1.2 The Asymptotic States and Fields

The first step towards transforming the interaction picture perturbation expansion for the S-matrix elements to the Heisenberg picture is to convert the initial and final states of the theory. Recall that due to the adiabatic hypothesis, the initial and final states, generically denoted $|\psi\rangle$, are the eigenstates of the free iP-Hamiltonian, as well as momentum, helicity and any other charge operators

$$H_0^{ip}|\psi\rangle = E|\psi\rangle$$

$$\vec{P}_0^{ip}|\psi\rangle = \vec{k}|\psi\rangle \qquad (1.2.1)$$

$$J_{0_3}^{ip}|\psi\rangle = s|\psi\rangle$$

$$Q_0^{ip}|\psi\rangle = q|\psi\rangle .$$

The subscript "o" on the momentum spin and charge operators are there just to remind us that they form a CSCO with H_0^{ip} the unperturbed iP Hamiltonian, and that they are the same form as in free field theory. In fact these operators are just the full operators in the iP, they have the same form as the free field operators that commute with H_0^{iP}

$$\vec{P}_0^{iP} = \vec{P}^{iP} = \int d^3x \, \pi^{iP} \vec{\nabla} \phi^{iP}$$

$$J_{03}^{iP} = J_3^{iP} = -\int d^3x \, \pi^{iP} \left[(\vec{x} \times \vec{\nabla})_3 + D^{12} \right] \phi^{iP}$$

$$Q_0^{iP} = Q^{iP} = i\int d^3x \, \pi^{iP} q \, \phi^{iP} \qquad (1.2.2)$$

for ϕ^{iP} in the spin and charge basis with charge q and Lorentz group representation matrix $D^{\mu\nu}$. Of course this is precisely not the case for the Hamiltonian,

$H_0^{iP} \neq H^{iP}$, the interaction Hamiltonian

H_I^{iP} is missing.

Note: Since P, J_3, Q generate symmetries we can view the above comments about their form as a result of the invariance of H_0^{iP} under these symmetries. For these operators we have

$$-i\frac{\partial A^{iP}(t)}{\partial t} = [H_0^{iP}, A^{iP}(t)] = \begin{cases} +i\vec{\nabla} H_0^{iP} & \text{for } \vec{A} = \vec{P}^{iP} \\ -s_H H_0^{iP} & \text{for } \vec{A}^{iP} = J_3^{iP} \\ -q_H H_0^{iP} & \text{for } \vec{A}^{iP} = Q^{iP} \end{cases} \quad (1.2.3)$$

So for H_0^{iP} a scalar (spinless, $s_H = 0$), charge free $q_H = 0$, and translation invariant $\vec{\nabla} H_0^{iP} = 0$, we have that $A^{iP}(t)$ is independent of time! Thus

$$\vec{P}_0^{iP} = \lim_{t \to \pm\infty} \vec{P}^{iP}(t) = \vec{P}^{iP}. \quad (1.2.4)$$

In order to transform the iP states to the Heisenberg picture, we operate with the time evolution operator, that is

$$\begin{aligned} U(0,t)|\Psi(t)\rangle_{iP} &= |\Psi\rangle_H \\ U(0,t)\phi^{iP}(x)\,U^{-1}(0,t) &= \phi^H(x) \\ U(0,t)\pi^{iP}(x)\,U^{-1}(0,t) &= \pi^H(x) \end{aligned} \quad (1.2.5)$$

where we have chosen the Heisenberg and interaction pictures to coincide at $t=0$ (otherwise we would use $U(t_0,t)$, then they coincide at $t=t_0$). If we have eigenstates of a CSCO $\{H, \vec{P}, J_3, Q\}$ in the Heisenberg picture, written generically as

$$A^H |\psi\rangle_H = a |\psi\rangle_H \qquad (1.2.6)$$

i.e. $H^H |\psi\rangle_H = E |\psi\rangle_H$, $\vec{P}^H |\psi\rangle_H = \vec{k} |\psi\rangle_H$, etc.

Then transforming to the iP yields

$$U^{-1}(0,t)\, A^{ip}_H(t)\, U(0,t)\, U^{-1}(0,t) |\psi(t)\rangle_{\cancel{ip}\,H}$$

$$= a\, U^{-1}(0,t) |\psi(t)\rangle_{\cancel{ip}\,H}.$$

That is

$$A^{ip}(t) |\psi(t)\rangle_{ip} = a |\psi(t)\rangle_{ip}. \qquad (1.2.7)$$

Thus

$$H^{ip}(t) |\psi(t)\rangle_{ip} = E |\psi(t)\rangle_{ip}$$

$$\vec{P}^{ip}(t) |\psi(t)\rangle_{ip} = \vec{k} |\psi(t)\rangle_{ip}$$

$$J_3^{ip}(t) |\psi(t)\rangle_{ip} = s |\psi(t)\rangle_{ip}$$

$$Q^{ip}(t) |\psi(t)\rangle_{ip} = q |\psi(t)\rangle_{ip}.$$

Now in the interaction representation

$$H^{ip}(t) = H_0^{ip}(t) + H_I^{ip}(t) = H_0^{ip} + H_I^{ip}(t)$$

where according to the adiabatic

hypothesis $H_I^{ip}(t \to \pm\infty) \to 0$. Thus we have that, using $U^{-1}(0,\pm\infty)\, H\, U(0,\pm\infty) = H_0^{ip}$,

$$H_0^{ip}\,|\Psi(\pm\infty)\rangle_{ip} = E\,|\Psi(\pm\infty)\rangle_{ip} \quad , (1.2.8)$$

But $|\Psi(\pm\infty)\rangle_{ip} = |\Psi\rangle$ the initial and final states. Since $\vec{P}^{ip}(t) = \vec{P}^{ip} = \vec{P}_0^{ip}$, $J_3^{ip}(t) = J_3^{ip}$ and $Q^{ip}(t) = Q^{ip}$ we have that the iP states obey the eigenvalue equations

$$\begin{aligned} H_0^{ip}|\Psi\rangle &= E|\Psi\rangle \\ \vec{P}^{ip}|\Psi\rangle &= \vec{k}|\Psi\rangle \\ J_3^{ip}|\Psi\rangle &= s|\Psi\rangle \\ Q^{ip}|\Psi\rangle &= q|\Psi\rangle \end{aligned} \qquad (1.2.9)$$

, with the same eigenvalues.

The initial and final states in the Heisenberg picture are called the "in-states" and "out-states", respectively.

