T\,\phi(y_1)\phi(y_2)$$

$$- i\int_{-\infty}^{+\infty} d^4y_2\, \cancel{\Delta(x_2-y_2)\,\delta^4(y_1-y_2)\,S} \Big\}$$

(Note: $K_{y_1}\Delta(x_2-y_1) = 0$ also!)

$$\boxed{\begin{aligned}&[\phi_{in}(x_2),[\phi_{in}(x_1),S]]\\ &\quad=\int_{-\infty}^{+\infty} d^4y_1 d^4y_2\,\Delta(x_1-y_1)\,\Delta(x_2-y_2)K_{y_1}K_{y_2}\\ &\qquad\times ST\phi(y_1)\phi(y_2)\end{aligned}}$$

Continuing we obtain the formula on page -110-

Now every operator can be expanded in terms of the in- or out- fields. Just as the in- and out- states are a complete set of states, the in- and out- fields are a complete set of operators, so

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\dots d^4x_n :\phi_{in}(x_1)\cdots\phi_{in}(x_n):\times$$

$$\times\,\mathcal{T}(x_1,\dots,x_n)$$

where we Normal order the in-fields So that

$$\langle 0in|S|0in\rangle = 1 \quad \text{as we choose}$$

and $\mathcal{T}(x_1,\dots,x_n)$ is a coefficient function to be determined. Calculating the n nested commutators of S with ϕ_{in} picks out the n^{th} coefficient

$$[\phi_{in}(x_n), [\cdots, [\phi_{in}(x_1), S]\cdots]$$

$$= \int_{-\infty}^{+\infty} d^4y_1 \cdots d^4y_n \, \Delta(x_1 - y_1) \cdots \Delta(x_n - y_n) \times$$

$$\times \, \sigma(y_1, \ldots, y_n) + O(:\phi_{in} \cdots \phi_{in}:)$$

where $\sigma(y_1, \ldots, y_n) = \sigma(y_{i_1}, \ldots, y_{i_n})$ is totally symmetric and the remaining terms are all normal ordered and contain at least one ϕ_{in}-field.

Thus taking the $\langle 0_{in}| \cdots |0_{in}\rangle$ matrix element and comparing to the matrix element of the nested commutator on p.-110- we find

$$\sigma(y_1, \ldots, y_n) = K_{y_1} \cdots K_{y_n} \times$$

$$\times \langle 0_{in}| S \, T \phi(y_1) \cdots \phi(y_n) |0_{in}\rangle$$

and since $\langle 0\,in| S = \langle 0\,out|$ we find the LSZ reduction formula on page -111- for the S-operator.

We can use the identity to express the Green functions of the interpolating fields in terms of the in-fields. Taking the vacuum expectation value of the operator identity on page -114-

$$S\,T\,\phi(x_1)\cdots\phi(x_n) = T\left[\phi_{in}(x_1)\cdots\phi_{in}(x_n)S\right]$$

we find using $\langle 0\,in|S = \langle 0\,out|$

$$\langle 0\,out|T\,\phi(x_1)\cdots\phi(x_n)|0\,in\rangle$$

$$= \frac{\langle 0\,in|T\,\phi_{in}(x_1)\cdots\phi_{in}(x_n)\,e^{-i\int_{-\infty}^{+\infty}dt\,H_I^{in}(\phi_{in},\pi_{in};t)}|0\,in\rangle}{\langle 0\,in|T\,e^{-i\int_{-\infty}^{+\infty}dt\,H_I^{in}(\phi_{in},\pi_{in};t)}|0\,in\rangle}$$

Since the RHS involves in-fields we have the equality to the corresponding out- and iP-field expression

$$\langle 0\,out | T\,\phi(x_1)\cdots\phi(x_n) | 0\,in\rangle$$

$$= \frac{\langle 0^{in}_{out} | T\,\phi_{\substack{in\\out}}(x_1)\cdots\phi_{\substack{in\\out}}(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(\phi_{\substack{in\\out}}, \pi_{\substack{in\\out}}; t)} | 0^{in}_{out}\rangle}{\langle 0^{in}_{out} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(\phi_{\substack{in\\out}}, \pi_{\substack{in\\out}}; t)} | 0^{in}_{out}\rangle}$$

$$= \frac{(0| T\,\phi^{iP}(x_1)\cdots\phi^{iP}(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{iP}(\phi^{iP}, \pi^{iP}; t)} |0)}{(0| T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{iP}(\phi^{iP}, \pi^{iP}; t)} |0)} .$$

We will use this expression shortly to given the time ordered functions as a Feynman diagram expression and use its similarity to the Feynman-Dyson expansion for S_{fi} to derive the LSZ-reduction formula. First we can further express the source of the field equation $j(x)$ in terms of the S-operator

In particular

$$S\, j(x) = i \frac{\delta S}{\delta \phi_{in}(x)}$$

where $\frac{\delta}{\delta\phi_{in}(x)}$ is the functional derivative with respect to $\phi_{in}(x)$. It gives the change in a functional of $\phi_{in}(x)$ if we change $\phi_{in}(x)$ at one point in space-time.

