ADVANCED QUANTUM FIELD THEORY

CHAPTER 1. THE LSZ AXIOMS

§1.1 Introduction: A review of scattering theory in the interaction picture

In the course "Introduction To Quantum Field Theory" scattering theory for relativistic quantum mechanical systems was developed. The S-matrixdescribing the transition probability amplitude was calculated in perturbation theory with the aid of a Feynman diagram expansion. Here we begin by recalling in a schematic fashion the general S-matrix formalism with the goal of distilling from it the essential relations and ideas of the approach. We will then reformulate this scattering theory in order to derive these essential relations from a minimal set of reasonable axioms. This approach is knownas the axiomatic approach to Quantum Field Theory. The particular formulation we will develop is that of Lehmann, Symanzik and Zimmermann, known as the LSZ formulation.

To begin with recall that in our introductory field theory course we asked for a method to find the transition probability amplitude for some initial state i to evolve into a particular final state f. This amplitude was called the S-matrix element. For calculational simplicity it was found convenient to separate the dynamics into a non-interacting (free) part and an interacting (perturbation) part, thus the Hamiltonian was given as the sum of these two sets of terms $H = H_0 + H_I$ with H_0 the free Hamiltonian and H_I the interaction Hamiltonian. We then worked in the interaction picture of quantum mechanics, here the state vectors, $|\psi(t)>_{iP}$, evolved in time according to the Schrödinger equation with the interaction Hamiltonian in the interaction picture

$$i\frac{\partial}{\partial t}|\psi(t)>_{iP} = H_I^{iP}(t)|\psi(t)>_{iP} . \tag{1.1.1}$$

Recall that the interaction picture operators, $A^{iP}(t)$, are related to the Schrödinger picture operators, A^S, and Heisenberg operators, $A(t)$, by

$$A^{iP}(t) = e^{+iH_0^S t}A^S e^{-iH_0^S t} = U(t,0)A(t)U^{-1}(t,0), \tag{1.1.2}$$

with the time evolution operator (in the interaction picture) given by

$$U(t,t_0) \equiv e^{+iH_0^S t}e^{-iH(t-t_0)}e^{-iH_0^S t_0}, \tag{1.1.3}$$

while the states in the pictures are related by

$$|\psi(t)>_{iP} = e^{+iH_0^S t}|\psi(t)>_S = U(t,0)|\psi> . \tag{1.1.4}$$

Hence the free Hamiltonian is the same in the Schrödinger and the interaction pictures, $H_0^S = H_0^{iP}$ (and is equal to the Heisenberg picture free Hamiltonian at $t = 0$, $H_0^S = H_0^{iP} = H_0^H(0)$) but the interaction Hamiltonian is not,

$$H_I^{iP} = e^{+iH_0^S t}H_I^S e^{-iH_0^S t} = H_I^{iP}(t). \tag{1.1.5}$$

Since $H_I^{iP}(t)$ depends on time the time evolution operator in the interaction picture, $U(t,t_0) = e^{+iH_0^S t}e^{-iH(t-t_0)}e^{-iH_0^S t_0}$, which takes the interaction picture states from time t_0 to a later time t, satisfies a non-trivial integral equation. Substituting the definition of the time evolution operator

$$|\psi(t)>_{iP} = U(t,t_0)|\psi(t_0)>_{iP} \tag{1.1.6}$$

into the interaction picture Schrödinger equation (1.1.1) we find that

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I^{iP}(t)U(t,t_0) \tag{1.1.7}$$

with the initial condition $U(t,t) = 1$. This can be re-cast as an integral equation

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt' H_I^{iP}(t')U(t',t_0). \tag{1.1.8}$$

This can be solved iteratively for $U(t,t_0)$, which, with the help of the time ordering operator T, can be written as

$$\begin{aligned} U(t,t_0) &= 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n T H_I^{iP}(t_1)H_I^{iP}(t_2)\cdots H_I^{iP}(t_n) \\ &\equiv Te^{-i\int_{t_0}^{t}dt' H_I^{iP}(t')}. \end{aligned} \tag{1.1.9}$$

Hence we developed a naturally perturbative solution for the time evolution operator in powers of the interaction strength. Thus we were able, in principle at least, to calculate the time evolution of states, and so could find the transition probability amplitude for some initial state to evolve into a particular state at a later time. That is we could imagine that the systemis in some initial state $|i(t_0)>_{iP}$ at time t_0, the probability amplitude for this state to evolve into the particular final state $|f(t)>_{iP}$, say, at time t is just given by

$${}_{iP}<f(t)|i(t)>_{iP} = {}_{iP}<f(t)|U(t,t_0)|i(t_0)>_{iP}\ . \tag{1.1.10}$$

Within this perturbative framework we naturally specified the initial states ($|i) \equiv |i(t_0 \to -\infty >_{iP}$) and the final states ($|f) \equiv |f(t \to +\infty >_{iP}$) by the eigenstates of the unperturbed Hamiltonian, H_0^{iP}. Actually in any scattering experiment the initial (final) state of the system consists of a definite number of spatially separated, localized particles which are not interacting with eachother, however they are physical particles which means that they are still surrounded by their own cloud of virtual particles. That we are still able to formulate scattering theory in terms of bare initial (final) states that are eigenstates of the free Hamiltonian is known as the Adiabatic Hypothesis. It requires that the interaction Hamiltonian vanishfor early and late times. This can be accomplised by multiplying the Hamiltonian by an exponential damping factor, $e^{-\frac{|t|}{T}}$, say, or by replacing all charges with time dependent charges that fall off to zerofor $|t| \gg T$. Indeed then for early and late times

the Hamiltonian becomes equal to just the free Hamiltonian and its eigenstates describe the initial and final states of the system.

In any scattering process then as time proceeds the interaction Hamiltonian is slowly turned on (slowly so that the process does not absorb or emit energy through the Fourier-transform of the Hamiltonian's cut-off time dependence) while the incoming bare particles are still not interacting with eachother. Only its own cloud of virtual particles surrounds each bare particle and interacts with it bringing the values of the bare mass and bare coupling constants up to their physical values. Then, as the now physical particles approach eachother, they exchange virtual quanta and interact resulting in the scattering of the particles, possibly the annihilation of some of them and the creation of other particles. As time progresses these physical particles become spatially separated, they are isolated physical particles again only interacting with their own virtual particle clouds. As the time proceeds far enough into the future even this interaction stops as the interaction Hamiltonian slowly goes to zero. The coupling constants vanish and the physical masses return to their bare values so that the final state is again described by an eigenstate of the free Hamiltonian. The adiabatic hypothesis is that this dressing process by which the initial bare particles become physical particles and the undressing process by which the physical particles return to final bare particles occurs at times so distant from the actual time of scattering of the physical particles that it does not interfere with it. Only at the end of the calculation do we allow $T \to \infty$. As we will see later the dressing of the bare particles to make them [Bphysical leads to important self-energy effects for single particle states in field theory that do not occur for finite range potentialsin the quantum mechanical formal theory of scattering.

