as well as

$$\langle f\,in|S|i\,in\rangle = \sum_k \langle f\,in|k\,in\rangle\langle k\,out|i\,in\rangle = \sum_k \delta_{fk}\langle k\,out|i\,in\rangle = \langle f\,out|i\,in\rangle \quad (1.2.93)$$

Since the scattering transition amplitudes are given in terms of the in- and out-states it would be convenient to have a method of constructing these states analogous to the Fock space construction of the iP initial and final states. The in-states and out-states are related to the iP states by operation of $U(0, \pm\infty)$, hence, these are the unitary transformations of the iP fields that allow us to build the in- and out-states from the action of creation operators on the vacuum. That is, cryptically, we have the Fourier expansion of the iP fields given by (1.1.15)

In the case of no external fields we have that

$$\langle 0\,out | 0\,in \rangle = 1 \qquad (\text{page } 49''')$$

That is $|0\,in\rangle = |0\,out\rangle$, the arbitrary phase factor between the states being defined to one!
Then

$$S_{fi} = \langle f\,out | i\,in \rangle = \langle f\,out | S | i\,out \rangle$$
$$= \langle f\,in | S | i\,in \rangle .$$

Thus the vacuum is stable. In the presence of external fields $\langle 0\,out | 0\,in \rangle$ is not simply a phase but a complex functional of the fields and so the in- and out-vacua are different. As usual, for particle state scattering, we normalize the transition amplitude by the in-vacuum to out-vacuum transition amplitude, so in general

$$S_{fi} = \frac{\langle f\,out | i\,in \rangle}{\langle 0\,out | 0\,in \rangle} .$$

$$\phi^{iP}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s \Big[b_s(\vec{k})\, u^{(s)}(\vec{k})\, e^{-ikx} + d_s^\dagger(\vec{k})\, v^{(s)}(\vec{k})\, e^{+ikx} \Big]. \qquad (1.2.94)$$

The iP initial and final states are then given by

$|0\rangle$ vacuum state

$b_s^\dagger(\vec{k})|0\rangle$ one-particle state

$d_s^\dagger(\vec{k})|0\rangle$ one-anti-particle state

$\vdots$

$b_{s_1}^\dagger(\vec{k}_1)\cdots b_{s_N}^\dagger(\vec{k}_N)\, d_{\bar{s}_1}^\dagger(\vec{k}_1)\cdots d_{\bar{s}_{\bar{N}}}^\dagger(\vec{k}_{\bar{N}})|0\rangle$

the (N-particle, $\bar{N}$-anti-particle) state (1.2.95)

and so on.

The in- or out-states are obtained by the action of $U(0,\pm\infty)$ on these states. Thus we find we can write

$$\mathcal{U}(0,\pm\infty)\left(b^{\dagger}_{s_1}(\vec{k}_1)\cdots d^{\dagger}_{\bar{s}_{\bar{N}}}(\vec{k}_{\bar{N}})|0\rangle\right)$$

$$=\left[\mathcal{U}(0,\pm\infty)\,b^{\dagger}_{s_1}(\vec{k}_1)\,\mathcal{U}^{-1}(0,\pm\infty)\right]\left[\mathcal{U}(0,\pm\infty)\,b^{\dagger}_{s_2}(\vec{k}_2)\,\mathcal{U}^{-1}(0,\pm\infty)\right]$$
$$\mathcal{U}(0,\pm\infty)\cdots\left[\mathcal{U}(0,\pm\infty)\,d^{\dagger}_{\bar{s}_{\bar{N}}}(\vec{k}_{\bar{N}})\,\mathcal{U}^{-1}(0,\pm\infty)\right]\mathcal{U}(0,\pm\infty)|0\rangle \qquad (1.2.96)$$

Defining in- and out- creation and annihilation operators as

$$b_{\substack{in\\out}\,s}(\vec{k})\equiv\mathcal{U}(0,\mp\infty)\,b_s(\vec{k})\,\mathcal{U}^{-1}(0,\mp\infty) \qquad (1.2.97)$$

So

$$b^{\dagger}_{\substack{in\\out}\,s}(\vec{k})=\mathcal{U}(0,\mp\infty)\,b^{\dagger}_s(\vec{k})\,\mathcal{U}^{-1}(0,\mp\infty) \qquad (1.2.98)$$

Similarly for the anti-particles

$$d_{\substack{in\\out}\,s}(\vec{k})\equiv\mathcal{U}(0,\mp\infty)\,d_s(\vec{k})\,\mathcal{U}^{-1}(0,\mp\infty)$$

and

$$d^{\dagger}_{\substack{in\\out}\,s}(\vec{k})=\mathcal{U}(0,\mp\infty)\,d^{\dagger}_s(\vec{k})\,\mathcal{U}^{-1}(0,\mp\infty) \qquad (1.2.99)$$

Hence the in- and out-states can be built up as Fock space states from the in- and out- vacua, for the initial (N-particle, $\bar{N}$-anti-particle) state we have

$$|(N,\bar{N})\,in\rangle = \frac{U(0,-\infty)\,|(N,\bar{N})\rangle}{\langle 0|U(0,-\infty)|0\rangle\, C_{in}}$$

$$= b^{\dagger}_{ins_1}(\vec{k}_1)\, b^{\dagger}_{ins_2}(\vec{k}_2)\cdots d^{\dagger}_{in\bar{s}_{\bar{N}}}(\vec{k}_{\bar{N}}) \times$$

$$\times \frac{U(0,-\infty)\,|0\rangle}{\langle 0|U(0,-\infty)|0\rangle\, C_{in}}$$

$$= b^{\dagger}_{ins_1}(\vec{k}_1)\cdots d^{\dagger}_{in\bar{s}_{\bar{N}}}(\vec{k}_{\bar{N}})\,|0\,in\rangle . \tag{1.2.100}$$

Similarly for the out-states, for the outgoing (N particle, $\bar{N}$ anti-particle) state we have

$$|(N,\bar{N})\,out\rangle = \frac{U(0,+\infty)\,|(N,\bar{N})\rangle}{\langle 0|U(0,+\infty)|0\rangle\, C_{out}}$$

$$= b^{\dagger}_{outs_1}(\vec{k}_1)\cdots d^{\dagger}_{out\bar{s}_{\bar{N}}}(\vec{k}_{\bar{N}})\,|0\,out\rangle . \tag{1.2.101}$$

Since the in- and out- operators are related to the iP operators by a unitary transformation they obey the same CCR or CAR

$$[b_{\substack{in\\out}\,s}(\vec{k}), b^{\dagger}_{\substack{in\\out}\,r}(\vec{p})]_{\pm}$$

$$= U(0,\mp\infty)[b_s(\vec{k}), b^{\dagger}_r(\vec{p})]_{\pm} U^{-1}(0,\mp\infty)$$

$$= (2\pi)^3 2\omega_k \delta_{rs} \delta^3(\vec{p}-\vec{k}) U(0,\mp\infty) U^{-1}(0,\mp\infty)$$

$$= (2\pi)^3 2\omega_k \delta_{rs} \delta^3(\vec{p}-\vec{k}). \quad (1.2.102)$$

Similarly $[d_{\substack{in\\out}\,s}(\vec{k}), d^{\dagger}_{\substack{in\\out}\,r}(\vec{p})]_{\pm}$

$$= (2\pi)^3 2\omega_k \delta_{rs} \delta^3(\vec{p}-\vec{k}) \quad (1.2.103)$$

with all other commutators or anti-commutators vanishing. Note that

the energy $\omega_k = +\sqrt{\vec{k}^2 + m^2}$ is the same for the in, out or iP states. It is the eigenvalue of the full Hamiltonian as we saw in the derivation of the Lippmann - Schwinger equation. Thus the initial and final particles of the in- and out states are physical particles they have the renormalized (physical) masses of the particles.