Namely if $\phi_{in}(x) \longrightarrow \phi_{in}(x) + \epsilon\, \delta^4(x-y)$

then $$\frac{\delta \phi_{in}(x)}{\delta \phi_{in}(y)} = \lim_{\epsilon\to 0} \frac{[\phi_{in}(x) + \epsilon\,\delta^4(x-y)] - \phi_{in}(x)}{\epsilon}$$

$$= \delta^4(x-y)$$

So $$\frac{\delta}{\delta\phi_{in}(y)} F[\phi_{in}(x)] = \frac{\partial F(x)}{\partial \phi_{in}(x)} \frac{\delta\phi_{in}(x)}{\delta \phi_{in}(y)}$$

$$= \frac{\partial F(x)}{\partial \phi_{in}(x)}\, \delta^4(x-y)$$

Also $$\frac{\delta}{\delta\phi_{in}(y)}\, \partial^\mu_x \phi_{in}(x) = \partial^\mu_x\, \delta^4(x-y) .$$

Now by our identity

$$S\, j(x) = T\left(j_{in}(x)\, S\right)$$

but we also have that

$$\frac{\delta S}{\delta \phi_{in}(x)} = e^{-i\theta} \frac{\delta}{\delta \phi_{in}(x)} T e^{-i\int_{-\infty}^{+\infty} d^4y\, \mathcal{H}_I^{in}(y)}$$

$$= e^{-i\theta} \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4y_1 \cdots d^4y_n \times$$

$$\times\, n\, T \frac{\delta \mathcal{H}_I^{in}(y_1)}{\delta \phi_{in}(x)} \mathcal{H}_I^{in}(y_2) \cdots \mathcal{H}_I^{in}(y_n)$$

where we relabelled the dummy integration variables to be $\mathcal{H}_I^{in}(y_1)$ that is always differentiated.

But $\frac{\delta \mathcal{H}_I^{in}(y)}{\delta \phi_{in}(x)} = \frac{\partial \mathcal{H}_I(y)}{\partial \phi_{in}(y)} \frac{\delta \phi_{in}(y)}{\delta \phi_{in}(x)}$

$$= j_{in}(y)\, \delta^4(x-y)$$

So re-labelling the indices again we have

$$\frac{\delta S}{\delta \phi_{in}(x)} = -i\, e^{-i\theta} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4y_1 \ldots d^4y_n \times$$

$$\times\, T\, j_{in}(x)\, \mathcal{H}_I^{in}(y_1) \ldots \mathcal{H}_I^{in}(y_n)$$

$$= -i\, T\big(j_{in}(x)\, S\big)$$

$$= -i S\, j(x)$$

Thus $\boxed{S j(x) = i \dfrac{\delta S}{\delta \phi_{in}(x)}}$ as desired.

Note, although messier we can continue this process to relate $S T j(x_1) \cdots j(x_n)$ to $i^n \dfrac{\delta^n S}{\delta \phi_{in}(x_1) \cdots \delta \phi_{in}(x_n)}$. However there are additional terms due to $\dfrac{\delta j_i(x_i)}{\delta \phi_{in}(x_j)}$ appearing. These will get cancelled against the equal time commutators of $\phi(x_i)$ and $j(x_j)$ when we convert the $S T j(x_1) \cdots j(x_n)$ to

$K_{x_1} \cdots K_{x_n} S T\phi(x_1)\cdots\phi(x_n)$ due to bringing time derivatives through T, as usual. So we obtain

$$K_{x_1} \cdots K_{x_n} S T\phi(x_1)\cdots\phi(x_n)$$
$$= i^n \frac{\delta^n S}{\delta\phi_{in}(x_1)\cdots\delta\phi_{in}(x_n)}$$

Hence we have a functional Taylor expansion for S. Again using the expansion

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n :\phi_{in}(x_1)\cdots\phi_{in}(x_n): \times$$
$$\times \sigma(x_1, \ldots, x_n)$$

we find

$$i^n \frac{\delta^n S}{\delta\phi_{in}(x_1)\cdots\delta\phi_{in}(x_n)} = \sigma(x_1,\ldots,x_n)$$
$$+ O(:\phi_{in}\cdots\phi_{in}:)$$

$$= K_{x_1} \cdots K_{x_n} S T\phi(x_1)\cdots\phi(x_n)$$

Taking $\langle 0\,\text{out}| \cdots |0\,\text{in}\rangle$ of this we have

$$\tau(x_1,\ldots,x_n) = K_{x_1}\cdots K_{x_n}\langle 0\,\text{out}|T\,\phi(x_1)\cdots\phi(x_n)|0\,\text{in}\rangle$$

as before.