The transition probability amplitude to find that an initial state $|i)$long before the scattering process has evolved into a particular final state long after the scattering has taken place is called the scattering matrix element or S-matrix element and is given by

$$S_{fi} = (f|S^{iP}|i) \equiv \lim_{t\to+\infty} \lim_{t_0\to-\infty} {}_{iP} < f(t)|U(t,t_0)|i(t_0) >_{iP}, \tag{1.1.11}$$

where the interaction picture scattering operator S^{iP} is just the time evolution operator for infinite times

$$S^{iP} = U(+\infty,-\infty) = Te^{-i\int_{-\infty}^{+\infty} dt H_I^{iP}(t)}. \tag{1.1.12}$$

It is understood that the interaction Hamiltonian above and in what follows has implicitly in it the adiabatic hypothesis damping factor and only after the S-matrix is calculated do we remove that factor (i.e. $T \to \infty$). Of course in lowest order in our perturbation expansion for any process the interaction is required solely to allow the transition to take place and not to dress the bare particles. Hence $T \to \infty$ from the beginning of the calculation.

The eigenstates of the free Hamiltonian are easily constructed in the interaction picture since the field operators evolve in time according to the free Hamiltonian Heisenberg equations of motion

$$-i\frac{\partial}{\partial t}\Phi^{iP}(x) = \left[H_0^{iP}, \Phi^{iP}(x)\right]$$

$$-i\frac{\partial}{\partial t}\Pi^{iP}(x) = \left[H_0^{iP}, \Pi^{iP}(x)\right], \tag{1.1.13}$$

where $\Phi^{iP}(x)$ denotes the quantum field operators in the interaction representation and $\Pi^{iP}(x)$ their canonically conjugate momenta. In addition the fields and their momenta obey the equal time commutation or anti-commutation relations, depending upon their spin being integer or half odd integer, since they a related to the Heisenberg fields and momenta by a unitary transformation

$$\begin{aligned}
\delta(x^0 - y^0)[\Pi^{iP}(x), \Phi^{iP}(y)]_{\pm} &= -i\delta^4(x-y) \\
\delta(x^0 - y^0)[\Phi^{iP}(x), \Phi^{iP}(y)]_{\pm} &= 0 \\
\delta(x^0 - y^0)[\Pi^{iP}(x), \Pi^{iP}(y)]_{\pm} &= 0.
\end{aligned} \tag{1.1.14}$$

Hence $\Phi^{iP}(x)$ and $\Pi^{iP}(x)$ can be Fourier expanded in terms of free particle creation and annihilation operators in momentum space. Cryptically writing this as

$$\Phi^{iP}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s [b_s(\vec{k})u^{(s)}(\vec{k})e^{-ikx} + d_s^\dagger(\vec{k})v^{(s)}(\vec{k})e^{+ikx}], \tag{1.1.15}$$

where $u^{(s)}(\vec{k})e^{-ikx}$ is the single particle wavefunction and $b_s(\vec{k})$ is its associated annihilation operator, with the hermitian conjugate operator $b_s^\dagger(\vec{k})$ the creation operator for a particle with momentum $\vec{k}$,energy $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$ and z-component of spin in therest frame (or helicity, depending on the definition of $u^{(s)}(\vec{k})$) givenby $(-1)^{s+1}$, while $d_s^\dagger(\vec{k})$ is the anti-particle creation operator with $\bar{v}^{(s)}(\vec{k})e^{-ikx}$ its single particle wavefunction. Analogous to the particle creation operator, $d_s^\dagger(\vec{k})$ creates an anti-particle with momentum $\vec{k}$, energy $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$ and z-component of spin in the rest frame (or helicity, depending on the definition of $\bar{v}^{(s)}(\vec{k})$) given by $(-1)^s$, and likewise the hermitian conjugate operator $d_s(\vec{k})$ is the anti-particle annihilation operator. If the field is real so that the particle is its own anti-particle then $b_s(\vec{k}) = d_s(\vec{k}) \equiv a_s(\vec{k})$ and $(u^{(s)}(\vec{k}))^* = v^{(s)}(\vec{k})$. Thus we have that the ETCR or ETAR imply that the momentum space creation and annihilation operators obey the CCR or CAR

$$\begin{aligned}
[b_r(\vec{p}\,), b_s^\dagger(\vec{k})]_{\pm} &= (2\pi)^3 2\omega_k \delta_{rs}\delta^3(\vec{p} - \vec{k}) \\
[b_r(\vec{p}\,), b_s(\vec{k})]_{\pm} &= 0 = [b_r^\dagger(\vec{p}\,), b_s^\dagger(\vec{k})]_{\pm} \\
[d_r(\vec{p}\,), d_s^\dagger(\vec{k})]_{\pm} &= (2\pi)^3 2\omega_k \delta_{rs}\delta^3(\vec{p} - \vec{k}) \\
[d_r(\vec{p}\,), d_s(\vec{k})]_{\pm} &= 0 = [d_r^\dagger(\vec{p}\,), d_s^\dagger(\vec{k})]_{\pm}.
\end{aligned} \tag{1.1.16}$$

The point of all this being that the Fock space of states made from the interaction picture fields are eigenstates of the number operator for each type of particle and anti-particle as well as H_0^{iP}, $\vec{\mathcal{P}}^{iP}$ and $\mathcal{J}_3^{iP}$ (or the helicity operator $\frac{\vec{\mathcal{P}}^{iP}\cdot\vec{\mathcal{J}}^{iP}}{|\vec{\mathcal{P}}^{iP}|}$). They are precisely the initial states $|i)$ and final states $|f)$. The general N-particle and $\bar{N}$-anti-particle initial

or final state is built up from the no particle or anti-particle state, that is the vacuum state $|0)$. So the vacuum stateis defined by

$$b_r(\vec{p}\,)|0) = 0 = d_r(\vec{p}\,)|0), \text{ for all } r, \vec{p}. \tag{1.1.17}$$

The multi-particle state is then given by the action of the creation operators on the vacuum state

$$\begin{aligned}&|(p_1, s_1), (p_2, s_2), \ldots, (p_N, s_N); (\bar{p}_1, \bar{s}_1), (\bar{p}_2, \bar{s}_2), \ldots, (\bar{p}_{\bar{N}}, \bar{s}_{\bar{N}}))\\ &\quad = b^\dagger_{s_1}(\vec{p}_1)\cdots b^\dagger_{s_N}(\vec{p}_N) d^\dagger_{\bar{s}_1}(\vec{\bar{p}}_1)\cdots d^\dagger_{\bar{s}_{\bar{N}}}(\vec{\bar{p}}_{\bar{N}})|0).\end{aligned} \tag{1.1.18}$$

The S-matrix element for the transition to such a final state $|f)$ from a particular of these Fock space initial states $|i)$ is simply given by

$$S_{fi} = (0|d_{\bar{s}_{\bar{N}}}(\vec{\bar{p}}_{\bar{N}})\cdots b_{s_1}(\vec{p}_1)\left(Te^{-i\int_{-\infty}^{+\infty} dt H_I^{iP}(t)}\right) b^\dagger_{r_1}(\vec{k}_1)\cdots d^\dagger_{\bar{r}_{\bar{M}}}(\vec{\bar{k}}_{\bar{M}})|0). \tag{1.1.19}$$