Since the asymptotic ip energy, momentum, spin and charge operators can be constructed in terms of the Fock space creation and annihilation operators, we, by the above unitary transformations to the in- and out- operators, can construct the Heisenberg pic operators in terms of the in- and out operators! That is recall

$$H = U(0,\pm\infty)\, H_0^{ip}\, U^{-1}(0,\pm\infty) \qquad (1.2.104)$$

where $H = H(\phi^H, \pi^H)$ and H_0^{ip} is simply

$$H_0^{ip} = H_0^{ip}(\phi^{ip}, \pi^{ip}) = H_0^{ip}(b, d)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s \omega_k \left(b_s^\dagger(\vec{k})\, b_s(\vec{k}) + d_s^\dagger(\vec{k})\, d_s(\vec{k}) \right)$$

(1.2.105)

So

$$H(\phi^H, \Pi^H) = U(0,\pm\infty) H_0^{ip} U^{-1}(0,\pm\infty)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s{}' \omega_k \, U(0,\pm\infty) \left[b_s^\dagger(\vec{k}) b_s(\vec{k}) + d_s^\dagger(\vec{k}) d_s(\vec{k}) \right] U^{-1}(0,\pm\infty)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s{}' \omega_k \left[b^\dagger_{\substack{out \\ in}\,s}(\vec{k}) \, b_{\substack{out \\ in}\,s}(\vec{k}) + d^\dagger_{\substack{out \\ in}\,s}(\vec{k}) \, d_{\substack{out \\ in}\,s}(\vec{k}) \right]$$

$$= H_0(b_{\substack{in \\ out}}, d_{\substack{in \\ out}}) \quad !!$$

$$\equiv H_0^{\substack{in \\ out}} \qquad (1.2.106)$$

The in- and out- operators have the amazing property of "diagonalizing" the full Hamiltonian. Also we have simply

$$\vec{P} = U(0,\mp\infty) \vec{P}^{ip} U^{-1}(0,\mp\infty)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s{}' \vec{k} \left[b^\dagger_{\substack{in \\ out}\,s}(\vec{k}) \, b_{\substack{in \\ out}\,s}(\vec{k}) + d^\dagger_{\substack{in \\ out}\,s}(\vec{k}) \, d_{\substack{in \\ out}\,s}(\vec{k}) \right]$$

$$\vec{\Phi} \equiv \vec{\Phi}^{\text{in}}_{\text{out}} \quad (1.2.107)$$

and likewise $J_3 = J_3^{\text{in}}_{\text{out}}$; $Q = Q^{\text{in}}_{\text{out}}$ with the "free" field expression for $J_3^{\text{in}}_{\text{out}}$, $Q^{\text{in}}_{\text{out}}$ as in the case of $\vec{\Phi}^{\text{in}}_{\text{out}}$, $H_0^{\text{in}}_{\text{out}}$ in terms of the in-and-out-operators.

Rather than continue working in momentum space with the creation and annihilation operators directly, we can Fourier transform to ~~space~~-time and define $\phi_{\text{in}\atop\text{out}}(x)$, $\Pi_{\text{in}\atop\text{out}}(x)$.

Thus, being somewhat pedantic, we have, with $k^0 = \omega_k$ as usual

$$\phi_{\text{in}\atop\text{out}}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s \left[b_{\text{in}\atop\text{out}\,s}(\vec{k})\, u^{(s)}(\vec{k})\, e^{-ikx} + d^{\dagger}_{\text{in}\atop\text{out}\,s}(\vec{k})\, v^{(s)}(\vec{k})\, e^{+ikx} \right]$$

but

$$= U(0,\mp\infty)\, \phi^{ip}(x)\, U^{-1}(0,\mp\infty) \quad (1.2.108)$$

Similarly

$$\Pi_{\text{in}\atop\text{out}}(x) = U(0,\mp\infty)\, \Pi^{ip}(x)\, U^{-1}(0,\mp\infty) \quad (1.2.109)$$

We also have the ETCR & ETAR for the space-time in- and out- fields,

$$\delta(x^0-y^0)\left[\Pi_{\substack{in\\out}}(x), \phi_{\substack{in\\out}}(y)\right]_{\pm} = -i\delta^4(x-y)$$

$$\delta(x^0-y^0)\left[\phi_{\substack{in\\out}}(x), \phi_{\substack{in\\out}}(y)\right]_{\pm} = 0$$

$$\delta(x^0-y^0)\left[\Pi_{\substack{in\\out}}(x), \Pi_{\substack{in\\out}}(y)\right]_{\pm} = 0 . \tag{1.2.110}$$

The in- and out- fields can be related to the Heisenberg and Schrödinger picture fields as usual

$$\phi_{\substack{in\\out}}(x) = U(0,\mp\infty)\,\phi^{IP}(x)\,U^{-1}(0,\mp\infty)$$

$$= U(0,\mp\infty)\,U(t,0)\,\phi(x)\,U^{-1}(t,0)\,U^{-1}(0,\mp\infty)$$

$$= U(0,\mp\infty)\,e^{+iH_0^S t}\,e^{-iHt}\,\phi(x)\times$$

$$\times e^{+iHt}\,e^{-iH_0^S t}\,U^{-1}(0,\mp\infty) \tag{1.2.111}$$

but $U(0,\pm\infty)\,e^{+iH_0^S t} = e^{+iHt}\,U(0,\pm\infty)$

Since $U(0,\pm\infty)\,H_0^S = H\,U(0,\pm\infty)$

So
$$\phi_{\substack{in\\out}}(x) = \left[e^{+iHt}\, U(0,\mp\infty)\, e^{-iHt}\right]\phi(x) \times \left[e^{+iHt}\, U^{-1}(0,\mp\infty)\, e^{-iHt}\right] \qquad (1.2.112)$$

defining
$$U^{(\pm)}(t) \equiv e^{+iHt}\, U(0,\mp\infty)\, e^{-iHt} \qquad (1.2.113)$$

we find that the in- and out-fields are related to the Heisenberg picture full (interacting) fields by a unitary transformation

$$\boxed{\phi_{\substack{in\\out}}(x) = U^{(\pm)}(t)\,\phi(x)\,U^{(\pm)^{-1}}(t)} \qquad (1.2.114)$$

Since $\phi(x)$ is a Heisenberg picture field we have
$$\phi(\vec{x},t) = e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt} = e^{iHt}\,\phi^{s}(\vec{x})\,e^{-iHt} \qquad (1.2.115)$$

Thus
$$\phi_{\substack{in\\out}}(x) = U^{(\pm)}(t)\,e^{iHt}\,\phi^{s}(\vec{x})\,e^{-iHt}\,U^{(\pm)^{-1}}(t) \qquad (1.2.116)$$

So
$$\boxed{\phi_{\substack{in\\out}}(\vec{x},0) = U(0,\mp\infty)\,\phi^{s}(\vec{x})\,U^{-1}(0,\mp\infty)} \qquad (1.2.117)$$