Finally, we would like to compare our perturbative expansion for S_{fi} and $\langle 0\,\text{out}|T\,\phi(x_1)\cdots\phi(x_n)|0\,\text{in}\rangle$ in order to derive the LSZ reduction formula.

The time ordered product of operators was defined by

$$T\,B_1(x_1)\cdots B_n(x_n) \equiv \sum_{(1,\ldots,n)\xrightarrow{P}(i_1,\ldots,i_n)} \theta(x^0_{i_1}-x^0_{i_2})\,\theta(x^0_{i_2}-x^0_{i_3})$$

$$\times\cdots\theta(x^0_{i_{n-1}}-x^0_{i_n})\,B_{i_1}(x_{i_1})\cdots B_{i_n}(x_{i_n})$$

where the B_i are any operators such as the Heisenberg picture fields ϕ or $\phi_{\substack{in\\out}}$, or any composite field made from them like $\mathcal{H}_I^{in}$.

The Green functions (or time-ordered functions) are then given as the in- to out- vacuum expectation value of the time ordered product of operators. As usual we normalize such a matrix element by the in-to-out-vacuum transition amplitude which is not one in the presence of external fields. So

$$G^{(n)}(x_1,\ldots,x_n) \equiv \frac{\langle 0\,\mathrm{out}|T\phi(x_1)\cdots\phi(x_n)|0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle}$$

where to be concrete we work with the fundamental interpolating fields $\phi(x)$. Using our (perturbative) identity

$$ST\phi(x_1)\cdots\phi(x_n) = T(\phi_{in}(x_1)\cdots\phi_{in}(x_n)S)$$

we obtain

<u>Note</u>: Since $|\Psi in\rangle \equiv \dfrac{U(0,-\infty)|\Psi\rangle}{C_{in}\langle 0|U(0,-\infty)|0\rangle}$

$$|\Psi out\rangle \equiv \frac{U(0,+\infty)|\Psi\rangle}{C_{out}\langle 0|U(0,+\infty)|0\rangle}$$

we have with $S \equiv \dfrac{U(0,-\infty)U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}$

$$\frac{\langle 0out|T\phi(x_1)\cdots\phi(x_n)|0in\rangle}{\langle 0out|0in\rangle}$$

$$= \frac{\left[\dfrac{C_{in}\langle 0|U(0,-\infty)|0\rangle}{C_{out}\langle 0|U(0,+\infty)|0\rangle}\right]^* \langle 0|U(+\infty,-\infty)|0\rangle \times \langle 0in|ST\phi(x_1)\cdots\phi(x_n)|0in\rangle}{\left[\dfrac{\langle 0|U(+\infty,-\infty)|0\rangle}{C_{out}^* C_{in}\langle 0|U(0,+\infty)|0\rangle^*\langle 0|U(0,-\infty)|0\rangle}\right]}$$

$$= \overbrace{|C_{in}|^2|\langle 0|U(0,-\infty)|0\rangle|^2}^{=1 \text{ by the definition of } C_{in}}\langle 0in|ST\phi(x_1)\cdots\phi(x_n)|0in\rangle$$

$$= \langle 0in|ST\phi(x_1)\cdots\phi(x_n)|0in\rangle$$

$$= \langle 0in|T(\phi_{in}(x_1)\cdots\phi_{in}(x_n)S)|0in\rangle .$$

The point being with $S \equiv \dfrac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}$

we have

$$S|0\,out\rangle = \frac{C_{in}\langle 0|U(0,-\infty)|0\rangle}{C_{out}\langle 0|U(0,+\infty)|0\rangle}\,\frac{|0\,in\rangle}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

$$= \left[\frac{C_{in}\langle 0|U(0,-\infty)|0\rangle}{C_{out}\langle 0|U(0,+\infty)|0\rangle}\,\frac{1}{\langle 0|U(+\infty,-\infty)|0\rangle}\right]|0\,in\rangle$$

For no external fields we showed that we can choose (see p. -31'''- & -42'''-)

$$C_{in} = C_{out} \text{ and}$$

$$\frac{\langle 0|U(0,-\infty)|0\rangle}{\langle 0|U(0,+\infty)|0\rangle} = \langle 0|U(+\infty,-\infty)|0\rangle .$$

That's Recall page -94- for $|0\,in\rangle = S|0\,out\rangle$ we should define $S = e^{-i(\theta_{in}-\theta_{out})}\,U(0,-\infty)\,U^{-1}(0,+\infty)$

$$= \left[e^{-i(\theta_{in}-\theta_{out})}\langle 0|U(+\infty,-\infty)|0\rangle\right]\frac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle} ;$$