As seen earlier, the S-operator is given in terms of the interaction Hamiltonian in the interaction picture, that is the interaction Hamiltonian iswritten in terms of the interaction picture fields and momenta. Since these are just the same free fields with which we built the initial and final states we can evaluate the S-matrix element with the help of Wick's Theorem

$$\begin{aligned}T\Phi^{iP}(x_1)\cdots\Phi^{iP}(x_n) &= N[\Phi^{iP}(x_1)\cdots\Phi^{iP}(x_n)]\\ &+ \sum_{\substack{1 \text{ contraction}\\ i<j}} (0|T\Phi^{iP}(x_i)\Phi^{iP}(x_j)|0) N\left[\frac{\Phi^{iP}(x_1)\cdots\Phi^{iP}(x_n)}{\Phi^{iP}(x_i)\Phi^{iP}(x_j)}\right]\\ &+ \sum_{\substack{2 \text{ contractions}\\ i<j,\ i<k,\ j<l}} (0|T\Phi^{iP}(x_i)\Phi^{iP}(x_j)|0)(0|T\Phi^{iP}(x_k)\Phi^{iP}(x_l)|0)\\ &\qquad \times N\left[\frac{\Phi^{iP}(x_1)\cdots\Phi^{iP}(x_n)}{\Phi^{iP}(x_i)\Phi^{iP}(x_j)\Phi^{iP}(x_k)\Phi^{iP}(x_l)}\right] + \cdots\\ &+ \sum_{\text{all contractions}} (0|T\Phi^{iP}(x_i)\Phi^{iP}(x_j)|0)\cdots(0|T\Phi^{iP}(x_r)\Phi^{iP}(x_s)|0)\\ &\qquad \times \begin{cases} 1, & \text{if } n = \text{even};\\ \Phi^{iP}(x_t), & \text{if } n = \text{odd}\end{cases},\end{aligned} \tag{1.1.20}$$

where $(0|T\Phi^{iP}(x)\Phi^{iP}(y)|0)$ is the free field Feynman propagator. That is given the Lagrangian density as the sum of the free and interaction parts, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, we can construct the Hamiltonian density in terms of free and interaction parts, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$. The canonically conjugate momentum is given as

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \tag{1.1.21}$$

and the Hamiltonian density is given by the Legendre transform

$$\mathcal{H} = \Pi\dot{\Phi} - \mathcal{L}. \tag{1.1.22}$$

Transforming to the interaction picture, we have that the free Hamiltonian Heisenberg equations of motion for Φ^{iP} andΠ^{iP} are equivalent to the free Euler-Lagrange equations of motionfor Φ^{iP}

$$\frac{\partial \mathcal{L}_0^{iP}}{\partial \Phi^{iP}} - \partial_\mu \frac{\partial \mathcal{L}_0^{iP}}{\partial \partial_\mu \Phi^{iP}} = 0. \tag{1.1.23}$$

Since the free Lagrangian is bilinear in the fields, the Euler-Lagrange equations have the form of a linear partial differential equation

$$D_x \Phi^{iP}(x) = 0, \tag{1.1.24}$$

in which D is typically the Klein-Gordon operator $K \equiv (\partial^2 + m^2)$for spin zero fields, or the Dirac operator $\not{D} \equiv (i\not{\partial} - m)$ for spin $\frac{1}{2}$ fields or the Maxwell-Proca-Stueckelberg gauge operator $K^{\mu\nu} \equiv (\partial^2 g^{\mu\nu} + \frac{(1-\alpha)}{\alpha}\partial^\mu\partial^\nu + \mu^2 g^{\mu\nu})$ for spin one fields. Then in accord withthe ETCR or ETAR the Feynman propagator is then given by the inverse of this operator times $-i$

$$D_x(0|T\Phi^{iP}(x)\Phi^{iP}(y)|0) = -i\delta^4(x-y)\mathbf{1}. \tag{1.1.25}$$

We then have a complicated perturbative expansion for the S-matrix in powers of the interaction strength. It involves convolutions of the Feynman propagators in coordinate space ending with the wavefunctions of the initial and final states' particles. From this we extracted graphical rules, Feynman rules, which allowed us to pictorially represent the terms in the perturbation expansion. The Feynman propagators were represented by lines which met at vertices given by the form of the interaction Hamiltonian. Cryptically written we converted the Feynman-Dyson expansion of the S-matrix, equation (1.1.19), with the use of Wick's Theorem, equation (1.1.20), to a sum over Feynman diagrams

$$\begin{aligned} S_{fi} &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4x_1 \cdots \int_{-\infty}^{+\infty} d^4x_n (0|d_{\bar{s}_{\bar{N}}}(\vec{\bar{p}}_{\bar{N}}) \cdots T[\mathcal{H}_I^{iP}(x_1) \cdots \mathcal{H}_I^{iP}(x_n)] \\ &\qquad \times b^\dagger_{r_1}(\vec{k}_1) \cdots d^\dagger_{\bar{r}_{\bar{M}}}(\vec{\bar{k}}_{\bar{M}})|0) \\ &= \sum_{\Gamma \in G_{fi}} S_{fi}^\Gamma, \end{aligned} \tag{1.1.26}$$

with G_{fi} the set of all possible Feynman diagrams with i-incoming lines and f-outgoing lines made from the vertices and lines of the theory according to the Feynman graphical rules and S_{fi}^Γ is the contribution to the S-matrix element from the particular Feynman diagram Γ calculated according to the Feynman rules relating the graphical elements to factors in the Feynman-Dyson expansion, that is the lines to the propagators and wavefunctions and the vertices to the interaction Hamiltonian with appropriate loop momentum integrals, combinatoric factors and signs and energy-momentum conserving Dirac delta functions.

Thus, at least in principle, we are able to determine the scattering matrix elements for any process. However, once applying the above expansion we found a few provisos due to the fact that we are dealing with systems with infinite degrees of freedom, that is a field theory. The first point was fairly innocuous and dealt with the adiabatic hypothesis applied to the vacuum state. In particular we find that the dressing and undressing of the Fock vacuum $|0)$ results in the fact that its phase may change. That is we can apply the perturbation expansion to the vacuum to vacuum transition amplitude, $(0|S^{iP}|0)$, to find this phase. Recall it must be a phase since the S-operator is translation invariant, $[\mathcal{P}_\mu^{iP}, S^{iP}] = 0$, hence it cannot change the energy-momentum of the initial state. Since $|0)$ is the zero energy-momentum eigenstate we have that $S^{iP}|0)$ is proportional to $|0)$, $S^{iP}|0) = \omega|0)$. Since the S-operator is unitary, $S^{iP\dagger}S^{iP} = 1$, we have that $|\omega|^2(0|0) = (0|S^{iP\dagger}S^{iP}|0) = (0|0)$. Hence $|\omega|^2 = 1$, so ω is a phase factor. Since each initial and final state is built up from the vacuum state they will have this additional phase factor. Further since it is the probability $|S_{fi}|^2$ that is observable, we might as well eliminate the unobservable phase factor ω from each S-matrix element by re-normalizing each element to the vacuum to vacuum transition amplitude. Thus we can re-define the S-operator to be

$$\begin{aligned} S &\equiv \frac{U(+\infty,-\infty)}{(0|U(+\infty,-\infty)|0)} \\ &= \frac{Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}}{(0|Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}|0)}, \end{aligned} \tag{1.1.27}$$

with S-matrix elements now given as

$$S_{fi} = \frac{(f|Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}|i)}{(0|Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}|0)}. \tag{1.1.28}$$

In perturbation theory we found this vacuum fluctuation phase factor, it is given by the sum of all vacuum bubble graphs (i.e. graphs with no external lines) and was in fact determined to be an infinite phase. In addition we explicitly factored this same phase factor out of each initial-final state transition amplitude of the numerator of S_{fi} in equation (1.1.28)showing that indeed the re-normalized S-matix elements are phase independent. Once shown we then considered only sets of Feynman diagrams, G_{fi}, excluding vacuum bubbles when calculating S_{fi}.