More relevant is the fact that from eq. (1.2.112)

$$\phi_{\substack{in\\out}}(x) = \left[e^{+iHt}\, U(0,\mp\infty)\, e^{-iHt}\right]\left[\phi(x)\right] \times \left[e^{+iHt}\, U^{-1}(0,\mp\infty)\, e^{-iHt}\right]$$

$$= e^{+iHt}\left(U(0,\mp\infty)\,\phi(\vec{x},0)\,U^{-1}(0,\mp\infty)\right)e^{-iHt} \quad (1.2.118)$$

but since $\phi(\vec{x},0) = \phi^S(\vec{x})$ we have from above

$$\phi_{\substack{in\\out}}(\vec{x},0) = U(0,\mp\infty)\,\phi(\vec{x},0)\,U^{-1}(0,\mp\infty) \quad (1.2.119)$$

So

$$\boxed{\phi_{\substack{in\\out}}(x) = e^{+iHt}\,\phi_{\substack{in\\out}}(\vec{x},0)\,e^{-iHt}} \quad (1.2.120)$$

Thus $\phi_{\substack{in\\out}}(x)$, as expected, are Heisenberg picture fields obeying the Heisenberg equation of motion

$$-i\frac{\partial}{\partial t}\phi_{\substack{in\\out}}(x) = \left[H, \phi_{\substack{in\\out}}(x)\right] \quad (1.2.121)$$

Likewise for the momentum

$$\Pi_{\substack{in\\out}}(x) = e^{+iHt}\, \Pi_{\substack{in\\out}}(\vec{x},0)\, e^{-iHt} \qquad (1.2.122)$$

So

$$-i\frac{\partial}{\partial t}\Pi_{\substack{in\\out}}(x) = \left[H, \Pi_{\substack{in\\out}}(x)\right]. \qquad (1.2.123)$$

Note that $H = H(\phi,\Pi)$ is the full Hamiltonian in terms of the full interacting Heisenberg picture fields above. However recall equation (1.2.106)

$$H(\phi,\Pi) = H_0(\phi_{\substack{in\\out}}, \Pi_{\substack{in\\out}}) \equiv H_0^{\substack{in\\out}} \qquad (1.2.124)$$

That is more carefully

$$H = e^{+iHt}\, H\, e^{-iHt}$$
$$= e^{+iHt}\left(U(0,\mp\infty) H_0^{ip} U^{-1}(0,\mp\infty)\right) e^{-iHt} \qquad (1.2.125)$$

but $H_0^{ip} = H_0^{s} = H_0(0)$

So
$$H = e^{+iHt}\, U(0,\mp\infty)\, H_0(0)\, U^{-1}(0,\mp\infty)\, e^{-iHt} \quad (1.2.126)$$

now we have $H_0(t) = e^{+iHt} H_0(0) e^{-iHt}$
since it is in the Heisenberg representation

$$H = U^{(\pm)}(t)\, H_0(t)\, U^{(\pm)-1}(t) \qquad (1.2.127)$$

Here $U^{(\pm)}(t) H_0(t) U^{(\pm)-1}(t)$ is just the free Hamiltonian written in terms of in- and out-operators, thus

$$H(\phi,\pi) = H = U^{(\pm)}(t)\, H_0(t)\, U^{(\pm)-1}(t)$$
$$= H_0(\phi_{\substack{in\\out}}, \pi_{\substack{in\\out}}) \equiv H_0^{\substack{in\\out}} \,. \quad (1.2.128)$$

Thus if $H(\phi,\pi)$ is expressed in terms of $\phi_{\substack{in\\out}}$ and $\pi_{\substack{in\\out}}$ by

$$\phi(x) = U^{(\pm)-1}(t)\, \phi_{\substack{in\\out}}(x)\, U^{(\pm)}(t)$$
$$\pi(x) = U^{(\pm)-1}(t)\, \pi_{\substack{in\\out}}(x)\, U^{(\pm)}(t) \qquad (1.2.129)$$

The result is just the free Hamiltonian

$$H(\phi,\pi) = H_0(\phi_{\substack{in\\out}}, \pi_{\substack{in\\out}}) \quad . \qquad (1.2.130)$$

The in- and out-fields are said to diagonalize the interacting (full) field Hamiltonian.

Further the in- and out-fields obey the Heisenberg equations of motion with the free Hamiltonian written in terms of in- and out-fields. Equations (1.2.121, 123) become

$$-i\frac{\partial}{\partial t}\phi_{\substack{in\\out}}(x) = [H_0^{\substack{in\\out}}, \phi_{\substack{in\\out}}(x)]$$

$$-i\frac{\partial}{\partial t}\pi_{\substack{in\\out}}(x) = [H_0^{\substack{in\\out}}, \pi_{\substack{in\\out}}(x)] \qquad (1.2.131)$$

Of course this result followed immediately from the Fourier transform of $b_{\substack{in\\out}}, d_{\substack{in\\out}}$, the creation and annihilation operators since

$$[H_0^{\substack{in\\out}}, b_{\substack{in\\out} s}(\vec{k})] = -\omega_k b_{\substack{in\\out} s}(\vec{k})$$

$$[H_0^{\substack{in\\out}}, d^\dagger_{\substack{in\\out} s}(\vec{k})] = +\omega_k d^\dagger_{\substack{in\\out} s}(\vec{k}) \qquad (1.2.132)$$

according to equations (1.2.102), (1.2.106)

As a matter of review, let's digress to refresh ourselves then with the various in- and out-field systems we studied in our introductory course. They described incoming and outgoing collections of spin $0, \frac{1}{2}, 1$ particles:

1) The Hermitian scalar field describing spin 0, mass m bosons. Denote the field $\phi_{in}(x)$, it obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi_{in}(x) = 0. \quad (1.2.133)$$

The Lagrangian for such a field is

$$N[\mathcal{L}_0^{in}] = N\left[\tfrac{1}{2}\partial_\mu\phi_{in}\partial^\mu\phi_{in} - \tfrac{1}{2}m^2\phi_{in}^2\right] \quad (1.2.134)$$

where $N[\cdots]$ denotes the Wick normal product.

The canonically conjugate in-momentum is defined as

$$\pi_{in} \equiv \frac{\partial\mathcal{L}_0^{in}}{\partial\dot{\phi}_{in}} = \dot{\phi}_{in} \quad (1.2.135)$$

with ETCR

$$\delta(x^0 - y^0)\left[\pi_{in}(x), \phi_{in}(y)\right] = -i\delta^4(x-y) \quad (1.2.136)$$

That is

$$\delta(x^0-y^0)[\dot\phi_{in}(x),\phi_{in}(y)]=-i\delta^4(x-y),$$

all others vanishing. (1.2.137)

Hence the Fourier decomposition for such a field is

$$\phi_{in}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\left[a_{in}(\vec{k})e^{-ikx}+a^\dagger_{in}(\vec{k})e^{+ikx}\right] \qquad (1.2.138)$$

where $\omega_k=+\sqrt{\vec{k}^2+m^2}$ and $k^0=\omega_k$ is understood

The Fourier expansion can be inverted to obtain

$$a_{in}(\vec{k})=i\int d^3x\, e^{ikx}\overleftrightarrow{\partial_0}\phi_{in}(x)$$

$$a^\dagger_{in}(\vec{k})=i\int d^3x\,\phi_{in}(x)\overleftrightarrow{\partial_0}e^{-ikx}. \qquad (1.2.139)$$

The ETCR then imply

$$[a_{in}(\vec{k}),a^\dagger_{in}(\vec{l})]=(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{l}),$$

all others vanishing. (1.2.140)