The Gell-Mann — Low expansion

$$\boxed{\begin{aligned} G^{(n)}(x_1,\dots,x_n) &= \frac{\langle 0\,\mathrm{out}|T\phi(x_1)\cdots\phi(x_n)|0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle} \\ &= \frac{\langle 0\,\mathrm{in}|T\phi_{in}(x_1)\cdots\phi_{in}(x_n)\,e^{-i\int_{-\infty}^{+\infty}dt\,H_I^{in}(t)}|0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{in}|T\,e^{-i\int_{-\infty}^{+\infty}dt\,H_I^{in}(t)}|0\,\mathrm{in}\rangle}. \end{aligned}}$$

Of course we could have derived a similar formula for the arbitrary in-, out- matrix element of the time ordered product of any set of interacting fields

$$\frac{\langle \chi\, out | T B_1(x_1) \cdots B_n(x_n) | \psi\, in \rangle}{\langle \chi\, out | \psi\, in \rangle}$$

$$= \frac{\langle \chi_{in} | T B_1^{in}(x_1) \cdots B_n^{in}(x_n) e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(t)} | \psi in \rangle}{\langle \chi_{in} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(t)} | \psi in \rangle}.$$

where B_i^{in} is the field B_i with all fields ϕ replaced by ϕ^{in};
$B_i^{in}(x) = B_i(\phi^{in}(x))$. That is if $B_i = B_i(\phi)$ then $B_i^{in} = B_i(\phi^{in})$.

Hence we see that time ordered functions have perturbative expansions very similar to those of the S-matrix the only difference being that the external lines entering and leaving the graph are given by free propagators rather than the in-coming or outgoing plane wave function of the particle.

To be concrete let's consider the example of a self-interacting Hermitian scalar field $\phi(x)$ with mass m; for short this is called $\lambda\phi^4$ theory.

The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 .$$

The Legendre transform to the Hamiltonian formulation is found first by calculating the momentum

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

Then
$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L}$$
$$= \frac{1}{2}\Pi^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

So that the unperturbed Hamiltonian is given by
$$\mathcal{H}_0 \equiv \frac{1}{2}\Pi^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}m^2\phi^2$$

with interaction Hamiltonian $\mathcal{H}_I = \frac{\lambda}{4!}\phi^4$.

The in- or out-fields are described by the free Hamiltonian $\mathcal{H}_{0in}$ or equivalently by that free Lagrangian
$$\mathcal{L}_{0in} = \frac{1}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \frac{1}{2}m^2\phi_{in}^2$$

From which the field equations follow as Euler-Lagrangian eq. of motion (or from $\mathcal{H}_{0in}$ as the Heisenberg eq. of motion)
$$\frac{\partial\mathcal{L}_{0in}}{\partial\phi_{in}} - \partial_\mu\frac{\partial\mathcal{L}_{0in}}{\partial\partial_\mu\phi_{in}} = 0$$
$$\Rightarrow \quad -(\partial^2 + m^2)\phi_{in}(x) = 0$$

And the ETCR result in

$$\delta(x^0-y^0)[\dot\phi_{in}(x),\phi_{in}(y)]=-i\delta^4(x-y).$$

This together with the field equation result in the in-field time ordered product being

$$\langle 0in|T\phi_{in}(x)\phi_{in}(y)|0in\rangle=\Delta_F(x-y)$$

$$=\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}$$

The interacting field time ordered functions are given by the Gell-Mann Low formula

$$G^{(n)}(x_1,\ldots,x_n)\equiv\frac{\langle 0out|T\phi(x_1)\ldots\phi(x_n)|0in\rangle}{\langle 0out|0in\rangle}$$

$$=\frac{\langle 0in|T\phi_{in}(x_1)\cdots\phi_{in}(x_n)e^{+i\int d^4x\,\mathcal{L}_{I\,in}(x)}|0in\rangle}{\langle 0in|Te^{+i\int d^4x\,\mathcal{L}_{I\,in}(x)}|0in\rangle}$$

(Note: we have no ext. fields so $\langle 0out|0in\rangle=1$ here)

$$G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0in| T\phi_{in}(x_1)\cdots\phi_{in}(x_n)\, e^{-\frac{i\lambda}{4!}\int dx\, \phi_{in}^4(x)} |0in\rangle}{\langle 0in| T e^{-\frac{i\lambda}{4!}\int dx\, \phi_{in}^4(x)} |0in\rangle}$$

As with the S-matrix Feynman-Dyson expansion we can use Wick's theorem in order to evaluate this perturbative expansion order by order.