As we will see later, the above discussion has assumed the absence of external fields or sources in the theory, when they are present the S-operator no longer conserves energy-momentum and then $S^{iP}|0)$ is not just proportional to the vacuum again but must be written as a sum over all Fock states. Hence ω is not just a phase but a complicated functional of the sources describing thestability of the vacuum state. In any particle scattering process we are interested in the particle state's transition amplitude only, not in the effects of an unstable vacuum state, which of course is interesting in its ownright. Since we have built the particle states from the Fock space vacuum state we continue to divide out the vacuum to vacuum transition amplitude in the definition of their S-matrix

elements. Alternately stated we continue to define a particle state scattering operator that leaves the vacuum invariant, hence we divide $U(+\infty,-\infty)$ by $(0|U(+\infty,-\infty)|0)$ even in the presence of external fields or sources.

The second point again deals with the adiabatic hypothesis but this time applied to the one particle states. The dressing of the states not only affects the stability of the vacuum, but as well, the probability for the single particle to remain in the same state after interacting with just its cloud of virtual particles is not one. As with the vacuum, the one-particle state does not remain stable due to its adiabatic dressing. The self-energy effects that arise from the bare particle's interaction with its cloud of virtual particles results in radiative corrections to the mass and normalization of the single particle state. That is consider the transition amplitude for the one-particlestate with momentum $\vec{p}$, $|\vec{p}\,)$, to remain a one-particle state withmomentum $\vec{p}\,'$, $|\vec{p}\,')$,

$$
\begin{aligned}
(\vec{p}\,'|S^{iP}|\vec{p}\,) &= \frac{(\vec{p}\,'|U(+\infty,-\infty)|\vec{p}\,)}{(0|U(+\infty,-\infty)|0)} = \frac{(\vec{p}\,'|Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}|\vec{p}\,)}{(0|Te^{-i\int_{-\infty}^{+\infty} dtH_I^{iP}(t)}|0)} \\
&= (2\pi)^3 2\omega_p \delta^3(\vec{p}-\vec{p}\,')\mathbf{1} - i\int_{-\infty}^{+\infty} dt(\vec{p}\,'|H_I^{iP}(t)|\vec{p}\,)_{\mathrm{NVB}} \\
&\quad + \frac{(-i)^2}{2!}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 (\vec{p}\,'|TH_I^{iP}(t_1)H_I^{iP}(t_2)|\vec{p}\,)_{\mathrm{NVB}} + \cdots .
\end{aligned}
\tag{1.1.29}
$$

The single particle state, in the absence of external fields, should remain stable since there is nothing with which it can scatter and this should be one. However, the particle's interaction with its cloud of virtual particles results in a mass shift from that of the bare(or unrenormalized) mass m_u to that of the physical mass m of the particle, which in a scattering process would then go on to interact. Hence, the amplitude for the bare particle to remain so is no longer one. This can be described by separating the Hamiltonian into free and perturbed terms differently. Hence we will introduce an additional interaction known as a mass [Bcounterterm in the interaction Hamiltonian (Lagrangian) to account for these radiative mass shifts. It will guarantee that the particle's mass is the physical mass, as it should be, from the beginning. Alternatively stated, we can add and subtract the required mass shift, δm, from the Hamiltonian. Then the free (unperturbed) Hamiltonian and the interaction Hamiltonian can be defined so that the free initial and final state particles have the physical particle's mass, $m = m_u + \delta m$. Of course we must compensate for this by subtracting the mass shift from the interaction Hamiltonian so that it is no longer present in the self-energy radiative corrections due to the original interaction.

For example, consider the electron-electron transition amplitude in QED through second order in the charge in perturbation theory as shown in Figure 1.1.1

$$
(\vec{p}\,', s'|S^{iP}|\vec{p}, s) = (2\pi)^3 2\omega_p \delta_{ss'}\delta^3(\vec{p}-\vec{p}\,') + (2\pi)^4\delta^4(p-p')\bar{u}^{(s')}(\vec{p}\,')\Sigma(p)u^{(s)}(\vec{p}\,), \tag{1.1.30}
$$

where the electron self-energy, $\Sigma(p)$, in the Feynman gauge, is given by

$$\begin{aligned}\Sigma(p) &= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\epsilon}(-ie\gamma^\mu)\frac{i}{(\not p+\not k)-m}(-ie\gamma^\nu) \\ &= e^2\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\frac{(2\not p+2\not k-4m)}{(p+k)^2-m^2+i\epsilon}.\end{aligned} \tag{1.1.31}$$

$(\vec{p}\,', s'|S^{iP}|\vec{p}, s) =$

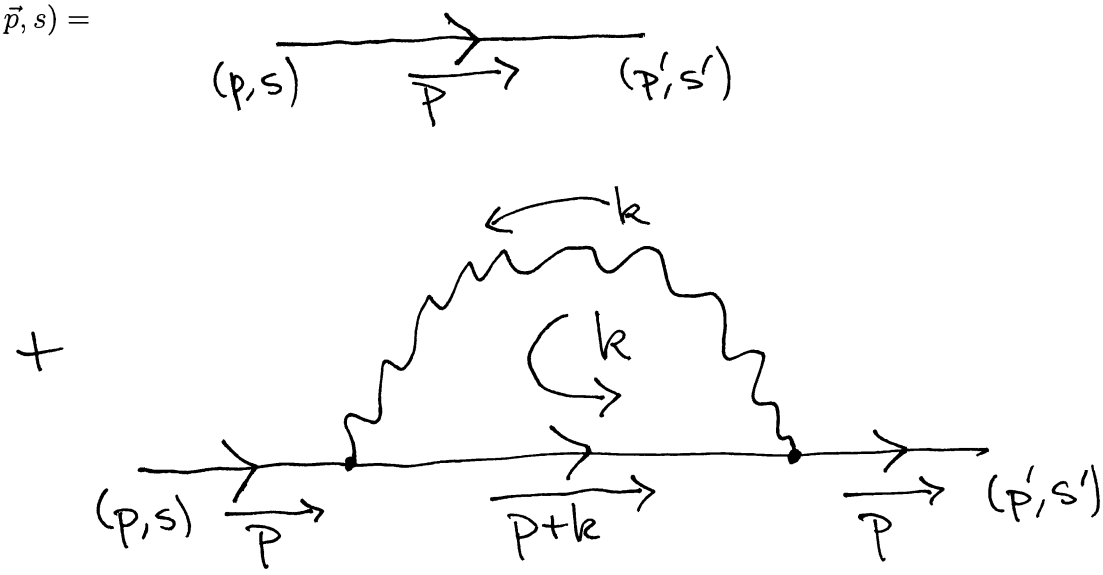


Figure 1.1.1

To see that this self-energy will cause the mass to shift from the unperturbed Hamiltonian's value, m, to what will be identified as the physical particle's value, denoted m_e, with the mass shift defined by $\delta m = m_e - m$, we Taylor expand the self-energy in momentum about the physical value (since this is the self-energy in lowest order in fact the masses, when inserted in the expression for it, are the same

$$\Sigma(p) = A + B(\not p - m_e) + \frac{1}{2}\sigma(p)(\not p - m_e)^2, \tag{1.1.32}$$