The all-time, or covariant, commutators of the field can be found using the creation and annihilation operator CCR

Defining the covariantly positive and negative frequency components of ϕ_{in}

$$\phi^+_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_{in}(\vec{k})\, e^{-ikx}$$

$$\phi^-_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger_{in}(\vec{k})\, e^{+ikx} \qquad (1.2.141)$$

so that $\phi_{in} = \phi^+_{in} + \phi^-_{in}$

we find

$$[\phi^+_{in}(x), \phi^-_{in}(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \equiv i\Delta^+(x-y)$$

$$[\phi^-_{in}(x), \phi^+_{in}(y)] = -i\Delta^+(y-x) = -\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+ik(x-y)}$$

$$\equiv +i\Delta^-(x-y) \,. \qquad (1.2.142)$$

Then the covariant commutator is simply the sum of Δ^+ and Δ^-

$$[\phi_{in}(x), \phi_{in}(y)] = i\Delta^+(x-y) + i\Delta^-(x-y)$$

$$\equiv i\Delta(x-y)$$

$$= -i\int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(k^2-m^2)\,\epsilon(k^0)\, e^{-ik(x-y)}$$

$$= \frac{i}{2\pi}\epsilon(x^0)\left\{\delta(x^2) - \frac{m^2}{2}\theta(x^2)\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right\} \quad (1.2.143)$$

The Fock space formed by the action of the creation operators $a^{\dagger}_{in}(\vec{k})$ on the in-vacuum $|0\,in\rangle$ forms a basis for the Hilbert space of states

$$\mathcal{H} = |0\,in\rangle \oplus |\vec{k}\,in\rangle = a^{\dagger}_{in}(\vec{k})|0\,in\rangle$$
$$\oplus\ |\vec{k}_1, \vec{k}_2\,in\rangle = a^{\dagger}_{in}(\vec{k}_1)a^{\dagger}_{in}(\vec{k}_2)|0\,in\rangle$$
$$\oplus \cdots \quad \text{is defined by} \qquad (11.2.144)$$

where $|0\,in\rangle$ ~~:~~ $a_{in}(\vec{k})|0\,in\rangle = 0$ with inner product determined by the 1-particle subspace, assuming $\langle 0^{in}|0\,in\rangle = 1$

$$\langle \vec{k}\,in|\vec{l}\,in\rangle = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{l}) \qquad (11.2.145)$$

and completeness given by the resolution of the identity on this subspace

$$1 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\vec{k}\,in\rangle\langle \vec{k}\,in| \quad . \qquad (11.2.146)$$

The states are eigenstates of P^{μ}, J_3

Since $\mathcal{P}^\mu = \mathcal{P}^\mu_{in}$; $\mathcal{J}_3 = \mathcal{J}_{3in}$

$$\mathcal{P}^\mu |\vec{k}\ in\rangle = k^\mu |\vec{k}\ in\rangle$$

$$\mathcal{J}_3 |\vec{k}=0\ in\rangle = 0 \qquad \text{with } k^\mu = (\omega_k, \vec{k}) .$$

(11.2.147)

2) Complex spinor (Dirac spinor) field describing spin $\frac{1}{2}$, mass m, charged fermions. Denote the field by $\Psi_{in}(x)$ (4-component), it obeys the Dirac eq.

$$(i\not{\partial}-m)\Psi_{in}=0, \qquad (1.2.148)$$

with $\not{\partial} \equiv \gamma^\mu \partial_\mu$ and γ^μ the 4×4 Dirac matrices.

The Lagrangian for such a field is

$$N[\mathcal{L}_{0in}]=N_0[\bar{\Psi}_{in}(\tfrac{i}{2}\overleftrightarrow{\not{\partial}}-m)\Psi_{in}]$$

$$=N_0[\bar{\Psi}_{in}(i\overrightarrow{\not{\partial}}-m)\Psi_{in}]$$

$$+N_0[-\tfrac{i}{2}\partial_\mu(\bar{\Psi}_{in}\gamma^\mu\Psi_{in})]$$

$$=N_0[\bar{\Psi}_{in}(-i\overleftarrow{\not{\partial}}-m)\Psi_{in}]$$

$$+N_0[\tfrac{i}{2}\partial_\mu(\bar{\Psi}_{in}\gamma^\mu\Psi_{in})]. \qquad (1.2.149)$$

The canonically conjugate momentum to Ψ_{in} is defined by the last Lagrangian with the total derivative ignored

$$\bar{\Pi}_{in}=\frac{\partial \mathcal{L}_{0in}}{\partial \dot{\Psi}_{in}}=-i\gamma^0\Psi_{in} \qquad (1.2.150)$$

with ETAR

$$\delta(x^0-y^0)\{\overline{\pi}_{in\,a}(x), \Psi_{in\,b}(y)\} = -i\,\delta_{ab}\,\delta^4(x-y) \quad (1.2.151)$$

that is

$$\delta(x^0-y^0)\{\Psi_{in\,a}(x), \overline{\Psi}_{in\,b}(y)\} = \gamma^0_{ab}\,\delta^4(x-y) \quad (1.2.152)$$

or using the definition of the adjoint field

$$\overline{\Psi}_{in} \equiv \Psi^\dagger_{in}\gamma^0 \quad (1.2.153)$$

we have

$$\delta(x^0-y^0)\{\Psi_{in\,a}(x), \Psi^\dagger_{in\,b}(y)\} = \delta_{ab}\,\delta^4(x-y) \quad (1.2.154)$$

all others vanishing.

Using the Dirac equation the Fourier transform for Ψ_{in} is

$$\Psi_{in}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \Big[b_{in\,s}(\vec{k})\,u^{(s)}(\vec{k})\,e^{-ikx} + d^\dagger_{in\,s}(\vec{k})\,v^{(s)}(\vec{k})\,e^{+ikx} \Big] \quad (1.2.155)$$

with $k^0 = +\omega_k$; $\omega_k = +\sqrt{\vec{k}^2 + m^2}$ understood.

Also $u^{(s)}(\vec{k})$ and $v^{(s)}(\vec{k})$ are the 4 independent momentum space plane wave solutions to the Dirac eq.

$$(\not{k} - m)\, u^{(s)}(\vec{k}) = 0$$

$$(\not{k} + m)\, v^{(s)}(\vec{k}) = 0 \qquad (1.2.156)$$

normalized so that

$$\bar{u}^{(r)}(\vec{k})\, u^{(s)}(\vec{k}) = \delta_{rs}\, 2m$$

$$\bar{v}^{(r)}(\vec{k})\, v^{(s)}(\vec{k}) = -\delta_{rs}\, 2m$$

$$\bar{u}^{(r)}(\vec{k})\, v^{(s)}(\vec{k}) = 0 = \bar{v}^{(r)}(\vec{k})\, u^{(s)}(\vec{k}) \qquad (1.2.157)$$

and complete so that

$$\sum_{s=1}^{2} u^{(s)}(\vec{k})\, \bar{u}^{(s)}(\vec{k}) = (\not{k} + m)$$

$$\sum_{s=1}^{2} v^{(s)}(\vec{k})\, \bar{v}^{(s)}(\vec{k}) = (\not{k} - m) \qquad (1.2.158)$$

hence $$\sum_{s=1}^{2} \frac{1}{2m} \left[u^{(s)}(\vec{k})\, \bar{u}^{(s)}(\vec{k}) - v^{(s)}(\vec{k})\, \bar{v}^{(s)}(\vec{k}) \right] = 1 \qquad (1.2.159)$$

To be concrete we can choose u, v to be eigenstates of γ^0 (Dirac eq.) and σ^{12} (the z-component of spin $J_3 = J^{12}$) at rest $\vec{k} = 0$, then u, v are completely determined. Another choice could be the u, v are solutions to the Dirac eq. i.e. eigenstates of k, and eigenstates of the helicity operator $h \equiv \frac{\vec{k} \cdot \vec{\sigma}}{|\vec{k}|}$, $\vec{\sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12})$. All such choices are linear combinations, boosts and rotations of eachother.