$$\langle 0in| T\phi_{in}(x_1)\cdots\phi_{in}(x_n)|0in\rangle$$

$$= \begin{cases} \displaystyle\sum_{\substack{(1,\dots,n)\\ \to (i_1,j_1)\dots(i_{n/2},j_{n/2})\\ i_a < i_{a+1};\ i_a<j_a}} \langle 0in|T\phi_{in}(x_{i_1})\phi_{in}(x_{j_1})|0in\rangle \cdots \langle 0in|T\phi_{in}(x_{i_{n/2}})\phi_{in}(x_{j_{n/2}})|0in\rangle & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{\text{pairs}} \prod_{a=1}^{n/2} \Delta_F(x_{i_a} - x_{j_a}) & , n = \text{even} \\ 0 & , n = \text{odd.} \end{cases}$$

Hence the numerator factors into vacuum bubbles (i.e. just terms involving contractions amongst $\mathcal{L}_{I\,in}$ fields only no external fields $\phi_{in}(x_1)\cdots\phi_{in}(x_n)$) times terms each of which involves contractions with at least one external field $\phi_{in}(x_1)\cdots\phi_{in}(x_n)$.

Then GML expansion becomes

$$G^{(n)}(x_1,\dots,x_n) = \sum_{l=0}^{\infty} \left(\frac{-i\lambda}{4!}\right)^l \int d^4y_1\cdots d^4y_l \times$$

$$\times \langle 0in | T\phi_{in}(x_1)\cdots\phi_{in}(x_n)\phi^4_{in}(y_1)\cdots\phi^4_{in}(y_l)|0in\rangle_{NVB}$$

where the subscript NVB indicates no vacuum bubbles contribute to the sum.

In order to develop the Feynman rules for the time ordered function let's consider an example, a specific second order contribution to the 4-point function, expanding first we have

$$G^{(4)}(x_1,\ldots,x_4) = \frac{\langle 0\text{out}|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\text{in}\rangle}{\langle 0\text{out}|0\text{in}\rangle}$$

$$= \langle 0\text{in}|T\phi_{in}(x_1)\phi_{in}(x_2)\phi_{in}(x_3)\phi_{in}(x_4)|0\text{in}\rangle$$

$$-\frac{i\lambda}{4!}\int d^4y\,\langle 0\text{in}|T\phi_{in}(x_1)\cdots\phi_{in}(x_4)\,\phi^4_{in}(y)|0\text{in}\rangle_{NVB}$$

$$+\frac{(-i\lambda)^2}{2(4!)^2}\int d^4y_1\int d^4y_2\,\langle 0\text{in}|T\phi_{in}(x_1)\cdots\phi_{in}(x_4)\,\phi^4_{in}(y_1)\,\phi^4_{in}(y_2)|0\text{in}\rangle_{NVB}$$

$$+\cdots$$

$$= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \text{permutations}$$

$$-\frac{i\lambda}{2}\int d^4y\,\Delta_F(x_1-x_2)\Delta_F(x_3-y)\Delta_F(x_4-y)\Delta_F(y-y)$$
$$+\text{perm.}$$

$$-i\lambda\int d^4y\,\Delta_F(x_1-y)\Delta_F(x_2-y)\Delta_F(x_3-y)\Delta_F(x_4-y)$$

$$+\frac{(-i\lambda)^2}{2\cdot 2}\int dy_1 dy_2\,\Delta_F(x_1-x_2)\Delta_F(x_3-y_1)\Delta_F(x_4-y_1)$$
$$[\Delta_F(y_1-y_2)]^2\Delta_F(y_2-y_2) + \text{perm.}$$

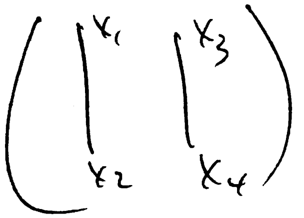
$+ \cdots$

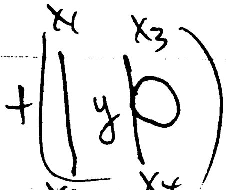
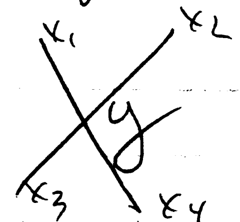
$$+ \frac{(-i\lambda)^2}{2} \int dy_1 dy_2 \, \Delta_F(x_1 - y_1) \Delta_F(x_2 - y_2) [\Delta_F(y_1 - y_2)]^2$$
$$\Delta_F(y_2 - x_3) \Delta_F(y_2 - x_4) \quad + \text{perm.}$$

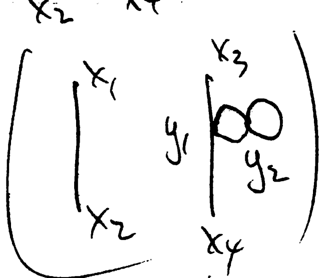
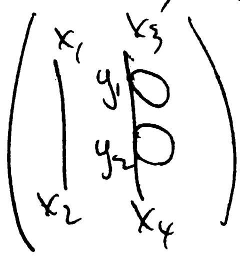
$\cdots$

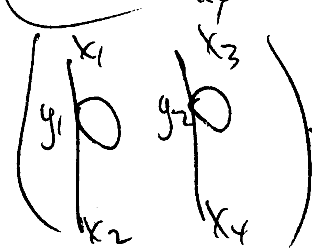
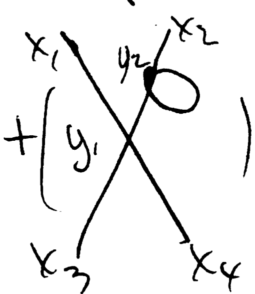
Where we have not listed all possible contractions to save time but the student is urged to work these details out in full!