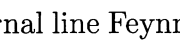
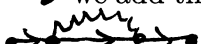
where A and B are constants while $\sigma = \sigma(p)$ is the remainder.Hence we have that

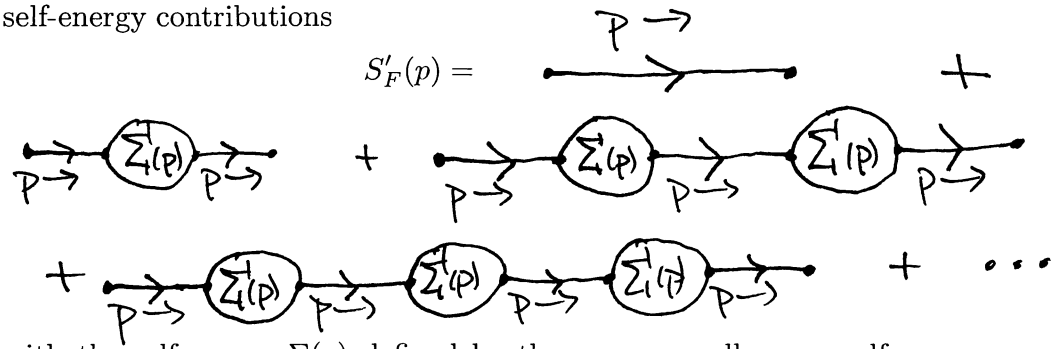
$$\bar{u}^{(s')}(\vec{p}\,')\Sigma(p)u^{(s)}(\vec{p}) = A\delta_{ss'} \tag{1.1.33}$$

since $(\not p - m)u(\vec{p}\,) = 0$. After some analysis(see for example Mandl and Shaw, QFT, chapter 10.2) we find that

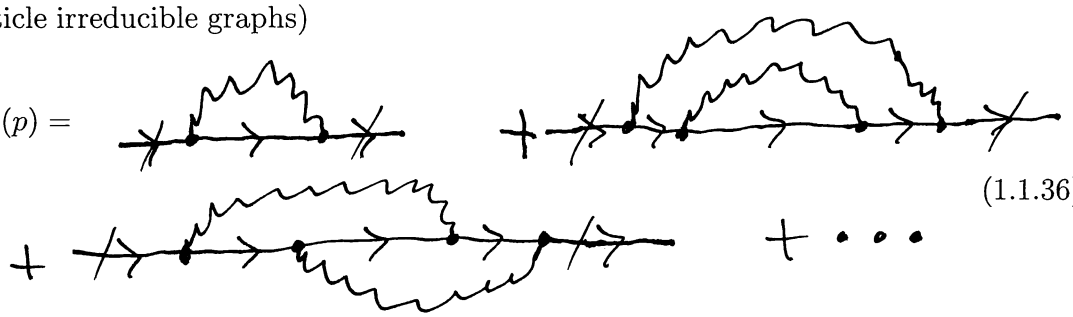
$$\begin{aligned}A &= \frac{-ime^2}{8\pi^2}\int_0^1 dz(2-z)\ln\left[1+\frac{\Lambda^2}{m^2}\frac{z}{(1-z)^2}\right] \\ &= \frac{-3ime^2}{8\pi^2}\ln\frac{\Lambda}{m} + O(1),\end{aligned} \tag{1.1.34}$$

where Λ is an ultra-violet cutoff for the loop momentum integral. The electron-electron transition amplitude is in fact logarithmically UV divergent. It is precisely the change in mass that results from the self-energy interactions.

To see this mass shift more clearly note that these same self-energy corrections occur for every internal line Feynman propagator. For every internal line  we add the self-energy graph for the next higher order correction to the propagator  . The full (orinteracting) Feynman propagator, denoted $S'_F(p)$, is given by the sum of all self-energy contributions

$S'_F(p) =$  (1.1.35)

with the self-energy $\Sigma(p)$ defined by the sum over all properself-energy graphs (graphs that cannot be separated into two parts by the removal of a single internal line, that is one-particle irreducible graphs)

$\Sigma(p) =$  (1.1.36)

Note that this sum has the form of a series which we can sum to obtainan equation for $S'_F(p)$ and is known as the Schwinger-Dyson equation

$$\begin{aligned} S'_F(p) &= S_F(p) + S_F(p)\Sigma(p)S_F(p) + S_F(p)\Sigma(p)S_F(p)\Sigma(p)S_F(p) \\ &\qquad + S_F(p)\Sigma(p)S_F(p)\Sigma(p)S_F(p)\Sigma(p)S_F(p) + \cdots \\ &= S_F(p) + S_F(p)\Sigma(p)[S_F(p) + S_F(p)\Sigma(p)S_F(p) \\ &\qquad + S_F(p)\Sigma(p)S_F(p)\Sigma(p)S_F(p) + \cdots] \\ &= S_F(p) + S_F(p)\Sigma(p)S'_F(p). \end{aligned} \tag{1.1.37}$$

Hence we find

$$\begin{aligned} S'_F(p) &= [1 - S_F(p)\Sigma(p)]^{-1} S_F(p) \\ &= \frac{1}{S_F^{-1}(p) - \Sigma(p)}. \end{aligned} \tag{1.1.38}$$

Now the free (unperturbed) propagator is simply $S_F^{-1}(p) = -i(\not{p} - m)$ where m is the bare mass, so

$$S'_F(p) = \frac{i}{\not{p} - m - i\Sigma(p)}. \tag{1.1.39}$$

Recalling the expansion for the self-energy, equation (1.1.32), the full propagator becomes

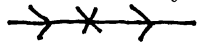
$$S'_F(p) = \frac{i}{\not p - m - iA - iB(\not p - m_e) - \frac{i}{2}\sigma(p)(\not p - m_e)^2}. \tag{1.1.40}$$