Using the properties of u, v we can invert the Fourier expansion to obtain

$$b_{\mathrm{in}\,s}(\vec{k}) = \int d^3x\, e^{ikx}\, \bar{u}^{(s)}(\vec{k}) \gamma^0 \Psi_{\mathrm{in}}(x)$$

$$d^{\dagger}_{\mathrm{in}\,s}(\vec{k}) = \int d^3x\, e^{-ikx}\, \bar{v}^{(s)}(\vec{k}) \gamma^0 \Psi_{\mathrm{in}}(x)$$

$$b^{\dagger}_{\mathrm{in}\,s}(\vec{k}) = \int d^3x\, \bar{\Psi}_{\mathrm{in}}(x) \gamma^0 u^{(s)}(\vec{k})\, e^{-ikx}$$

$$d_{\mathrm{in}\,s}(\vec{k}) = \int d^3x\, \bar{\Psi}_{\mathrm{in}}(x) \gamma^0 v^{(s)}(\vec{k})\, e^{+ikx} . \tag{1.2.160}$$

The ETAC then imply

$$\{b_{in\,r}(\vec{p}), b^{\dagger}_{in\,s}(\vec{k})\} = (2\pi)^3 (2\omega_k)\delta_{rs}\delta^3(\vec{p}-\vec{k})$$

$$\{d_{in\,r}(\vec{p}), d^{\dagger}_{in\,s}(\vec{k})\} = (2\pi)^3 (2\omega_k)\delta_{rs}\delta^3(\vec{p}-\vec{k})$$

(1.2.161)

all others vanishing.

As in the scalar case the covariant anti-commutators can be found with the help of the above CAR.

Defining the covariant decomposition of Ψ_{in} into positive and negative frequency components

$$\Psi^{+}_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} b_{in\,s}(\vec{k})\, u^{(s)}(\vec{k})\, e^{-ikx}$$

$$\Psi^{-}_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} d^{\dagger}_{in\,s}(\vec{k})\, v^{(s)}(\vec{k})\, e^{+ikx}$$

(1.2.162)

so that

$$\Psi_{in} = \Psi^{+}_{in} + \Psi^{-}_{in}.$$

We find

$$\{\Psi^{+}_{in}(x), \overline{\Psi}^{-}_{in}(y)\} = (i\not{\partial}_x + m)\, i\Delta^{+}(x-y)$$
$$\equiv i S^{+}(x-y) \qquad (1.2.163)$$

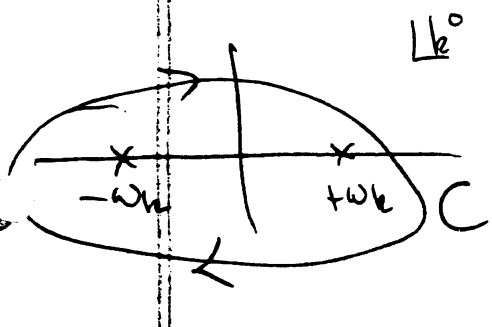
and

$$\{\Psi^{-}_{in}(x), \overline{\Psi}^{+}_{in}(y)\} = (i\not{\partial}_x + m)\, i\Delta^{-}(x-y)$$
$$\equiv i S^{-}(x-y). \qquad (1.2.164)$$

Hence the covariant anti-commutator becomes

$$\{\Psi_{in}(x), \overline{\Psi}_{in}(y)\} = i(i\not{\partial}_x + m)\Delta(x-y)$$
$$= i S^{+}(x-y) + i S^{-}(x-y)$$
$$\equiv i S(x-y).$$

$$= \int_C \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \frac{1}{\not{k} - m} \qquad (1.2.165)$$



In addition to describing spin $\frac{1}{2}$ particles, the Dirac field describes charged particles.

The charge operator is defined as

$$Q \equiv \int d^3x \, N\left[\bar{\Psi}_{in} \gamma^0 \Psi_{in} \right]$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left[b^{\dagger}_{in\,s}(\vec{k}) b_{in\,s}(\vec{k}) - d^{\dagger}_{in\,s}(\vec{k}) d_{in\,s}(\vec{k}) \right]$$

(1.2.166)

Hence

$$[Q, b_{in\,s}(\vec{k})] = -b_{in\,s}(\vec{k})$$
$$[Q, b^{\dagger}_{in\,s}(\vec{k})] = +b^{\dagger}_{in\,s}(\vec{k})$$
$$[Q, d_{in\,s}(\vec{k})] = +d_{in\,s}(\vec{k})$$
$$[Q, d^{\dagger}_{in\,s}(\vec{k})] = -d^{\dagger}_{in\,s}(\vec{k}) \, .$$

(1.2.167)

The Hilbert space of in-states is spanned by the Fock space made from $d^\dagger_{in\,s}$ and $b^\dagger_{in\,s}$ acting on the in-vacuum $|0\,in\rangle$ defined by

$$b_{in_s}(\vec{k})\,|0\,in\rangle = 0 = d_{in\,s}(\vec{k})\,|0\,in\rangle \qquad (1.2.168)$$

So that

$$\mathcal{H} = |0\,in\rangle \oplus |\vec{k}, \tfrac{(-1)^{s+1}}{2}, +)\,in\rangle \equiv b^\dagger_{s\,in}(\vec{k})\,|0\,in\rangle$$
$$\oplus\ |\vec{k}, \tfrac{(-1)^{s}}{2}, -)\,in\rangle \equiv d^\dagger_{in\,s}(\vec{k})\,|0\,in\rangle \qquad (1.2.169)$$
$$\oplus \ \cdots$$

where the single particle states have the normalization

$$\langle(\vec{k}, \tfrac{(-1)^{r+1}}{2}, +)\,in\,|(\vec{k}', \tfrac{(-1)^{r'+1}}{2}, +)\,in\rangle$$
$$= (2\pi)^3 (2\omega_k)\,\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')$$

$$\langle(\vec{k}, \tfrac{(-1)^{r}}{2}, -)\,in\,|(\vec{k}', \tfrac{(-1)^{r'}}{2}, -)\,in\rangle$$
$$= (2\pi)^3\, 2\omega_k\,\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')$$

$$\langle(\vec{k}, \pm\tfrac{1}{2}, +)\,in\,|(\vec{k}', \pm\tfrac{1}{2}, -)\,in\rangle = 0 \qquad (1.2.170)$$

while the 1 particle subspace resolution of the identity reflects this inner product

$$1 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \Big[|(\vec{k}, \tfrac{(-1)^{r+1}}{2}, +)\, in\rangle\langle(\vec{k}, \tfrac{(-1)^{r+1}}{2}, +)\, in| + |(\vec{k}, \tfrac{(-1)^{r}}{2}, -)\, in\rangle\langle(\vec{k}, \tfrac{(-1)^{r}}{2}, -)\, in| \Big]$$