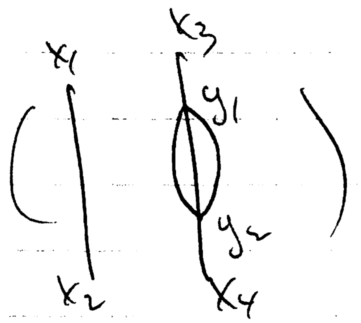
We can represent these contributions graphically in coordinate space

$G^{(4)}(x_1, \ldots, x_4) =$  + perm

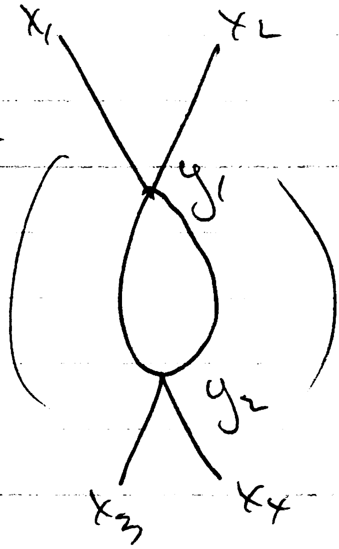
\+  + perm + 

\+  + perm. +  + perm.

\+  + perm +  + perm

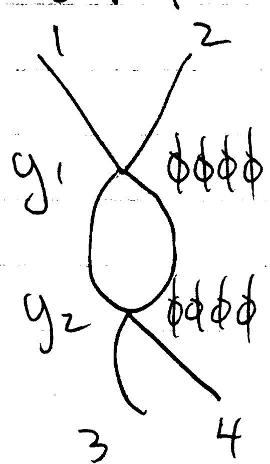
$+$ 

$+$ perm

$+$ 

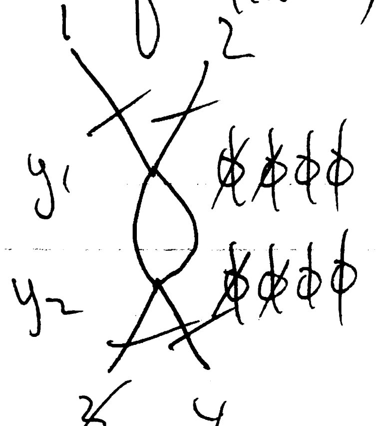
$+$ perm. $+ O(\lambda^3)$

As usual the combinatoric factors in the above mathematical expressions are calculated directly from Wick's theorem — they are just the number of ways the contractions can be made. For instance consider the last graph above



0) label external lines
1) there are 2! ways to label interaction vertices, pick one and label

2) at y_1 vertex there are 4 choices $\phi_{in}^4(y_1)$ for $\phi_{in}(x_1)$ to contract with and 3 for $\phi_{in}(x_2)$; similarly 4 for $\phi_{in}(x_3)$ and 3 for $\phi_{in}(x_4)$



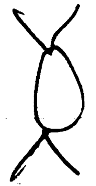
3) This leaves $\phi_{in}^2(y_1)$ contracted with $\phi_{in}^2(y_2)$ there are only 2 ways for this to occur.

4) So altogether we have

$$\frac{1}{2!}\frac{1}{(4!)^2}(2)(4\cdot 3)(4\cdot 3)(2) = \frac{1}{2}$$

The overall combinatoric factor for this contribution to $G^{(4)}$ is $\frac{1}{2}$.

It is much more useful to calculate these graphs in momentum space rather than coordinate space and to develop the GML expansion as a momentum space Feynman diagram expansion.