The in-states are denoted by $|\psi\, in\rangle$ and are defined as the Heisenberg picture initial states

$$|\psi\, in\rangle \equiv U(0,-\infty)|\psi\rangle \qquad (1.2.10)$$

while the out-states are denoted by $|\psi\, out\rangle$ and they are defined as the Heisenberg picture final states

$$|\psi\, out\rangle \equiv U(0,+\infty)|\psi\rangle \,. \qquad (1.2.11)$$

These are eigenstates of the full Hamiltonian, momentum, spin (or helicity) and charge operators in the Heisenberg picture

$$
\begin{aligned}
H|\psi\, {}^{in}_{out}\rangle &= E|\psi\, {}^{in}_{out}\rangle \\
\vec{P}|\psi\, {}^{in}_{out}\rangle &= \vec{k}|\psi\, {}^{in}_{out}\rangle \\
J_3|\psi\, {}^{in}_{out}\rangle &= s|\psi\, {}^{in}_{out}\rangle \\
Q|\psi\, {}^{in}_{out}\rangle &= q|\psi\, {}^{in}_{out}\rangle \,.
\end{aligned}
\qquad (1.2.12)
$$

Due to the adiabatic hypothesis we have

$$H = U(0,\pm\infty)\, H_0^{i.p.}\, U^{-1}(0,\pm\infty) \,. \qquad (1.2.13)$$

To repeat, we have $H_0^{ip}|\psi\rangle = E|\psi\rangle$ operating with $\mathcal{U}(0,\pm\infty)$ yields

$$\left[\mathcal{U}(0,\pm\infty) H_0^{ip}\, \mathcal{U}^{-1}(0,\pm\infty)\right]\left[\mathcal{U}(0,\pm\infty)|\psi\rangle\right]$$

$$= E\left[\mathcal{U}(0,\pm\infty)|\psi\rangle\right] \qquad (1.2.14)$$

but $\mathcal{U}(0,\pm\infty) H_0^{ip}\, \mathcal{U}^{-1}(0,\pm\infty) = H$, so

$$H|\psi^{in}_{out}\rangle = E|\psi^{in}_{out}\rangle. \qquad (1.2.15)$$

Similarly for the other operators, for instance,

$\vec{\Phi}^{ip}|\psi\rangle = \vec{k}|\psi\rangle$ becomes

$$\left[\mathcal{U}(0,\pm\infty)\vec{\Phi}^{ip}\, \mathcal{U}^{-1}(0,\pm\infty)\right]\left[\mathcal{U}(0,\pm\infty)|\psi\rangle\right]$$

$$= \vec{k}\left[\mathcal{U}(0,\pm\infty)|\psi\rangle\right] \qquad (1.2.16)$$

but $\mathcal{U}(0,\pm\infty)\vec{\Phi}^{ip}\, \mathcal{U}^{-1}(0,\pm\infty) = \mathcal{U}(0,\pm\infty)\vec{\Phi}^{ip}(\pm\infty)\,\mathcal{U}^{-1}(0,\pm\infty)$

$= \vec{\Phi}^{H}(\pm\infty) = \vec{\Phi}^{H}$ for $\vec{\Phi}$ time indep.

So $\quad \vec{\Phi}|\psi^{in}_{out}\rangle = \vec{k}\,|\psi^{in}_{out}\rangle. \qquad (1.2.17)$

Note: We can again consider the symmetry generating aspects of $\vec{\Phi}^{ip}, J_3^{ip}, Q^{ip}$. If they are in the CSCO with the total Hamiltonian H^{ip} they must commute with H_I^{ip} also since they commute with H_0^{ip}. Alternatively if H_I^{ip} is translation invariant, spinless and chargeless we have

$$[\vec{\Phi}^{ip}, H_I^{ip}] = -i\vec{\nabla} H_I^{ip} = 0$$

$$[J_3^{ip}, H_I^{ip}] = s_I H_I^{ip} = 0 \qquad (1.2.18)$$

$$[Q^{ip}, H_I^{ip}] = q_I H_I^{ip} = 0 .$$

The total spin and charge of H_I^{ip} is zero and it is independent of position $\vec{x}$.

Hence these operators commute with the time evolution operator $U(t,t_0)$

$$[\vec{\Phi}^{ip}, U(t,t_0)] = 0$$

$$[J_3^{ip}, U(t,t_0)] = 0 \qquad (1.2.19)$$

$$[Q^{ip}, U(t,t_0)] = 0 .$$

Then
$$\vec{\Phi}^{ip}|\Psi\,{}^{out}_{in}\rangle = \vec{\Phi}^{ip}\,U(0,\pm\infty)|\Psi\rangle$$
$$= U(0,\pm\infty)\vec{\Phi}^{ip}|\Psi\rangle = U(0,\pm\infty)\,\vec{k}\,|\Psi\rangle$$
$$= \vec{k}\,|\Psi\,{}^{out}_{in}\rangle, \qquad (1.2.20)$$

but $\vec{\Phi}^{ip} = \vec{\Phi}^{ip}(t) = \vec{\Phi}^{ip}(0) = \vec{\Phi}^{H}(0) = \vec{\Phi}^{H} = \vec{\Phi}$ since we choose the Heisenberg and interaction pictures to coincide at $t=0$.

Since the iP initial & final states were eigenstates of a CSCO they are a complete set of states. Similarly in the Heisenberg pix. the set of in-states is a complete set of states. Since the out-states are eigenstates of the same CSCO, they also form a complete set of states. Since the in-states and out-states are both basis sets for the Hilbert space they must be related by a unitary operator change of basis. To see this consider any in-state $|i\ in\rangle$ expanded in terms of out-states $|f\,out\rangle$

$$|i\ in\rangle = \sum_f \langle f\,out|i\ in\rangle\,|f\,out\rangle$$
$$(1.2.21)$$

where $\sum_f |f\,out\rangle\langle f\,out| = 1$, (1.2.22)

But $\langle f\,out| = \langle f|U^{-1}(0,+\infty) = \langle f|U(+\infty,0)$

while $|i\,in\rangle = U(0,-\infty)|i\rangle$, so

$$\langle f\,out|i\,in\rangle = \langle f|U(+\infty,0)\,U(0,-\infty)|i\rangle$$

$$= \langle f|U(+\infty,-\infty)|i\rangle$$

(ignoring vacuum bubbles for the moment) $= \langle f|S^{ip}|i\rangle = S_{fi}$ (1.2.23)

Thus $|i\,in\rangle = \sum_f S_{fi}\,|f\,out\rangle$, where (1.2.24)

S_{fi} are the S-matrix elements. Thus

we have that the S-operator is the unitary transformation between the in- and out-basis vectors, for each out-state $|i\,out\rangle$ there corresponds an in-state $|i\,in\rangle$

where $|i\,in\rangle = S|i\,out\rangle$ (1.2.25)

Since we assume the in-states & out-states are normalized

$$\langle i\,in|j\,in\rangle = \delta_{ij} = \langle i\,out|j\,out\rangle$$ (1.2.26)

we have that $|i\,in\rangle = S|i\,out\rangle$ & $\langle f\,out| = \langle f\,in|S$
So

$$S_{fi} = \langle f\,out|i\,in\rangle = \langle f\,out|S|i\,out\rangle$$
$$= \langle f\,in|S|i\,in\rangle. \quad (1.2.27)$$