(1.2.171)

These states are eigenstates of the CSCO $\{H, \vec{P}, J_3, Q\}$

$$P^\mu |(\vec{k}, \pm\tfrac{1}{2}, \pm)\, in\rangle = k^\mu |(\vec{k}, \pm\tfrac{1}{2}, \pm)\, in\rangle$$

$$Q\, |(\vec{k}, \pm\tfrac{1}{2}, \pm)\, in\rangle = \pm\, |(\vec{k}, \pm\tfrac{1}{2}, \pm)\, in\rangle$$

$$J_3 |(\vec{0}, \tfrac{(-1)^{s+1}}{2}, +)\, in\rangle = \frac{(-1)^{s+1}}{2} |(\vec{0}, \tfrac{(-1)^{s+1}}{2}, +)\, in\rangle$$

$$J_3 |(\vec{0}, \tfrac{(-1)^{s}}{2}, -)\, in\rangle = \frac{(-1)^{s}}{2} |(\vec{0}, \tfrac{(-1)^{s}}{2}, -)\, in\rangle.$$

(1.2.172)

3) Hermitian, 4-vector Maxwell field describing spin 1, mass 0 particles "photons". Denote the field by $A^{\mu}_{in}(x)$, it obeys the Maxwell equations in the Stueckelberg gauge

$$\partial_\mu F^{\mu\nu}_{in} + \frac{1}{\alpha}\partial^\nu \partial_\lambda A^{\lambda}_{in} = 0 \qquad (1.2.173)$$

with $F^{\mu\nu}_{in} \equiv \partial^\mu A^\nu_{in} - \partial^\nu A^\mu_{in}$ and α a arbitrary positive real #.

The Stueckelberg gauge Lagrangian is given by

$$N[\mathcal{L}_{0in}] = N\left[-\frac{1}{4}F_{in\mu\nu}F^{\mu\nu}_{in} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda_{in})^2\right] \qquad (1.2.174)$$

hence the momentum canonically conjugate to A^μ_{in} is

$$\Pi^\mu_{in} \equiv \frac{\partial \mathcal{L}_{0in}}{\partial \dot{A}_{in\mu}} = F^{\mu 0}_{in} - \frac{1}{\alpha}g^{\mu 0}\partial_\lambda A^\lambda_{in}. \qquad (1.2.175)$$

with ETCR

$$\delta(x^0-y^0)\left[\Pi^\mu_{in}(x), A^\nu_{in}(y)\right] = -i\,\delta^\mu{}_\nu\, \delta^4(x-y),$$

all others vanishing. (1.2.176)

That is, after some algebra,

(7μ)
$$\delta(x^0-y^0)\left[\dot{A}^\mu_{in}(x), A^\nu_{in}(y)\right] = i g^{\mu\nu}\left[1+(\alpha-1)g^{\mu 0}\right]\delta^4(x-y)$$

$$\delta(x^0-y^0)\left[\dot{A}^0_{in}(x), \dot{A}^i_{in}(y)\right] = i\,(1-\alpha)\,\partial_x^i\, \delta^4(x-y)$$

all others vanishing. (1.2.177)

Without loss of generality we can work in the Feynman gauge $\alpha \equiv 1$, then the field equations become

$$\partial^2 A^\mu_{in} = 0 \qquad (1.2.178)$$

and the ETCR become

$$\delta(x^0-y^0)\left[\dot{A}^\mu_{in}(x), A^\nu_{in}(y)\right] = i g^{\mu\nu}\delta^4(x-y). \qquad (1.2.179)$$

This looks like 4 massless Klein-Gordon fields except the $\dot{A}^0_{in}$ commutator has the "wrong" sign.

The photon field can be Fourier decomposed to yield

$$A^{\mu}_{in}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}$$
$$\times \sum_{\lambda=0}^{3} \left[\epsilon^{\mu}(k,\lambda)\, a_{in\,(\lambda)}(\vec{k})\, e^{-ikx} + \epsilon^{*\mu}(k,\lambda)\, a^{\dagger}_{in\,(\lambda)}(\vec{k})\, e^{+ikx} \right] \quad (1.2.180)$$

with $k^0 = +\omega_k$; $\omega_k = +|\vec{k}|$ and $\epsilon^{\mu}(k,\lambda)$ the 4 polarization 4 vectors which are orthonormal

$$\epsilon^{\mu}(k,\lambda)\, \epsilon_{\mu}(k,\rho) = g_{\lambda\rho} \quad (1.2.181)$$

and complete

$$\sum_{\lambda=0}^{3} \epsilon^{\mu}(k,\lambda)\, \epsilon^{\nu}(k,\lambda)\, g_{\lambda\lambda} = +g^{\mu\nu}$$

$$\left(\text{or } \sum_{\lambda=0}^{3} \epsilon^{\mu}(k,\lambda)\, \epsilon^{\nu}(k,\lambda) = \delta^{\mu\nu} \right) \quad (1.2.182)$$

Such a choice is

$$\epsilon^\mu(k,\lambda) = \begin{cases} n^\mu \equiv (1,0,0,0) & \lambda = 0 \\ (0, \vec{\epsilon}_\lambda(\vec{k})) & \lambda = 1,2 \text{ (the direction of plane polarization)} \\ \dfrac{k^\mu - (n\cdot k) n^\mu}{\sqrt{(n\cdot k)^2 - k^2}} = \left(0, \dfrac{\vec{k}}{|\vec{k}|}\right) & \text{on shell.} \end{cases}$$

(1.2.183)

The Fourier transforms can be inverted to yield

$$a_{in(\lambda)}(\vec{k}) = i\int d^3x\, \epsilon^\nu(k,\lambda) g_{\lambda\lambda} e^{ikx} \overleftrightarrow{\partial_0} A_{in\,\nu}(x)$$
$$a^\dagger_{in(\lambda)}(\vec{k}) = i\int d^3x\, A^\nu_{in}(x) \overleftrightarrow{\partial_0} \epsilon_\nu(k,\lambda) g_{\lambda\lambda} e^{-ikx}.$$

(1.2.184)

The ETCR for Feynman gauge $\alpha = 1$ yield the indefinite metric CCR

$$[a_{in(\lambda)}(\vec{k}), a^\dagger_{in(\rho)}(\vec{l})] = -g_{\lambda\rho}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{l})$$

(1.2.185)

all others vanishing.