Defining the pole of the full propagator to be at the physical mass, m_e, we have that $\delta m = iA$ so that $m_e = m + iA$. The full propagator then becomes

$$S'_F(p) = \frac{i}{(1 - iB)(\not p - m_e) - \frac{i}{2}\sigma(p)(\not p - m_e)^2}, \tag{1.1.41}$$

clearly having the pole at m_e. Rather than sum up the self-energy mass insertions for each internal line in a process we can account for the mass shift in the propagator initially by moving the mass shift from the interaction Hamiltonian to the unperturbed Hamiltonian. This is accomplished by simply adding and subtracting the mass shift in the (Hamiltonian)Lagrangian,$\mathcal{L} = \mathcal{L} + \delta m\bar{\Psi}\Psi - \delta m\bar{\Psi}\Psi$. We then include one of the mass shift terms in the free Lagrangian so that the free propagator now has its pole at the mass of the physical particle.However the interaction Lagrangian now has an additional mass counterterm guaranteeing the absence of the radiative mass shift due to the originalinteraction. This new counterterm gives rise to an additional interaction vertex in the Feynman rules. That is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 + i\bar{\Psi}\not\partial\Psi - (m + \delta m)\bar{\Psi}\Psi \\ \mathcal{L}_I &= -e\bar{\Psi}\gamma^\mu\Psi A_\mu + \delta m\bar{\Psi}\Psi, \end{aligned} \tag{1.1.42}$$

with the anti-symmetric field strength tensor defined by $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.The new graphical element is the interaction vertex  , known as a two vertex or a mass insertion, with the Feynman rule factor $+i\delta m$ associated with it. So the self-energy in second order will now be given by

$$\begin{aligned} \Sigma(p) &= +i\delta m + A + B(\not p - m_e) + \frac{1}{2}\sigma(p)(\not p - m_e)^2 \\ &= B(\not p - m_e) + \frac{1}{2}\sigma(p)(\not p - m_e)^2, \end{aligned} \tag{1.1.43}$$

as shown in Figure 1.1.2.

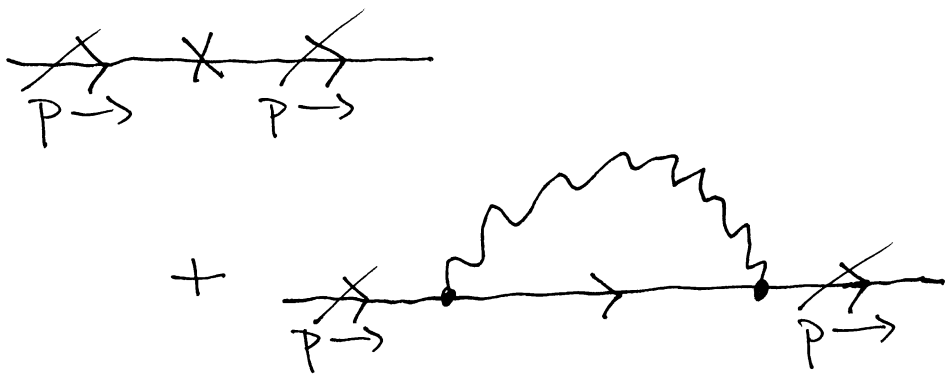
$\Sigma(p) =$ 

Figure 1.1.2

Hence we obtain the same full propagator $S'_F(p)$ as in equation (1.1.41).

Returning to the question of the one-particle state's stability. Including the mass counterterm in the self-energy, which **defines** the pole of the propagator to be at the physical particle mass, we establish the required stability as seen in Figure 1.1.3 and given analytically by

$$\begin{aligned}(\vec{p}\,', s'|S^{iP}|\vec{p}, s) &= (2\pi)^3 2\omega_p \delta_{ss'}\delta^3(\vec{p}-\vec{p}\,') + (2\pi)^4\delta^4(p-p')\bar{u}^{(s')}(\vec{p}\,')[+i\delta m + A]u^{(s)}(\vec{p}\,) \\ &= (2\pi)^3 2\omega_p \delta_{ss'}\delta^3(\vec{p}-\vec{p}\,').\end{aligned} \tag{1.1.44}$$

$(\vec{p}\,', s'|S^{iP}|\vec{p}, s) =$

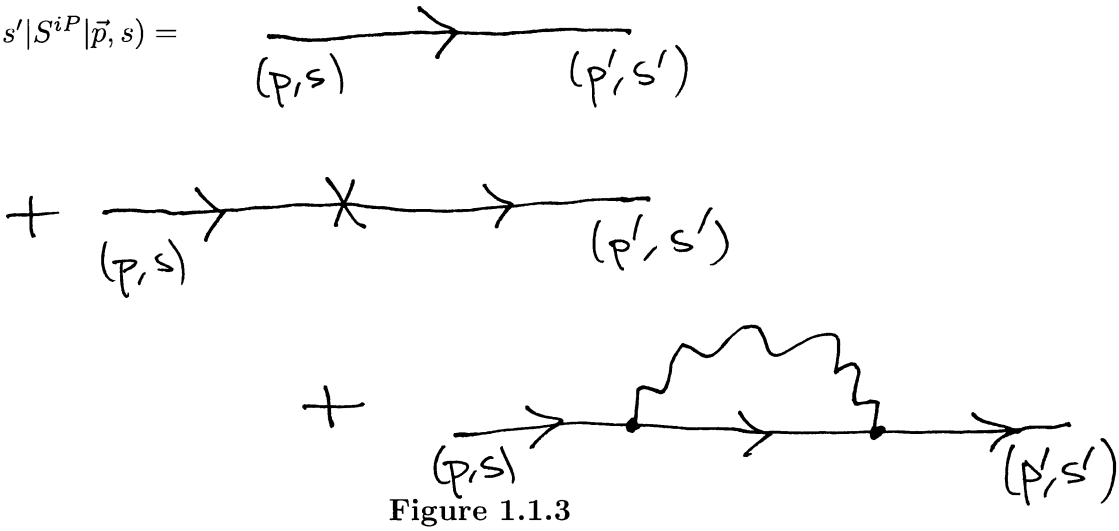


Figure 1.1.3

Note that although the mass renormalization is necessary for a consistent fundamental field theory, the energy scale at which the cut-off Λ contributes a mass shift the order of the mass of the electron is quite large, $\Lambda = me^{\frac{8\pi^2}{3e^2}} = me^{\frac{2\pi}{3\alpha}} \approx 10^{121} GeV$.