As discussed in the introduction vacuum bubble graph $\langle 0|U(+\infty,-\infty)|0\rangle$, contributions to the S-matrix lead to it having an infinite phase. Since these appeared in every process and further factored out we were able to eliminate such a factor by normalizing the $\langle f|U(+\infty,-\infty)|i\rangle$ matrix elements by $\langle 0|U(+\infty,-\infty)|0\rangle$. Since $\langle f\,out|i\,in\rangle = \langle f|U(+\infty,-\infty)|i\rangle$ we are faced with a similar situation. The in- and out-states so defined contain vacuum bubble contributions which are singular. Since the graphical structure of $\langle f|U(0,\pm\infty)|i\rangle$ is the same as in the S-matrix case, the only difference being in the time integrals in U running over the half-line rather than R^1. Hence each matrix element will factor into graphs with external lines and graphs without external lines, vacuum bubbles. The vacuum bubble graphs can be summed to result in the analytic factor $\langle 0|U(0,\pm\infty)|0\rangle$ times graphs with external lines. Thus

$$\langle f|U(0,\pm\infty)|i\rangle = \langle 0|U(0,\pm\infty)|0\rangle \langle f|U(0,\pm\infty)|i\rangle_{NVB} \quad (1.2.28)$$

where the "NVB" subscript on the last term indicates no vacuum bubble graphs are included in its evaluation. The point of all this being that we should remove this factor from the definition of the in- and out-states a priori just as we removed it from the definition of the S-operator. Thus the correct definition of the in- and out-states is

$$c_{in}|i\ in\rangle \equiv \frac{U(0,-\infty)|i\rangle}{\langle 0|U(0,-\infty)|0\rangle}$$

$$c_{out}|f\ out\rangle \equiv \frac{U(0,+\infty)|f\rangle}{\langle 0|U(0,+\infty)|0\rangle} \quad (1.2.29)$$

— See -31'-

In the presence of external fields these denominator factors are complicated functionals of the fields!

where $C_{\substack{in\\out}}$ are normalization factors so that $\langle i^{in}_{out} | j^{in}_{out} \rangle = (i|j) = \delta_{ij}$.

That is

$$|C_{\substack{in\\out}}|^2 \langle i^{in}_{out} | j^{in}_{out} \rangle = \frac{(i|j)}{(0|U(0,\mp\infty)|0)^* \, (0|U(0,\mp\infty)|0)}$$

Thus $$\boxed{C_{\substack{in\\out}} = \frac{e^{\pm i\varphi_{\substack{in\\out}}}}{|(0|U(0,\mp\infty)|0)|}}$$ with $\varphi_{\substack{in\\out}}$ arbitrary

Now for the vacuum

$$C_{\substack{in\\out}} (0|0^{in}_{out}\rangle = \frac{(0|U(0,\mp\infty)|0)}{(0|U(0,\mp\infty)|0)} = 1$$

Thus $$\boxed{C_{\substack{in\\out}} = \frac{1}{(0|0^{in}_{out}\rangle}}$$

Since $(0|U(0,\mp\infty)|0)$ is complex

$$(0|U(0,\mp\infty)|0) \equiv \frac{1}{C_{\substack{in\\out}}} e^{\pm i\theta_{\substack{in\\out}}} = |(0|U(0,\mp\infty)|0)| e^{\pm i(\theta_{\substack{in\\out}} - \varphi_{\substack{in\\out}})}$$

we have that, as in the $(0|U(+\infty,-\infty)|0)$ case, the phase $\theta_{\substack{in\\out}}$ is divergent.

So we have

$$|i\,in\rangle = e^{-i\theta_{in}}\, U(0,-\infty)|i)$$

$$|f\,out\rangle = e^{-i\theta_{out}}\, U(0,+\infty)|f) \quad .$$

Now

$$\langle f\,out|i\,in\rangle = e^{-i(\theta_{in}-\theta_{out})}\,(f|U(+\infty,-\infty)|i)$$

but $\langle 0\,out|0\,in\rangle = e^{-i(\theta_{in}-\theta_{out})}\,(0|U(+\infty,-\infty)|0)$

hence

$$\frac{\langle f\,out|i\,in\rangle}{\langle 0\,out|0\,in\rangle} = \frac{(f|U(+\infty,-\infty)|i)}{(0|U(+\infty,-\infty)|0)}$$

$$= S_{fi} \, .$$

Since $\sum_n' |n\,out\rangle\langle n\,out| = 1$ we have

$$(0|0_{in}\rangle = \frac{1}{c_{in}} = \sum_n' (0|n\,out\rangle\langle n\,out|0\,in\rangle$$

but $\langle n\,out|0\,in\rangle = 0, n \neq 0$ by energy-momentum conservation in the absence of external fields

$$\frac{1}{c_{in}} = (0|0\,out\rangle\langle 0\,out|0\,in\rangle = \frac{1}{c_{out}}\langle 0\,out|0\,in\rangle$$

But $\langle 0_{out}|0_{in}\rangle$ is a phase only in the absence of external fields so that $|\langle 0_{out}|0_{in}\rangle|=1$

$$\Rightarrow \boxed{\frac{1}{|C_{in}|}=\frac{1}{|C_{out}|}}$$ i.e. $|\langle 0|U(0,+\infty)|0\rangle|$
$=|\langle 0|U(0,-\infty)|0\rangle|$

Hence we have

$$e^{-i\varphi_{in}}=e^{-i\varphi_{out}}\langle 0_{out}|0_{in}\rangle \Rightarrow$$

$$e^{-i(\varphi_{in}-\varphi_{out})}=e^{-i(\theta_{in}-\theta_{out})}\langle 0|U(+\infty,-\infty)|0\rangle$$

Now we can choose $\boxed{\varphi_{in}=\varphi_{out}}$

So that $\langle 0_{out}|0_{in}\rangle=1$ i.e. $|0_{out}\rangle$
$=|0_{in}\rangle$

Then $$e^{+i(\theta_{in}-\theta_{out})}=\langle 0|U(+\infty,-\infty)|0\rangle.$$

So $$\boxed{C_{in}=C_{out}\equiv|C|e^{+i\varphi}}$$
$$\boxed{\equiv C}$$

The singularity of the vacuum bubble factor can be exhibited by more carefully dealing with the adiabatic hypothesis. That is by making explicit the adiabatic cut-off of the interaction we can regulate the singularities in $\langle 0|U(0,\pm\infty)|0\rangle$ and then study in- and out-states equations as the limit the cut-off is removed. Alternately stated, the initial and final states are energy eigenstates we can remove the ambiguity of working with plane wave states by guaranteeing that the time integrals converge absolutely. For in-states the range of time integration is always negative, hence a convergence factor of $e^{+\epsilon t}$, $\epsilon > 0$, should be inserted in each integral with the understanding that $\epsilon \to 0^+$ at the end of the calculation. Likewise for out-states the time integration is always positive so we insert a $e^{-\epsilon t}$, $\epsilon > 0$ convergence factor in each integral taking $\epsilon \to 0^+$ at the end of the calculation. The insertion of these factors is equivalent to replacing $H_I^{ip}(t)$ with $e^{-\epsilon|t|} H_I^{ip}(t)$

in each integral in $U(0,\pm\infty)$. This is nothing but the adiabatic hypothesis (see Merzbacher, QM Chapter 21).