As in the scalar case the covariant commutator can be formed using the positive and negative frequency decomposition of the photon field

$$A_{in}^{\mu +}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \epsilon^{\mu}(k,\lambda)\, a_{in(\lambda)}(\vec{k})\, e^{-ikx}$$

and

$$A_{in}^{\mu -}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \overset{*}{\epsilon}{}^{\mu}(k,\lambda)\, a_{in(\lambda)}^{\dagger}(\vec{k})\, e^{+ikx}$$

(1.2.186)

with

$$A_{in}^{\mu} = A_{in}^{\mu +} + A_{in}^{\mu -}.$$

We then find the usual relations

$$[A_{in}^{\mu +}(x), A_{in}^{\nu -}(y)] = -g^{\mu\nu}\, i\Delta^{+}(x-y)$$

$$[A_{in}^{\mu -}(x), A_{in}^{\nu +}(y)] = -g^{\mu\nu}\, i\Delta^{-}(x-y) \quad (1.2.187)$$

So that

$$[A_{in}^{\mu}(x), A_{in}^{\nu}(y)] = -g^{\mu\nu}\, i\Delta(x-y).$$

(1.2.188)

The Fock space constructed from the creation operators acting on the in-vacuum $|0\,in\rangle$

$$\mathcal{V} = \begin{cases} |0\,in\rangle \\ |\vec{k},\lambda\,in\rangle = a^{\dagger}_{in(\lambda)}(\vec{k})|0\,in\rangle \\ \vdots \text{ etc.} \end{cases} \qquad (1.2.189)$$

describes 4 types of state

two physical photons for $\lambda = 1,2$

with helicity ± 1 i.e.

$$a^{\dagger}_{in\pm}(\vec{k}) \equiv \mp\frac{1}{\sqrt{2}}\left[a^{\dagger}_{in(1)}(\vec{k}) \mp i\, a^{\dagger}_{in(2)}(\vec{k})\right] \qquad (1.2.190)$$

and

$$|k,\pm\, in\rangle \equiv a^{\dagger}_{in\pm}(\vec{k})|0\,in\rangle \qquad (1.2.191)$$

and two unphysical degrees of freedom the scalar photon

$$|\vec{k},0\,in\rangle \equiv a^{\dagger}_{in(0)}(\vec{k})\,|0\,in\rangle \qquad (1.2.192)$$

and the longitudinal photon

$$|\vec{k}, 3\ \text{in}\rangle \equiv a^{\dagger}_{\text{in}(3)}(\vec{k})|0\ \text{in}\rangle \qquad (1.2.193)$$

These have helicity 0.

So
$$\frac{\vec{S}^{0\vec{k}}_{\text{in}}}{|\vec{k}|}|\vec{k}, \lambda\ \text{in}\rangle = \begin{cases} \pm\ |\vec{k}, \lambda\ \text{in}\rangle & \lambda = \pm \\ 0 & \lambda = 0, 3 \end{cases}.$$

(1.2.194)

The "inner" product of this space is indefinite

$$\langle \vec{k}, \lambda\ \text{in}|\vec{l}, \rho\ \text{in}\rangle = -g_{\lambda\rho} \overset{(2\pi)^3}{\vee} 2\omega_k\, \delta^3(\vec{k} - \vec{l}). \qquad (1.2.195)$$

The Gupta-Bleuler subsidiary condition picks out the physical Hilbert subspace, $\mathcal{H}_{phys.}$, from $\mathcal{H}$ in a covariant manner

$$\partial_\mu A^{\mu +}_{\text{in}}(x)\,|\phi\ \text{in}\rangle = 0, \qquad (1.2.196)$$

if $|\phi\ \text{in}\rangle \in \mathcal{H}_{phys}$.

$|\phi\ \text{in}\rangle$ has the general form

$$|\phi_{in}\rangle = |\phi_{tr\,in}\rangle |\hat{\phi}_{in}\rangle \quad (1.2.197)$$ where

$|\phi_{tr\,in}\rangle$ consists only of transverse photon modes i.e. $\lambda = 1,2$ or $\lambda = \pm$, while $|\hat{\phi}_{in}\rangle$ is $\ni$

$$G_{in}(\vec{k}) \equiv [a_{in(0)}(\vec{k}) - a_{in(3)}(\vec{k})]|\hat{\phi}_{in}\rangle = 0. \quad (1.2.198)$$

Thus we found that

$$|\hat{\phi}_{in}\rangle = \Big[1 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} C(\vec{k}) G^\dagger_{in}(\vec{k}) + \dots + \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2\omega_{k_n}} C(\vec{k}_1, \dots, \vec{k}_n) G^\dagger_{in}(\vec{k}_1) \cdots G^\dagger_{in}(\vec{k}_n) + \dots \Big] |0\,in\rangle. \quad (1.2.199)$$

Then for $|\phi\,in\rangle, |\psi\,in\rangle \in \mathcal{H}_{phys}$.

$$\langle \phi\,in|\psi\,in\rangle = \langle \phi_{tr}\,in|\psi_{tr}\,in\rangle.$$

$$\langle \phi\,in|P^\mu_{in}|\psi\,in\rangle = \langle \phi_{tr}\,in|P^\mu_{in\,tr}|\psi_{tr}\,in\rangle$$

$$\langle \phi\,in|A^{\mu_1}_{in}(x_1)\cdots A^{\mu_n}_{in}(x_n)|\psi\,in\rangle = \langle \phi_{tr}\,in|A^{\mu_1}_{in\,tr}(x_1)\cdots A^{\mu_n}_{in\,tr}(x_n)|\psi_{tr}\,in\rangle. \quad (1.2.200)$$

4) <u>Homework</u>: Hermitian, 4-vector Proca field describing spin 1, mass m particles like the $W^{\pm}, Z^0$ intermediate vector bosons mediating the weak interactions. Denote the field by $B^{\mu}_{in}(x)$, it obeys the Proca equation

$$\partial_\mu F^{\mu\nu}_{in} + m^2 B^{\nu}_{in} = 0 \;, \qquad (1.2.201)$$

~~Use~~ with $F^{\mu\nu}_{in} \equiv \partial^\mu B^\nu_{in} - \partial^\nu B^\mu_{in}$.

Use the condition that

$$\cancelto{0}{\partial_\nu \partial_\mu F^{\mu\nu}_{in}} + m^2 \partial_\nu B^\nu_{in} = 0 \qquad (1.2.202)$$

inorder to analyze the particle states described by this field.

So much for our review of in- and out- states and fields.

Finally we notice that since the in- and out- fields and Fock space have the same mathematical structure, i.e. creation and annihilation operators on a vacuum state as the iP initial and final states and fields. They are free fields. Thus we immediately recognize that the Feynman-Dyson perturbative formula for the S-matrix elements in terms of the iP states and fields has the same mathematical structure if we replace the iP quantities with the in- or out- quantities, that is

$$S_{fi} = \frac{(f|U^{ip}(+\infty,-\infty)|i)}{(0|U^{ip}(+\infty,-\infty)|0)}$$

$$= \frac{(f|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{ip}(\phi^{ip},\pi^{ip};t)}|i)}{(0|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{ip}(\phi^{ip},\pi^{ip};t)}|0)}$$

$$= \frac{\langle f\,{}^{in}_{out}|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{{}^{in}_{out}}(\phi_{{}^{in}_{out}},\pi_{{}^{in}_{out}};t)}|i\,{}^{in}_{out}\rangle}{\langle 0^{in}_{out}|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{{}^{in}_{out}}(\phi_{{}^{in}_{out}},\pi_{{}^{in}_{out}};t)}|0^{in}_{out}\rangle},$$

The mechanics of evaluating the free in- or out-field matrix elements is identical to that of the iP case. Hence we arrive at the same Feynman diagram expansion, just our viewpoint has changed, we are working with Heisenberg picture states and fields.