In addition to the mass renormalization the dressing of the bare electron to become the physical electron also involves a rescaling of the fields and charge as well as a wavefunction renormalization of the one particle wavefunction with the same factor. As we will see a careful application of the adiabatic hypothesis will be required to determine the correct wavefunction renormalization factor for the external lines. To see this we once again turn to the full fermion propagator, after mass renormalization we have

$$\begin{aligned}S'_F(p) &= \frac{i}{(1-iB)(\not{p}-m_e) - \frac{i}{2}\sigma(p)(\not{p}-m_e)^2} \\ &= \frac{iZ}{(\not{p}-m_e)[1-\frac{iZ}{2}\sigma(p)(\not{p}-m_e)]} \\ &= \frac{iZ}{(\not{p}-m_e)}[1+\frac{1}{2}\sigma(p)(\not{p}-m_e)^2\frac{iZ}{\not{p}-m_e}],\end{aligned} \tag{1.1.45}$$

with $Z \equiv \frac{1}{1-iB} = 1+iB$ to this lowest order. As with δm we can show that $B = \frac{\partial\Sigma(p)}{\partial\not{p}}|_{\not{p}=m_e}$ is also logarithmically ultraviolet divergent. As seen above, it now appears thatthe full

propagator is given by a perturbation expansion in terms of a "free" propagator having the physical mass m_e but now with residue iZ. We can return this residue to i by re-scaling the fields of the Lagrangian $\Psi_R \equiv Z^{-\frac{1}{2}}\Psi$ where Ψ_R is called the renormalized field while Ψ is the bare or unrenormalized field. The Lagrangian in terms of the renormalized fieldthen has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 + Z\bar{\Psi}_R(i\partial\!\!\!/ - m_e)\Psi_R + Z\delta m\bar{\Psi}_R\Psi_R - Ze\bar{\Psi}_R\gamma^\mu\Psi_R A_\mu. \quad (1.1.46)$$

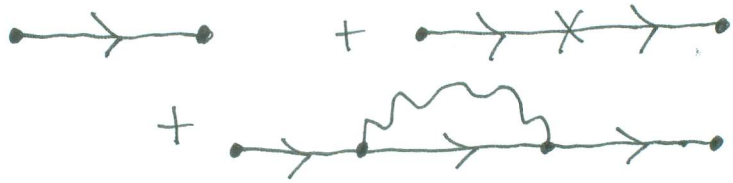
The unperturbed Lagrangian in terms of the renormalized field and the physical mass is then given as usual by

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 + \bar{\Psi}_R(i\partial\!\!\!/ - m_e)\Psi_R \quad (1.1.47)$$

with the interaction Lagrangian given by

$$\begin{aligned}\mathcal{L}_I &= \mathcal{L} - \mathcal{L}_0 \\ &= -Ze\bar{\Psi}_R\gamma^\mu\Psi_R A_\mu + Z\delta m\bar{\Psi}_R\Psi_R + (Z-1)\bar{\Psi}_R(i\partial\!\!\!/ - m_e)\Psi_R.\end{aligned} \quad (1.1.48)$$

The full propagator for the theory in terms of the renormalized field is then simply

$S'_F(p) =$  (1.1.49)

$$= \frac{i}{p\!\!\!/ - m_e}[1 + \Sigma(p)\frac{i}{p\!\!\!/ - m_e}]$$

where now the self-energy is calculated with $\mathcal{L}_I$ above, that is with the additional 2-vertex,  ,with the corresponding Feynman rule factor (including the mass counterterm)

$$iZ\delta m + i(Z-1)[p\!\!\!/ - m_e] \quad (1.1.50)$$

coming from the bilinear $\bar{\Psi}_R\Psi_R$ terms in $\mathcal{L}_I$. So we have

$\Sigma(p) =$  (1.1.51)

$$\begin{aligned}&= +i\delta m + i(Z-1)(p\!\!\!/ - m_e) + A + B(p\!\!\!/ - m_e) + \frac{1}{2}\sigma(p)(p\!\!\!/ - m_e)^2 \\ &= \frac{1}{2}\sigma(p)(p\!\!\!/ - m_e)^2.\end{aligned}$$

At this point we note that there are two directions in which we can proceed. One is to use the bare fields not the re-scaled renormalized fields. The infinite wavefunction

renormalization factors Z for each internal line will be absorbed into a renormalization of the coupling constant factor at each vertex where the line begins or ends. Two lines meet at each vertex so we have that the renormalized charge $e_R = Ze$ and the perturbation series is in terms of the renormalized, physical electric charge e_R. However for external lines, only one end of the line meets at the vertex the other end is a wavefunction. Hence we must make sure that we only obtain a factor of $Z^{1/2}$ for the external line not Z so that the coupling constants of vertices attached external lines result in finite renormalized electric charges $e_R = Ze$. So we note at this point that we have defined a renormalized, physical charge $e_R = Ze$, then the finite perturbation expansion would be given in terms of e_R rather than e as all the Z factors cancel. Also, even though the mass shift and wave function renormalization factors are infinite here we would still need to renormalize the mass and residue if they were finite since the mass does shift as a result of self interactions, as well as the probability that the bare particle is in the physical particle state, which is what Z measures. The second approach to renormalized perturbation theory is to explicitly rescale the fields as we did above, $\Psi_R \equiv Z^{-\frac{1}{2}}\Psi$, and to renormalize the coupling constants as in the bare case, $e_R = Ze$, as well as to use the renormalized mass term. The Lagrangian then is written in terms of these renormalized quantities. Then the free Lagrangian is in terms of Ψ_R and m_e. The interaction Lagrangian then contains new terms corresponding to the $(Z-1)$, δm and Ze factors that are now explicit from the rewriting of the Lagrangian in terms of the renormalized fields, masses and coupling constants. These are now new vertices in the Feynman rules as we had above–they are called counter terms. They are not new terms, we are just re-writing the bare Lagrangian in terms of new variables, $\Psi_R = Z^{-\frac{1}{2}}\Psi$, $m_e = m + \delta m$ and $e_R = Ze$ and then splitting it into free and interaction pieces. The counter-terms are then used to cancel the infinities that occur in the perturbation expansion. More precisely we define the physical parameters of the theory by a) specifying the location of the pole of the propagator to be at the physical mass:

$$S'_F{}^{-1}(p)|_{\not{p}=m_e} = 0.$$

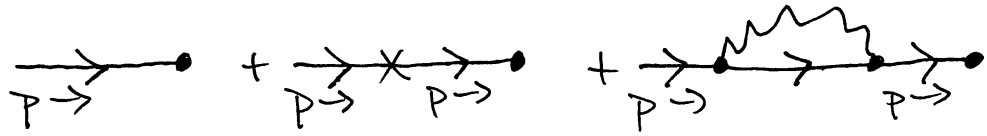
b) specifying that the residue of the propagator is i

$$(\not{p} - m_e)S'_F(p)|_{\not{p}=m_e} = i$$

c) specifying the value of the coupling constant e_R through some scattering process at a particular momentum value (we will specify this later). These expressions will define the counter-terms as we have seen for example $\delta m = iA$ and $Z = 1 + iB$. The infinities that arise in the Feynman integrals are then cancelled by the counter-terms yielding a finite perturbation expansion for the S-matrix elements. This second proceedure is the one we will in general adopt. The details of these renormalization methods needed to show that they indeed render the perturbation expansion finite in every order of expansion require quite technical proofs.

Finally we come back to the point concerning the external lines of the theory. So far we have been considering the internal lines, the full propagators. They carry a factor of Z since they connect different vertices, that is they are associated with two fields, one at each vertex and hence the re-scaling of the fields $\Psi_R = Z^{-\frac{1}{2}}\Psi$ in order to eliminate this factor.