Thus making explicit the adiabatic cut-off we have the time evolution operator $U(t,t_0) \longrightarrow U_\epsilon(t,t_0)$ where $U_\epsilon(t,t_0)$ is defined as before

$$i\frac{\partial}{\partial t}U_\epsilon(t,t_0) = H_I^{ip}(t)e^{-\epsilon|t|}\,U_\epsilon(t,t_0) \qquad (1.2.30)$$

where now the adiabatic switching on and off of H_I^{ip} is made explicit. Since $U_\epsilon(t,t) = 1$ we have the solution

$$U_\epsilon(t,t_0) = T e^{-i\int_{t_0}^{t} dt'\, e^{-\epsilon|t'|} H_I^{ip}(t')} \qquad (1.2.31)$$

hence in particular

$$U_\epsilon(0,-\infty) = T e^{-i\int_{-\infty}^{0} dt\, e^{+\epsilon t} H_I^{ip}(t)} \qquad (1.2.32)$$

and

$$U_\epsilon(0,+\infty) = T e^{+i\int_{0}^{\infty} dt\, e^{-\epsilon t} H_I^{ip}(t)} \qquad (1.2.33)$$

Note that the above ~~exponentials~~ are given by

$$U_\epsilon(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, e^{-\epsilon(|t_1|+\cdots+|t_n|)} \times$$

$$\times T H_I^{ip}(t_1) \cdots H_I^{ip}(t_n) . \quad (1.2.34)$$

So we can calculate the vacuum bubble singularity in ϵ now, for example in QED in second order we find

$$\langle 0 | U_\epsilon(0,-\infty) | 0 \rangle = 1 +$$ 

$$= 1 - \frac{(-ie)^2}{2} \int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \int_{-\infty}^{+\infty} d^3x_1 \, d^3x_2 \, e^{+\epsilon(t_1+t_2)} \times$$

$$\times \Delta_{F_{\mu\nu}}(x_1-x_2) \, \mathrm{Tr}\left[\gamma^\mu S_F(x_1-x_2) \times\right.$$

$$\left.\times \gamma^\nu S_F(x_2-x_1)\right] \quad (1.2.35)$$

performing the Fourier transforms we can check that the second term goes like $\frac{1}{\epsilon}$ after doing the time integrals

$$\langle 0 | U_\epsilon(0,-\infty) | 0 \rangle = 1 + \frac{1}{\epsilon} D . \quad (1.2.36)$$

The mathematically unambiguous definition of the in- and out-states is then given as the adiabatic limit explicitly as

$$C_{in}|i\,in\rangle \equiv \lim_{\epsilon \to 0^+} \frac{U_\epsilon(0,-\infty)|i\rangle}{\langle 0|U_\epsilon(0,-\infty)|0\rangle}$$

(1.2.37)

$$C_{out}|f\,out\rangle \equiv \lim_{\epsilon \to 0^+} \frac{U_\epsilon(0,+\infty)|f\rangle}{\langle 0|U_\epsilon(0,+\infty)|0\rangle} .$$

(Gell-Mann & Low Phys. Rev. 84 (1951) 350.)

Since $U_\epsilon(0,\pm\infty)|i\rangle$ has $\frac{1}{\epsilon}$ singularities we should review our proof that the in- & out-states are eigenstates of H. (Since $\vec{P}^{ip}$, J_3^{ip}, Q^{ip} still transform to the Heisenberg picture the same way the in- & out-states are eigenstates of $\vec{P}$, J_3, Q (alternately $\vec{P}^{ip}$, J_3^{ip}, Q^{ip} still commutes with H_0^{ip}, $H_I^{ip}(t)$ hence also with $U_\epsilon(t,t_0)$, so $|\Psi^{in}_{out}\rangle$ are still eigenstates of $\vec{P}$, J_3, Q).

That is we must now consider for instance

$$H_0^{iP}\, U_\epsilon(0,-\infty)|\Psi\rangle = U_\epsilon(0,-\infty) H_0^{iP}|\Psi\rangle$$

$$+ \left[H_0^{iP}, U_\epsilon(0,-\infty)\right]|\Psi\rangle$$

$$= E\, U_\epsilon(0,-\infty)|\Psi\rangle \qquad (1.2\text{-}38)$$

$$+ \left[H_0^{iP}, U_\epsilon(0,-\infty)\right]|\Psi\rangle .$$

Hence we must analyze $[H_0^{iP}\, U_\epsilon(0,-\infty)]$. To do this we just use the Heisenberg equations of motion for the iP fields

$$-i \frac{\partial}{\partial t} A^{iP}(t) = [H_0^{iP}, A^{iP}(t)] \,,$$

$$(1.2\text{-}39)$$

So

$$[H_0^{iP}, U_\epsilon(0,-\infty)]$$

$$= -i \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} dt_1 \cdots dt_n\, e^{\epsilon(t_1 + \cdots + t_n)} \times$$

$$\times \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n}\right) T H_I^{iP}(t_1) \cdots H_I^{iP}(t_n)$$

differentiating by parts we find (1.2-40)

$$[H_0^{ip}, U_\epsilon(0,-\infty)]$$

$$= -i\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{0} dt_1\cdots dt_n\left(\frac{\partial}{\partial t_1}+\cdots+\frac{\partial}{\partial t_n}\right)\times$$

$$\times\left[e^{\epsilon(t_1+\cdots+t_n)}\,T\,H_I^{ip}(t_1)\cdots H_I^{ip}(t_n)\right]$$

$$+\,i\epsilon\sum_{n=1}^{\infty} n\frac{(-i)^n}{n!}\int_{-\infty}^{0} dt_1\cdots dt_n\, e^{\epsilon(t_1+\cdots+t_n)}\,T H_I^{ip}(t_1)\cdots H_I^{ip}(t_n)$$

(1.2-41)

The total derivative can be evaluated at the end points, at $t_i = -\infty$ the $e^{\epsilon t_i}$ damps the result to zero, at $t_i = 0$ $H_I^{ip}(t_i) = H_I^{ip}(0)$ the latest time operator, hence it can be brought out of the time ordered product since it is always the furthest left operator. So

$$[H_0^{ip}, U_\epsilon(0,-\infty)] = -iH_I^{ip}(0)\sum_{n=1}^{\infty} n\frac{(-i)^n}{n!}\int_{-\infty}^{0} dt_1\cdots dt_{n-1}\,\times$$

$$\times\, e^{\epsilon(t_1+\cdots+t_{n-1})}\; T H_I^{ip}(t_1)\cdots H_I^{ip}(t_{n-1})$$

$$+\, i\epsilon\sum_{n=1}^{\infty} n\frac{(-i)^n}{n!}\int_{-\infty}^{0} dt_1\cdots dt_n\, e^{\epsilon(t_1+\cdots+t_n)}\; T H_I^{ip}(t_1)\cdots H_I^{ip}(t_n)$$