To proceed, consider the last contribution to $G^{(4)}$; call is $G^{(4)}_\Gamma$ where $\Gamma =$ 

$$G^{(4)}_\Gamma = \frac{(-i\lambda)^2}{2}\int dy_1\, dy_2\, \Delta_F(x_1-y_1)\Delta_F(x_2-y_1)$$

$$\Delta_F(y_1-y_2)\Delta_F(y_1-y_2)\Delta_F(x_3-y_2)\Delta_F(x_4-y_2)$$

$$= \frac{(-i\lambda)^2}{2}\int dy_1\, dy_2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}$$

$$e^{-ip_1(x_1-y_1)}\, e^{-ip_2(x_2-y_1)}\, e^{+ip_3(y_2-x_3)}\, e^{-ip_4(x_4-y_2)}$$

$$e^{-ik_1(y_1-y_2)}\, e^{-ik_2(y_1-y_2)}\, \frac{i}{p_1^2-m^2+i\epsilon}$$

$$\cdots \frac{i}{p_4^2-m^2+i\epsilon}\, \frac{i}{k_1^2-m^2+i\epsilon}\, \frac{i}{k_2^2-m^2+i\epsilon}$$

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_4}{(2\pi)^4} e^{-ip_ix_i} \frac{i}{p_1^2-m^2+i\epsilon} \cdots \frac{i}{p_4^2-m^2+i\epsilon}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} dy_1 dy_2\, e^{-i(k_1+k_2-p_1-p_2)y_1} e^{+i(p_3+p_4+k_1+k_2)y_2}$$

$$\times \frac{i}{k_1^2-m^2+i\epsilon} \frac{i}{k_2^2-m^2+i\epsilon}$$

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_4}{(2\pi)^4} e^{-ip_ix_i} \frac{i}{p_1^2-m^2+i\epsilon} \cdots \frac{i}{p_4^2-m^2+i\epsilon}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2) (2\pi)^4 \delta^4(k_1+k_2+p_3+p_4)$$

$$\frac{i}{k_1^2-m^2+i\epsilon} \frac{i}{k_2^2-m^2+i\epsilon}$$

$$= \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_4}{(2\pi)^4} e^{-ip_ix_i} (2\pi)^4 \delta^4(p_1+p_2+p_3+p_4) \frac{(-i\lambda)^2}{2}$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2-m^2+i\epsilon]} \frac{i}{[(p_1+p_2-k)^2-m^2+i\epsilon]}$$

$$\frac{i}{p_1^2-m^2+i\epsilon} \cdots \frac{i}{p_4^2-m^2+i\epsilon}$$

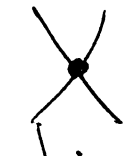
We can glean the following Feynman rules for calculating the Green functions from the above example.
(For $G^{(n)}(x_1, \cdots, x_n)$)

1) List all Feynman diagrams contributing to the n-point function (excluding vacuum bubbles). The diagrams consist of

1) n external lines entering the diagram at one vertex or 2 fields making a straight through line attached to no vertex

2) internal lines joining two vertices (or possibly the same vertex at both ends)

3) vertices which are points where the lines meet. For $\lambda\phi^4$ theory four lines meet at each vertex



The number of vertices = the order of perturbation theory. $= V$

The number of external lines $= n$
The number of internal lines, L, is determined since 4 lines meet at each vertex.

Thus, the total number of fields meeting at V vertices is

4V. This comes from each end of an internal line and one end of an external line. Then $4V = 2L + n$
(this excludes straight through lines)

2) Label all momentum flow through the diagram with each external line carrying its momentum p_i <u>into</u> the graph by convention. There's overall E-M conservation as well as conservation at each vertex.

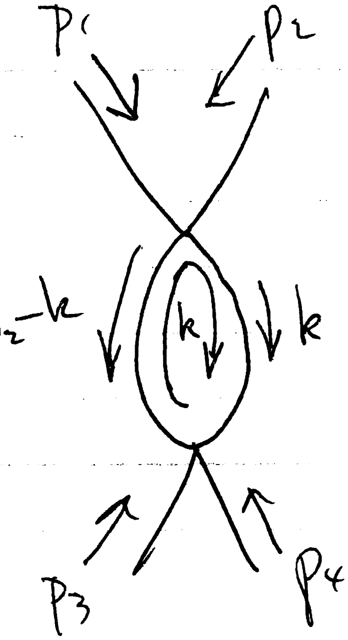
each internal line has momentum of the form

$$l = q + k$$

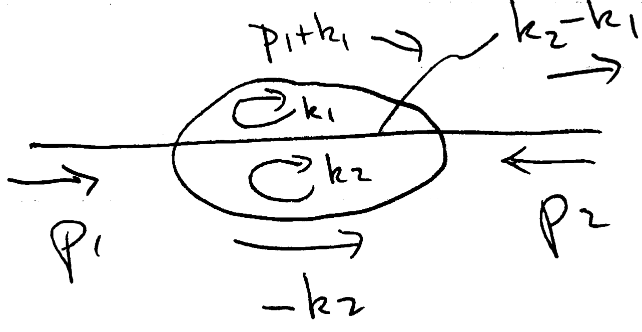
where q is the sum of external momentum flowing thru the line and k is the sum of internal loop momenta flowing through line. For each independent loop in the graph there is a loop momentum flowing around the loop.

examples

1)



2)



3) For each external line we have a Fourier transform factor

$$\int \frac{d^4 p_i}{(2\pi)^4}\, e^{-ip_i x_i}$$

4) For each connected subdiagram we have a $(2\pi)^4 \delta^4(\sum_i p_i)$ an associated over E-M conserving δ-function