This result of equivalence between matrix elements of course follows from the fact that it is a unitary transformation that relates the iP fields and states to the in- and out-fields and states, that is

$$c_{\substack{in \\ out}} |\psi_{\substack{in \\ out}}\rangle = \frac{U(0,\mp\infty)|\psi\rangle}{\langle 0|U(0,\mp\infty)|0\rangle}$$

$$\phi_{\substack{in \\ out}}(x) = U(0,\mp\infty)\, \phi^{iP}(x)\, U^{-1}(0,\mp\infty)$$

$$\pi_{\substack{in \\ out}}(x) = U(0,\mp\infty)\, \pi^{iP}(x)\, U^{-1}(0,\mp\infty)$$

So for any in- or out- matrix element we have

$$\langle \chi_{\substack{in\\out}} | \phi_{\substack{in\\out}}(x_1) \cdots \phi_{\substack{in\\out}}(x_n) | \psi_{\substack{in\\out}} \rangle$$

$$= \langle \chi_{\substack{in\\out}} | U(0,\mp\infty) \phi^{ip}(x_1) \cdots \phi^{ip}(x_n) U^{-1}(0,\mp\infty) | \psi_{\substack{in\\out}} \rangle$$

$$= (\chi | \phi^{ip}(x_1) \cdots \phi^{ip}(x_n) | \psi) \text{ , where we}$$

have used

$$|\psi_{\substack{in\\out}}\rangle = e^{-i\Theta_{\substack{in\\out}}} U(0,\mp\infty)|\psi).$$

Hence we have immediately that

$$\langle f_{\substack{in\\out}} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(\phi_{\substack{in\\out}}, \pi_{\substack{in\\out}}; t)} | i_{\substack{in\\out}} \rangle$$

$$= (f | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{ip}(\phi^{ip}, \pi^{ip}; t)} | i) .$$

And ~~the~~ so we secure the above formula for S_{fi}. In fact we can obtain an expression for the S-operator in terms of the in- and out- fields by directly relating the S-operator in the interaction picture to the Heisenberg picture in- and out- fields. That's recalling

The definition of the Heisenberg picture S-operator

$$|\Psi_{in}\rangle = S|\Psi_{out}\rangle$$

but

$$|\Psi_{in}\rangle = e^{-i\theta_{in}}\,U(0,-\infty)|\Psi\rangle$$

$$= e^{-i\theta_{in}}\,U(0,-\infty)\,e^{+i\theta_{out}}\,U^{-1}(0,+\infty)|\Psi_{out}\rangle$$

$$= e^{-i(\theta_{in}-\theta_{out})}\,U(0,-\infty)\,U^{-1}(0,+\infty)\,|\Psi_{out}\rangle$$

$$= \frac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}\,|\Psi_{out}\rangle$$

Comparing the two, we have

$$\boxed{S = \frac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}}$$

Now

$$U(0,-\infty)\,U^{-1}(0,+\infty) = U(0,-\infty)\,U(+\infty,-\infty)\times$$

$$\times\,U^{-1}(+\infty,-\infty)\,U^{-1}(0,+\infty)$$

$$= U(0,-\infty)\,U(+\infty,-\infty)\,U^{-1}(0,-\infty)\,U(0,-\infty)\times$$

$$\times\, U(-\infty,+\infty)\,U(+\infty,0)$$

$$= U(0,-\infty)\,U(+\infty,-\infty)\,U^{-1}(0,-\infty)\times$$

$$\times\,\underbrace{U(0,-\infty)\,U(-\infty,+\infty)\,U(+\infty,0)}_{=U(0,0)=1}$$

So then

$$\boxed{U(0,-\infty)\,U^{-1}(0,+\infty) = U(0,-\infty)\,U(+\infty,-\infty)\,U^{-1}(0,-\infty)}$$

But this is just the transformation of the iP S-operator to in-fields

$$U(0,-\infty)\,U(+\infty,-\infty)\,U^{-1}(0,-\infty)$$

$$= U(0,-\infty)\,T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{ip}(\phi_{ip},\pi_{ip};t)}\,U^{-1}(0,-\infty)$$

$$= T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(\phi_{in},\pi_{in};t)}$$

Likewise

$$\langle 0|U(+\infty,-\infty)|0\rangle =$$

$$= \langle 0\,in|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(\phi_{in},\pi_{in};t)}|0\,in\rangle$$

So

$$\boxed{\begin{aligned} S &= U(0,-\infty)\, S^{iP}\, U^{-1}(0,-\infty) \\ &= \frac{T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(\phi_{in},\pi_{in};t)}}{\langle 0\,in|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{in}(\phi_{in},\pi_{in};t)}|0\,in\rangle} \end{aligned}}$$

Similarly for the out-fields

$$\boxed{\begin{aligned} S &= U(0,+\infty)\, S^{iP}\, U^{-1}(0,+\infty) \\ &= \frac{T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{out}(\phi_{out},\pi_{out};t)}}{\langle 0\,out|T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{out}(\phi_{out},\pi_{out};t)}|0\,out\rangle} \end{aligned}}$$

Of course the identification of the S-operator above follows directly from the equality of matrix elements

$$S_{fi} = \frac{\langle f^{in}_{out} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(t)} | i^{in}_{out} \rangle}{\langle 0^{in}_{out} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(t)} | 0^{in}_{out} \rangle}$$

$$\equiv \langle f^{in}_{out} | S | i^{in}_{out} \rangle$$

$$\Rightarrow \quad S = \frac{T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(t)}}{\langle 0^{in}_{out} | T e^{-i\int_{-\infty}^{+\infty} dt\, H_I^{\substack{in\\out}}(t)} | 0^{in}_{out} \rangle}$$

As we have seen above the in-matrix elements of in-operators equals that of out-matrix elements of out operators (equals the iP matrix elements of iP operators)

$$\langle \chi\, in | \phi_{in}(x_1) \cdots \phi_{in}(x_n) | \psi\, in \rangle$$

$$= \langle \chi\, out | \phi_{out}(x_1) \cdots \phi_{out}(x_n) | \psi\, out \rangle$$

But $|\Psi_{in}\rangle = S|\Psi_{out}\rangle$ So

$$\langle \chi\, in|\phi_{in}(x_1)\cdots\phi_{in}(x_n)|\Psi_{in}\rangle$$

$$= \langle \chi_{out}|S^{-1}\phi_{in}(x_1)SS^{-1}\phi_{in}(x_2)SS^{-1}\cdots$$

$$\cdots SS^{-1}\phi_{in}(x_n)S|\Psi_{out}\rangle$$

this must equal

$$= \langle \chi_{out}|\phi_{out}(x_1)\cdots\phi_{out}(x_n)|\Psi_{out}\rangle$$

hence we have that the S operator not only relate the in- and out-states but alter the operators

$$\boxed{\phi_{out}(x) = S^{-1}\phi_{in}(x)S}$$

Of course we have this directly through the operators in the iP

$$\phi_{out}(x) \equiv U(0,+\infty)\,\phi^{iP}(x)\,U^{-1}(0,+\infty)$$

$$= U(0,+\infty)\,U^{-1}(0,-\infty)\,\phi_{in}(x)\,U(0,-\infty)\times$$

$$\times U^{-1}(0,+\infty)$$

but recall p.-94-

$$S = \frac{U(0,-\infty)\,U^{-1}(0,+\infty)}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

So $$S^{-1} = U(0,+\infty)\,U^{-1}(0,-\infty)\,\langle 0|U(+\infty,-\infty)|0\rangle$$

and

$$\phi_{out}(x) = S^{-1}\,\phi_{in}(x)\,S$$

So far we have been concentrating on relating the S-operator to the in- and out-fields. We see that it is essentially just the time evolution operator in terms of the in- or out-fields. Again this is a perturbative expression for S and has all the drawbacks that S_{fi} does. It is nothing more than $U(0,\pm\infty)$, we would like to eliminate any explicit appearance of $U(0,\pm\infty)$ since it will involve approximations and our interpretive difficulties. Since it is $U(t,t_0)$ that contains the dynamical evolution information for Heisenberg we need another quantity that contains that at intermediate