(1.2.42)

Re-labelling $m = n-1$ in the first term we just recover $U_\epsilon(0,-\infty)$. So

$$[H_0^{ip}, U_\epsilon(0,-\infty)]$$

$$= -H_I^{ip}(0)\, U_\epsilon(0,-\infty) + i\epsilon \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\, n \int_{-\infty}^{0} dt_1 \cdots dt_n\, e^{\epsilon(t_1 + \cdots + t_n)} \times T H_I^{ip}(t_1) \cdots H_I^{ip}(t_n) \tag{1.2.43}$$

As a matter of convenience we can write the last term as a derivative wrt a coupling constant. That is imagine $H_I^{ip} \to g H_I^{ip}$ with g a "coupling constant" parameter then

$$g\frac{\partial}{\partial g} U_\epsilon(0,-\infty) = g\frac{\partial}{\partial g} \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} g^n \int_{-\infty}^{0} dt_1 \cdots dt_n\, e^{\epsilon(t_1 + \cdots + t_n)}\, T H_I^{ip}(t_1) \cdots H_I^{ip}(t_n)$$

$$= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\, n g^n \int_{-\infty}^{0} dt_1 \cdots dt_n\, e^{\epsilon(t_1 + \cdots + t_n)}\, T H_I^{ip}(t_1) \cdots H_I^{ip}(t_n) \tag{1.2.44}$$

So we obtain

$$\boxed{[H_0^{ip}, U_\epsilon(0,-\infty)] = -H_I^{ip}(0)\, U_\epsilon(0,-\infty) + i\epsilon g\frac{\partial}{\partial g} U_\epsilon(0,-\infty) \quad (1.2.45)}$$

Note this yields

$$H_0^{ip} U_\epsilon(0,-\infty) + H_I^{ip}(0) U_\epsilon(0,-\infty)$$

$$= U_\epsilon(0,-\infty) H_0^{ip} + i\epsilon g \frac{\partial}{\partial g} U_\epsilon(0,-\infty) \quad (1.2.46)$$

But $H_0^{ip} = H_0^S$ and all pictures coincide at $t=0$ So $H_I^{ip}(0) = H_I^S$

Thus $H_0^{ip} + H_I^{ip}(0) = H_0^S + H_I^S = H^S = H.$

Then we obtain

$$\boxed{H U_\epsilon(0,-\infty) = U_\epsilon(0,-\infty) H_0^{ip} + i\epsilon g \frac{\partial}{\partial g} U_\epsilon(0,-\infty)} \quad (1.2.47)$$

Now this yields

$$H_0^{ip} U_\epsilon(0,-\infty)|\Psi\rangle = E\, U_\epsilon(0,-\infty)|\Psi\rangle$$

$$- H_I^{ip}(0) U_\epsilon(0,-\infty)|\Psi\rangle$$

$$+ i\epsilon g \frac{\partial}{\partial g} U_\epsilon(0,-\infty)|\Psi\rangle \quad (1.2.48)$$

So

$$H\, U_\epsilon(0,-\infty)|\psi\rangle = E\, U_\epsilon(0,-\infty)|\psi\rangle$$
$$+ i\epsilon g \frac{\partial}{\partial g} U_\epsilon(0,-\infty)|\psi\rangle \qquad (1.2.49)$$

If $U_\epsilon(0,-\infty)|\psi\rangle$ was non-singular in the limit $\epsilon \to 0^+$ then the last term would vanish and we would obtain the desired result. But as we argued we must divide $U_\epsilon(0,-\infty)|\psi\rangle$ by $\langle 0|U_\epsilon(0,-\infty)|0\rangle$ to remove the $\epsilon \to 0^+$ singularity. The above becomes

$$c_{in} H|\psi in\rangle_\epsilon = E|\psi in\rangle_\epsilon c_{in}$$
$$+ i\epsilon \frac{1}{\langle 0|U_\epsilon(0,-\infty)|0\rangle} g \frac{\partial}{\partial g} U_\epsilon(0,-\infty)|\psi\rangle \qquad (1.2.50)$$

where $c_{in}|\psi in\rangle_\epsilon \equiv \dfrac{U_\epsilon(0,-\infty)|\psi\rangle}{\langle 0|U_\epsilon(0,-\infty)|0\rangle}$.

writing the last term as

$$\frac{1}{\langle 0|U_\epsilon(0,-\infty)|0\rangle} \frac{\partial}{\partial g} U_\epsilon(0,-\infty) = \frac{\partial}{\partial g}\left(\frac{U_\epsilon(0,-\infty)}{\langle 0|U_\epsilon(0,-\infty)|0\rangle}\right)$$
$$+ \frac{U_\epsilon(0,-\infty)}{\langle 0|U_\epsilon(0,-\infty)|0\rangle^2} \frac{\partial}{\partial g}\left(\langle 0|U_\epsilon(0,-\infty)|0\rangle\right) \qquad (1.2.51)$$

we find

$$\boxed{\begin{aligned} H|\Psi in\rangle_\epsilon &= E|\Psi in\rangle_\epsilon + i\epsilon g\frac{\partial}{\partial g}|\Psi in\rangle_\epsilon \\ &\quad + i\epsilon|\Psi in\rangle_\epsilon \left(g\frac{\partial}{\partial g}\ln\langle 0|U_\epsilon(0,-\infty)|0\rangle\right) \end{aligned}} \quad (1.2.52)$$

Now taking the scalar product of eq. (1.2.49) with $\langle 0|$ we find

$$\begin{aligned} \langle 0|H U_\epsilon(0,-\infty)|\Psi\rangle &= E\langle 0|U_\epsilon(0,-\infty)|\Psi\rangle \\ &\quad + i\epsilon g\frac{\partial}{\partial g}\langle 0|U_\epsilon(0,-\infty)|\cancel{0}\Psi\rangle \end{aligned} \quad (1.2.53)$$

So for $|\Psi\rangle = |0\rangle$ and $H_0^{ip}|0\rangle = E_0|0\rangle$ we have

$$\begin{aligned} \langle 0|H U_\epsilon(0,-\infty)|0\rangle &= E_0\langle 0|U_\epsilon(0,-\infty)|0\rangle \\ &\quad + i\epsilon g\frac{\partial}{\partial g}\langle 0|U_\epsilon(0,-\infty)|0\rangle \end{aligned} \quad (1.2.54)$$

dividing by $\langle 0|U_\epsilon(0,-\infty)|0\rangle$ we obtain

$$\boxed{\begin{aligned} & i\epsilon g\frac{\partial}{\partial g}\ln\langle 0|U_\epsilon(0,-\infty)|0\rangle \\ &= \langle 0|(H-E_0)|0\, in\rangle_\epsilon\, c_{in} \end{aligned}} \quad (1.2.55)$$