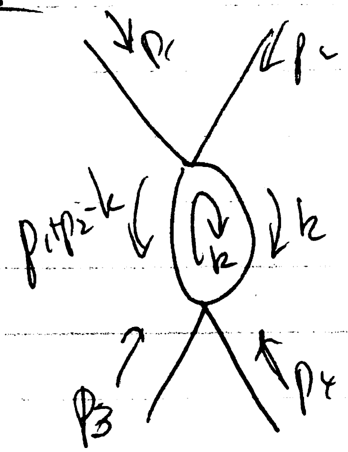
5) A combinatoric factor called the symmetry number $\alpha(\Gamma)$ of the graph Γ in question calculated directly from Wick's theorem.

6) An integration factor for each loop momentum $\int \frac{d^4 k}{(2\pi)^4}$

7) A Feynman propagator for each line. $\xrightarrow{\quad p \quad} \longleftrightarrow \frac{i}{p^2 - m^2 + i\epsilon}$

8) $(-i\lambda)$ a coupling constant for each vertex.

Examples

$\Gamma =$ 

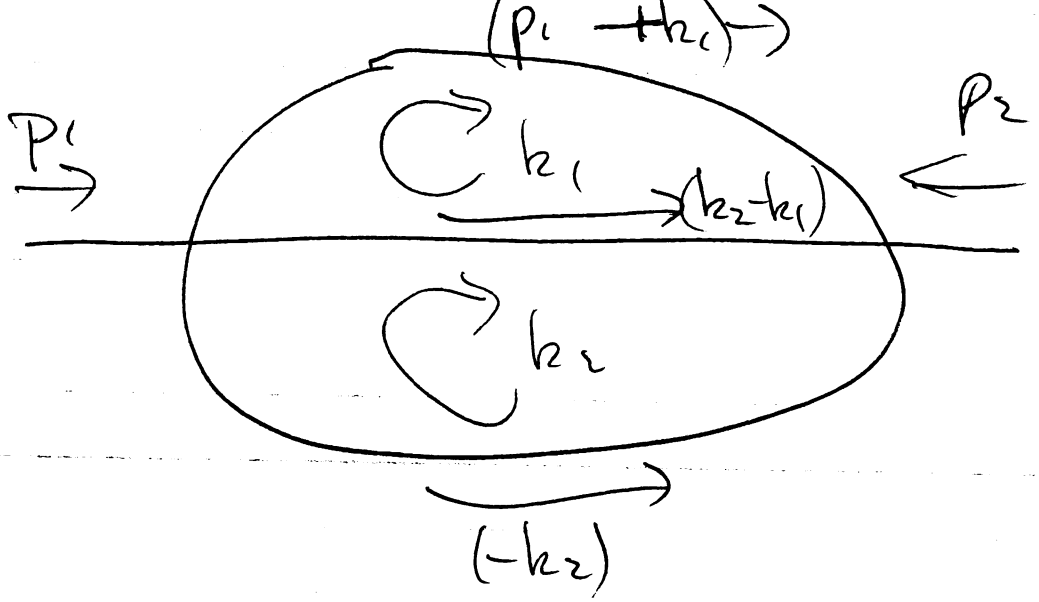
The corresponding contribution to $G^{(n)}$ is denoted

$$G^{(4)}_\Gamma \quad \text{or} \quad \langle 0\,\text{out} | T\,\phi(x_1)\cdots\phi(x_4)|0\,\text{in}\rangle_\Gamma$$

$$G^{(4)}_\Gamma(x_1,\dots,x_4) = \frac{(-i\lambda)^2}{2}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}\, e^{-ip_i x_i}(2\pi)^4\delta(p_1+p_2+p_3+p_4)$$

$$\int\frac{d^4k}{(2\pi)^4}\frac{i}{p_1^2-m^2+i\epsilon}\cdots\frac{i}{p_4^2-m^2+i\epsilon}$$

$$\frac{i}{k^2-m^2+i\epsilon}\;\frac{i}{[(p_1+p_2-k)^2-m^2+i\epsilon]}$$

$\Gamma =$ 

$$G_\Gamma^{(2)}(x_1,x_2) = \frac{(-i\lambda)^2}{6}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} e^{-ip_1x_1} e^{-ip_2x_2} (2\pi)^4\delta^4(p_1+p_2)$$

$$\frac{i}{p_1^2-m^2+i\epsilon}\,\frac{i}{p_2^2-m^2+i\epsilon} \times \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{[(p_1+k_1)^2-m^2+i\epsilon]}$$

$$\frac{i}{[(k_2-k_1)^2-m^2+i\epsilon]}\,\frac{i}{[k_2^2-m^2+i\epsilon]}$$