So

$$\boxed{\begin{aligned} H|\Psi_{in}\rangle_\epsilon &= E|\Psi_{in}\rangle_\epsilon + i\epsilon g\frac{\partial}{\partial g}|\Psi_{in}\rangle_\epsilon \\ &\quad + C_{in}\,\langle 0|(H-E_0)|0_{in}\rangle_\epsilon V|\Psi_{in}\rangle_\epsilon \end{aligned}}$$

(1.2.56)

where note that

$$C_{in}|0\,in\rangle_\epsilon = \frac{U_\epsilon(0,-\infty)|0\rangle}{\langle 0|U_\epsilon(0,-\infty)|0\rangle}$$

(1.2.57)

thus $C_{in}\langle 0|0_{in}\rangle_\epsilon = 1.$ (1.2.58)

Then in the limit $\epsilon \to 0^+$ we find

$$H|\Psi_{in}\rangle = \left[E + C_{in}\langle 0|(H-E_0)|0_{in}\rangle\right]|\Psi_{in}\rangle$$

(1.2.59)

Now

$$C_{in}\langle 0|H|0_{in}\rangle = \langle 0|H\,U_\epsilon(0,-\infty)|0\rangle\, e^{-i\theta_{in}}C_{in}$$

$$= \langle 0|U_\epsilon(0,-\infty)H_0^{ip}|0\rangle e^{-i\theta_{in}}C_{in}$$

$$+ i\epsilon\, e^{-i\theta_{in}}C_{in}\left(g\frac{\partial}{\partial g}\langle 0|U_\epsilon(0,-\infty)|0\rangle\right)$$

So

$$C_{in}\langle 0|H|0\,in\rangle = E_0\, C\langle 0|U_\epsilon(0,-\infty)|0\rangle e^{-i\theta_{in}} C_{in}$$
$$+ i\epsilon C_{in}\left(g\partial_g \frac{1}{C_{in}}\right) + i\epsilon g\frac{\partial}{\partial g}(i\theta_{in})$$

as $\epsilon \to 0^+$ we have

$$C_{in}\langle 0|H|0\,in\rangle = E_0 - g\frac{\partial}{\partial g}(\epsilon\theta_{in})$$
$$\equiv E_0 + \Delta E_{in}$$

Hence

$$\boxed{H|\Psi\,in\rangle = (E+\Delta E_{in})|\Psi\,in\rangle} \qquad (1.2.59')$$

Thus $|\Psi\,in\rangle$ is an eigenstate of the full Hamiltonian with eigenvalue $(E+\Delta E_{in})$

That is the in-vacuum ~~has~~ an energy eigenvalue $E_0+\Delta E_{in}$

$$H|0\,in\rangle = (E_0+\Delta E_{in})|0\,in\rangle,$$

The dressing of the bare vacuum results

in a constant energy shift for all states. The spectrum of H is just that of $H_0 + \Delta E_{in}$.

Similarly we can derive the same results for the out-states

$$H|\psi_{out}\rangle = (E + \Delta E_{out})|\psi_{out}\rangle \quad (1.2.60)$$

with

$$C_{out}\langle 0|H|0_{out}\rangle = E_0 + g\frac{\partial}{\partial g}(-\theta_{out})$$

$$\equiv E_0 + \Delta E_{out}$$

and $\quad H|0_{out}\rangle = (E_0 + \Delta E_{out})|0_{out}\rangle$.

But we have that

$$\langle\psi_{out}|H|\psi_{in}\rangle = (E + \Delta E_{in})\langle\psi_{out}|\psi_{in}\rangle$$

H acting to the right,

$$= (E + \Delta E_{out})\langle\psi_{out}|\psi_{in}\rangle$$

for H acting to the left.

but $\langle\psi_{out}|\psi_{in}\rangle = e^{-i(\theta_{in} - \theta_{out})}(\psi|U(+\infty,-\infty)|\psi)$

$$\neq 0$$

$\Rightarrow$

$$\boxed{\Delta E_{out} = \Delta E_{in} \equiv \Delta E}$$

$$\Rightarrow g\frac{\partial}{\partial g}(\epsilon\,\theta_{out}) = -g\frac{\partial}{\partial g}(\epsilon\,\theta_{in})$$

since $\theta_{\substack{in\\out}}$ has no g independent terms

$$\boxed{\theta_{in} = -\theta_{out} \equiv \frac{\theta}{2}}$$

Thus, in the absence of external fields,

$$\boxed{\begin{aligned} (0|U(0,+\infty)|0) &= \frac{1}{c}e^{-i\frac{\theta}{2}} \\ (0|U(0,-\infty)|0) &= \frac{1}{c}e^{+i\frac{\theta}{2}} \end{aligned}}$$

Recall p. -31'''-

$$(0|U(+\infty,-\infty)|0) = e^{+i(\theta_{in}-\theta_{out})} = e^{+i\theta}.$$

and So $\boxed{\langle f\,out|i\,in\rangle = \dfrac{(f|U(+\infty,-\infty)|i)}{(0|U(+\infty,-\infty)|0)}}$

$$\langle 0\,out|0\,in\rangle = 1 \Rightarrow \boxed{|0\,out\rangle = |0\,in\rangle}$$

We can further clarify the relation of $|\Psi\rangle$ and $|\Psi(^{in}_{out})\rangle$ by explicitly doing the time integrals in the definition of $U_\epsilon(0,\pm\infty)|\Psi\rangle$.

That is

$$U_\epsilon(t,-\infty)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t}dt_1\cdots dt_n e^{+\epsilon(t_1+\cdots+t_n)}\, T\, H_I^{ip}(t_1)\cdots H_I^{ip}(t_n)$$

$$=\sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n\, e^{+\epsilon(t_1+\cdots+t_n)}\, H_I^{ip}(t_1)\cdots H_I^{ip}(t_n)$$

(1.2-61)

Thus

$$U_\epsilon(0,-\infty)|\Psi\rangle=\lim_{t\to 0^-}\Big[|\Psi\rangle-i\int_{-\infty}^{t}dt_1\, e^{\epsilon t_1}H_I^{ip}(t_1)|\Psi\rangle$$

$$+(-i)^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\, e^{\epsilon(t_1+t_2)}H_I^{ip}(t_1)H_I^{ip}(t_2)|\Psi\rangle+\cdots\Big]$$

(1.2-62)

In order to perform the time integrals we would like to eliminate the time dependence of $H_I^{ip}(t)$; we can do this by transforming to the Schrödinger picture

Recall that $|\psi(t)\rangle_{ip} = e^{+iH_0^s t}|\psi(t)\rangle_s$

$$\phi^{ip}(t,\vec{x}) = e^{+iH_0^s t}\phi^s(\vec{x})e^{-iH_0^s t} \qquad (1.2.63)$$

So

$$H_I^{ip}(t) = e^{+iH_0^{ip} t} H_I^s e^{-iH_0^{ip} t} \qquad (1.2.64)$$

Further we have that $H_0^{ip}|\psi\rangle = E|\psi\rangle$ so that

$$e^{-iH_0^{ip}t}|\psi\rangle = e^{-iEt}|\psi\rangle .$$

Considering the first integral on the RHS of equation (1.2.62) we find

$$\int_{-\infty}^{t} dt_1 e^{+\epsilon t_1} H_I^{ip}(t_1)|\psi\rangle$$

$$= \int_{-\infty}^{t} dt_1 e^{+\epsilon t_1} e^{+iH_0^{ip}t_1} H_I^s e^{-iH_0 t_1}|\psi\rangle$$

$$= \int_{-\infty}^{t} dt_1 e^{+\epsilon t_1 + iH_0^{ip}t_1} H_I^s e^{-iEt_1}|\psi\rangle$$

$$= \int_{-\infty}^{t} dt_1 e^{-i(E-H_0^s+i\epsilon)t_1} H_I^s|\psi\rangle . \qquad (1.2.65)$$

So the time integral can now be performed

$$= e^{-i(E-H_0^s+i\epsilon)t} \frac{i}{E-H_0^s+i\epsilon} H_I^s |\psi\rangle .$$

Similarly we can analyze the second order term

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, e^{+\epsilon(t_1+t_2)} H_I^{ip}(t_1) H_I^{ip}(t_2) |\psi\rangle$$

using the above result for the t_2 integral we have

$$= \int_{-\infty}^{t} dt_1\, e^{+\epsilon t_1} H_I^{ip}(t_1)\, e^{-i(E-H_0^s+i\epsilon)t_1} \frac{i}{E-H_0^s+i\epsilon} H_I^s |\psi\rangle$$

$$= \int_{-\infty}^{t} dt_1\, e^{+\epsilon t_1} \left(e^{+iH_0^{ip}t_1} H_I^s e^{-iH_0^{ip}t_1} \right) e^{+iH_0^{ip}t_1} e^{-i(E+i\epsilon)t_1} \times$$

$$\times \frac{i}{E-H_0^s+i\epsilon} H_I^s |\psi\rangle$$

$$= \int_{-\infty}^{t} dt_1\, e^{-i(E-H_0^s+2i\epsilon)t_1} H_I^s \frac{i}{E-H_0^s+i\epsilon} H_I^s |\psi\rangle \qquad (1.2.66)$$

performing the integral

$$= e^{-i(E-H_0^s+2i\epsilon)t} \frac{i}{E-H_0^s+2i\epsilon} H_I^s \frac{i}{E-H_0^s+i\epsilon} H_I^s |\psi\rangle \qquad (1.2.67)$$

Hence we can continue to evaluate the terms order by order, the n^{th} order term yielding

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \, e^{+\epsilon(t_1+\cdots+t_n)} H_I^{ip}(t_1) \cdots H_I^{ip}(t_n) |\psi\rangle$$

$$= i^n e^{-i(E - H_0^S + ni\epsilon)t} \times$$

$$\times \frac{1}{E - H_0^S + in\epsilon} H_I^S \frac{1}{E - H_0^S + i(n-1)\epsilon} H_I^S \cdots \frac{1}{E - H_0^S + i\epsilon} H_I^S |\psi\rangle .$$

(1.2.68)

Then letting $t \to 0^-$ we find

$$\boxed{U_\epsilon(0,-\infty)|\psi\rangle = |\psi\rangle + \frac{1}{E - H_0^S + i\epsilon} H_I^S |\psi\rangle + \frac{1}{E - H_0^S + 2i\epsilon} H_I^S \frac{1}{E - H_0^S + i\epsilon} H_I^S |\psi\rangle + \cdots}$$

(1.2.69)

In the limit $\epsilon \to 0^+$ this becomes

$$U_\epsilon(0,-\infty)|\psi\rangle = |\psi\rangle + \frac{1}{E - H_0^S + i\epsilon} H_I^S |\psi\rangle + \frac{1}{E - H_0^S + i\epsilon} H_I^S \frac{1}{E - H_0^S + i\epsilon} H_I^S |\psi\rangle + \cdots + R$$

(1.2.70)

where we have written $n\epsilon = \epsilon + (n-1)\epsilon$ and
$$\frac{1}{E-H_0^s+in\epsilon} = \frac{1}{E-H_0^s+i\epsilon}\left(1-\frac{(n-1)i\epsilon}{E-H_0^s+i\epsilon}+\cdots\right)$$
$$= \frac{1}{E-H_0^s+i\epsilon} + \epsilon r \qquad (1.2.71)$$

Thus R appears from these terms $(n-1)\epsilon$ and will not contain the infinite phase factor as $\epsilon \to 0^+$. So

$$U_\epsilon(0,-\infty)|\Psi\rangle = \frac{1}{\left[1-\frac{1}{E-H_0^s+i\epsilon}H_I^s\right]}|\Psi\rangle + R \qquad (1.2.72)$$

where we used $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots$,

Thus in the limit $\epsilon \to 0^+$ we have

$$C_{in}|\Psi\, in\rangle = \lim_{\epsilon\to 0^+}\left(\frac{\left[1-\frac{1}{E-H_0^s+i\epsilon}H_I^s\right]^{-1}|\Psi\rangle}{\langle 0|\left[1-\frac{1}{E_0-H_0^s+i\epsilon}H_I^s\right]^{-1}|0\rangle}\right) \qquad (1.2.73)$$

where $H_0^s|0\rangle = E_0|0\rangle$.

Alternately we can write the expression for $|\Psi in\rangle$ in terms of $|\Psi\rangle$ as an integral equation. Using

$$\frac{1}{1-x} = \frac{1+x-x}{1-x} = 1 + x\frac{1}{1-x} \qquad (1.2.74)$$

i.e. $\frac{1}{1-x} = 1+x+x^2+x^3+\dots = 1+x(1+x+x^2+x^3+\dots)$

$$= 1 + x\frac{1}{1-x} \quad (1.2.75)$$

So for $X = \frac{1}{E-H_0^s+i\epsilon} H_I^s$ we have, ~~for~~ $\epsilon \to 0^+$

$$U_\epsilon(0,-\infty)|\Psi\rangle = |\Psi\rangle + \frac{1}{E-H_0^s+i\epsilon} H_I^s\, U_\epsilon(0,-\infty)|\Psi\rangle \quad (1.2.76)$$

That is we secure the integral equation (ignoring the vacuum bubbles)

$$|\Psi in\rangle = |\Psi\rangle + \frac{1}{E-H_0^s+i\epsilon} H_I^s\, |\Psi in\rangle . \quad (1.2.77)$$

This is the Lippmann-Schwinger equation for the Heisenberg picture in-state valid in the limit $\epsilon \to 0^+$ (after we factor out the vacuum bubble phase)

Similarly we could proceed for the out-states. The only difference being that we have $-i\epsilon$ instead of $+i\epsilon$ since the region of time integration is positive and $e^{-\epsilon t}$ is the needed convergence factor. So

$$|\Psi_{out}\rangle = |\Psi\rangle + \frac{1}{E - H_0^s - i\epsilon} H_I^s |\Psi_{out}\rangle \tag{1.2.78}$$

That is we find

$$U_\epsilon(0,+\infty)|\Psi\rangle = |\Psi\rangle + \frac{1}{E-H_0^s - i\epsilon} H_I^s |\Psi\rangle$$

$$+ \frac{1}{E-H_0^s-2i\epsilon} H_I^s \frac{1}{E-H_0^s-i\epsilon} H_I^s |\Psi\rangle + \cdots \tag{1.2.79}$$

Referring to our experience in the non-relativistic formal theory of scattering the Lippmann-Schwinger equation in-state represents an exact solution to the Schrödinger equation which asymptotically consists of a superposition of an incoming plane wave and outgoing spherical wave. The $|\Psi in\rangle$ here are the relativistic generalization of such states. They are eigenstates of the total Hamiltonian which represent not only the N-incoming particles but also all possible outgoing particles that result from the scattering process such as pair production, annihilation, new particles, etc.. The out-states enjoy an analogous interpretation.

So then, in the Heisenberg picture formulation of scattering theory, the in-states form a complete set of states which correspond, in the remote past, to the initial non-interacting particle states of the theory. Likewise the out-states form another complete set of states of the system which correspond, in the remote future, to the final non-interacting particle states of the theory. The transition probability to go from some initial state to a particle final state is then just given by the S-matrix elements

$$S_{fi} = \frac{\langle f\,out \mid i\,in\rangle}{\langle 0\,out \mid 0\,in\rangle} \qquad (1.2.80')$$

where we have normalized the S-matrix by the in-vacuum to out-vacuum transition amplitude. Clearly this is just the same S-matrix as was defined in the interaction picture formulation, even in the presence of external fields.

$$S_{fi} \equiv \frac{\langle f | U^{ip}(+\infty, -\infty) | i \rangle}{\langle 0 | U^{ip}(+\infty, -\infty) | 0 \rangle} \qquad (1.2.81)$$

but $\quad U^{ip}(+\infty, -\infty) = U^{ip}(+\infty, 0) \, U^{ip}(0, -\infty)$

$$= U^{ip\dagger}(0, +\infty) \, U^{ip}(0, -\infty) \qquad (1.2.82)$$

So

$$S_{fi} = \frac{\langle f | U^{ip\dagger}(0, +\infty) \; U^{ip}(0, -\infty) | i \rangle}{\langle 0 | U^{ip\dagger}(0, +\infty) \; U^{ip}(0, -\infty) | 0 \rangle} \qquad (1.2.83)$$

Now multiplying the top and bottom by the adiabatic hypothesis vacuum bubble factors,

$$1 = \left[\frac{\frac{1}{\langle 0 | U^{ip\dagger}(0, +\infty) | 0 \rangle}}{\frac{1}{\langle 0 | U^{ip\dagger}(0, +\infty) | 0 \rangle}} \right] \left[\frac{\frac{1}{\langle 0 | U^{ip}(0, -\infty) | 0 \rangle}}{\frac{1}{\langle 0 | U^{ip}(0, -\infty) | 0 \rangle}} \right], \qquad (1.2.84)$$

we have

$$S_{fi} = \frac{\left(\frac{\langle f | U^{ip\dagger}(0, +\infty)}{\langle 0 | U^{ip\dagger}(0, +\infty) | 0 \rangle} \right) \left(\frac{U^{ip}(0, -\infty) | i \rangle}{\langle 0 | U^{ip}(0, -\infty) | 0 \rangle} \right)}{\left(\frac{\langle 0 | U^{ip\dagger}(0, +\infty)}{\langle 0 | U^{ip\dagger}(0, +\infty) | 0 \rangle} \right) \left(\frac{U^{ip}(0, -\infty) | 0 \rangle}{\langle 0 | U^{ip}(0, -\infty) | 0 \rangle} \right)} \qquad (1.2.85)$$

which using equation (1.2.37) just yields the desired result

$$S_{fi} = \frac{\langle f\ \text{out} | i\ \text{in} \rangle}{\langle 0\ \text{out} | 0\ \text{in} \rangle} \tag{1.2.86}$$

Since the in-state and out-states are just 2 different sets of basis states for the Hilbert space of the theory, they are unitarily related. That is the S-operator transforms from one basis to the other

$$|\psi\ \text{in}\rangle = S|\psi\ \text{out}\rangle \tag{1.2.87}$$

where, for normalized states, $\langle i\ \text{in} | j\ \text{in}\rangle = \delta_{ij}$ and $\langle i\ \text{out} | j\ \text{out}\rangle = \delta_{ij}$, we have

$$\langle i\ \text{in} | j\ \text{in}\rangle = \langle i\ \text{out} | S^\dagger S | j\ \text{out}\rangle = \delta_{ij} \tag{1.2.88}$$

implying $S^\dagger S = 1$ as necessary.

Thus the S-matrix elements are given by

$$\boxed{\begin{aligned} S_{fi} &= \frac{\langle f\,\text{out} | i\,\text{in}\rangle}{\langle 0\,\text{out} | 0\,\text{in}\rangle} \\ &= \frac{\langle f\,\text{out}|S|i\,\text{out}\rangle}{\langle 0\,\text{out}|S|0\,\text{out}\rangle} \end{aligned}} \qquad (1.2.89)$$

since $\langle \psi\,\text{in}| = \langle \psi\,\text{out}|S^{\dagger} \Rightarrow \langle \psi\,\text{out}| = \langle \psi\,\text{in}|S$

we also have

$$\boxed{S_{fi} = \frac{\langle f\,\text{in}|S|i\,\text{in}\rangle}{\langle 0\,\text{in}|S|0\,\text{in}\rangle}} \qquad (1.2.90)$$

From the operator point of view then S is just given by the sum over all states

$$S = \sum_i |i\,\text{in}\rangle\langle i\,\text{out}| \,, \qquad (1.2.91)$$

Indeed

$$\begin{aligned} \langle f\,\text{out}|S|i\,\text{out}\rangle &= \sum_k \langle f\,\text{out}|k\,\text{in}\rangle\langle k\,\text{out}|i\,\text{out}\rangle \\ &= \sum_k \langle f\,\text{out}|k\,\text{in}\rangle\,\delta_{ik} = \langle f\,\text{out}|i\,\text{in}\rangle \end{aligned}$$

(1.2.92)

as well as

$$\langle f\,in|S|i\,in\rangle = \sum_k \langle f\,in|k\,in\rangle\langle k\,out|i\,in\rangle$$
$$= \sum_k \delta_{fk}\langle k\,out|i\,in\rangle = \langle f\,out|i\,in\rangle$$
(1.2.93)

Since the scattering transition amplitudes are given in terms of the in- and out-states it would be convenient to have a method of constructing these states analogous to the Fock space construction of the iP initial and final states. The in-states and out-states are related to the iP states by operation of $U(0,\pm\infty)$, hence these are the unitary transformations of the iP fields that allow us to build the in- and out-states from the action of creation operators on the vacuum. That is, cryptically, we have the Fourier expansion of the iP fields given by (1.1.15)