But the external lines of a process only connect to one vertex, hence we expect that they carry a factor of $Z^{\frac{1}{2}}$. That is we will calculate the full external line contribution to the one particle wavefunction to check that it carries the expected $Z^{1/2}$ factor: $u_R(\vec{p}\,) = Z^{\frac{1}{2}} u(\vec{p}\,)$ for the external electrons. To see this consider the self-energy contributions to an external electron line (before we have renormalized the fields, that is in the bare field method) which we denote $u_R(\vec{p}\,)$

$u_R(\vec{p}\,) =$ 

$$
\begin{aligned}
&= \left(1 + \frac{i}{\not p - m_e}\Sigma(p)\right) u(\vec{p}\,) \qquad (1.1.52)\\
&= \left(1 + \frac{i}{\not p - m_e} B(\not p - m_e) + \frac{i}{\not p - m_e}\frac{1}{2}\sigma(p)(\not p - m_e)^2\right) u(\vec{p}\,).
\end{aligned}
$$

Now the last term vanishes

$$\frac{1}{\not p - m_e}(\not p - m_e)^2 u(\vec{p}\,) = (\not p - m_e) u(\vec{p}\,) = 0,$$

while the second term is of the form $\frac{0}{0}$ and must be determinedby a limiting procedure since it is indeterminant as it stands (at best it naively appears to be $(1 + iB)u(\vec{p}\,) = Zu(\vec{p}\,)$ which has the wrong renormalization factor). In order to determine the external line renormalization factor we must dress the particle according to the adiabatic hypothesis. In particular the interaction Hamiltonian $H_I^{iP}(t)$should be taken to vanish for early and late times, that is as $t \to \pm\infty$. This can be implemented by allowing the charge to become a function of time, $e \to ef(t)$ where $f(t)$ goes to zero as $t \to \pm\infty$ and is equal to 1 for times comparable to the interaction times being considered but is otherwise arbitrary. $f(t)$ has the Fourier transform

$$f(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{f}(E) \qquad (1.1.53)$$

and we choose $\tilde{f}$ to be normalized so that

$$f(0) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \tilde{f}(E) \equiv 1, \qquad (1.1.54)$$

as illustrated in Figure 1.1.4.

As $\tilde{f}(E) \to 2\pi\delta(E)$ we have $f(t) \to 1$ and the original theory is restored.The interaction Hamiltonian becomes

$$\mathcal{H}_I = ef(t)\bar{\Psi}\gamma^\mu\Psi A_\mu - \delta m(f(t)^2)\bar{\Psi}\Psi \qquad (1.1.55)$$

where $\delta m = \delta m(f^2)$ since the mass shift begins in second order.

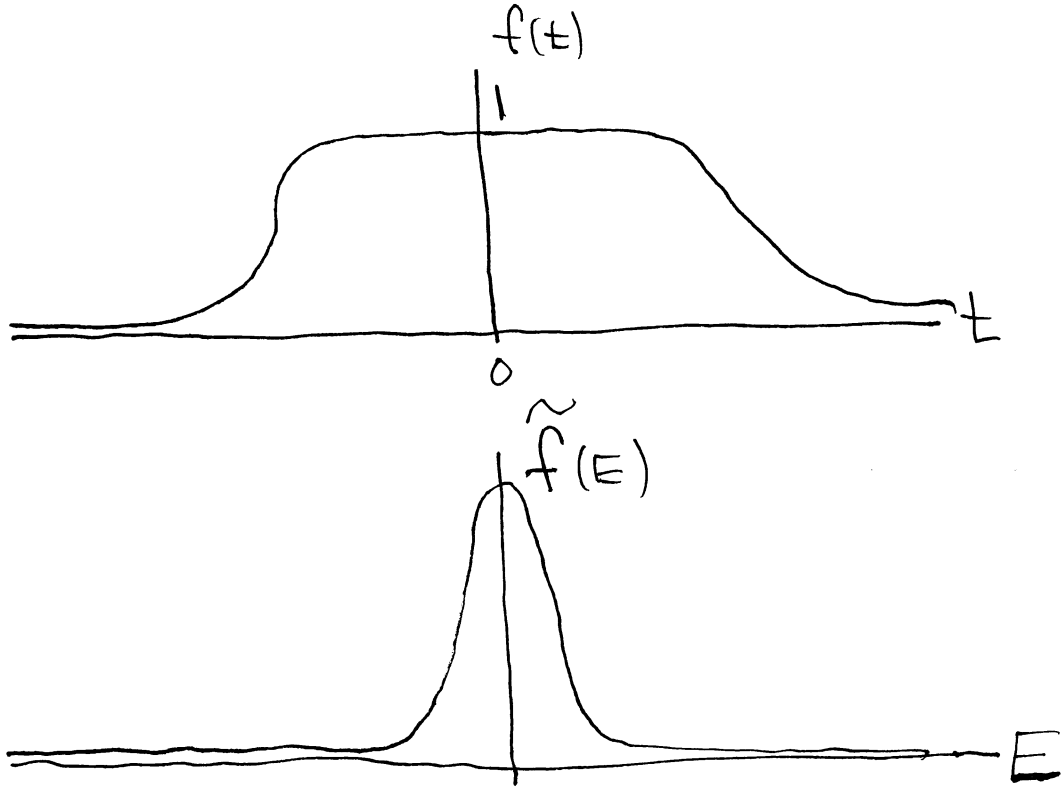


Figure 1.1.4

The full external line is given by the same Feynman graphs as before except now the presence of $f(t)$ acts as an external field at the interaction vertices supplying additional energy to the process at each vertex in the Feynman rules as is illustrated in Figure 1.1.5 and is given by

$$u_R(\vec{p}\,) = [1 + \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi}\tilde{f}(E)\tilde{f}(E')(\frac{i}{\not p + \not q + \not q' - m_e}B(\not p + \not q - m_e)$$

$$+\frac{i}{\not p + \not q + \not q' - m_e}\frac{1}{2}\sigma(p,q,q')(\not p + \not q - m_e)^2)]u(\vec{p}\,) \qquad (1.1.56)$$

where we have defined $q^\mu = (E, \vec{0}\,)$ and $q'^\mu = (E', \vec{0}\,)$. Since $(\not p - m)u(\vec{p}) = 0$ we have that $(\not p + \not q - m)u(\vec{p}) = [\not p + \not q - m - \frac{1}{2}(\not p - m)]u(\vec{p}) = \frac{1}{2}(\not p + 2\not q - m)u(\vec{p})$.

Since the rest of the integrand is $E \leftrightarrow E'$ symmetric, we have $2\not q \to \not q + \not q'$ yielding

$$u_R(\vec{p}\,) = [1 + \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi}\tilde{f}(E)\tilde{f}(E')(\frac{iB}{\not p + \not q + \not q' - m_e}\frac{1}{2}(\not p + \not q + \not q' - m_e)$$

$$+\frac{i}{4}\sigma(p,q,q')\not q)]u(\vec{p}\,). \qquad (1.1.57)$$

So we see that the propagator factor cancels unambiguously now in the secondterm while as before the remainder vanishes as $f \to 1$. Hence using the normalization $f(0) = 1$ this becomes $u_R(\vec{p}\,) = [1 + \frac{1}{2}iB]u(\vec{p}\,)$, which is independent of f which we now let go to 1. So as advertised we find that the external lines are renormalized by a factor of $Z^{\frac{1}{2}}$, $u_R(\vec{p}\,) = Z^{\frac{1}{2}}u(\vec{p}\,)$. Hence, as seen by the bare perturbation theory the infinite wavefunction

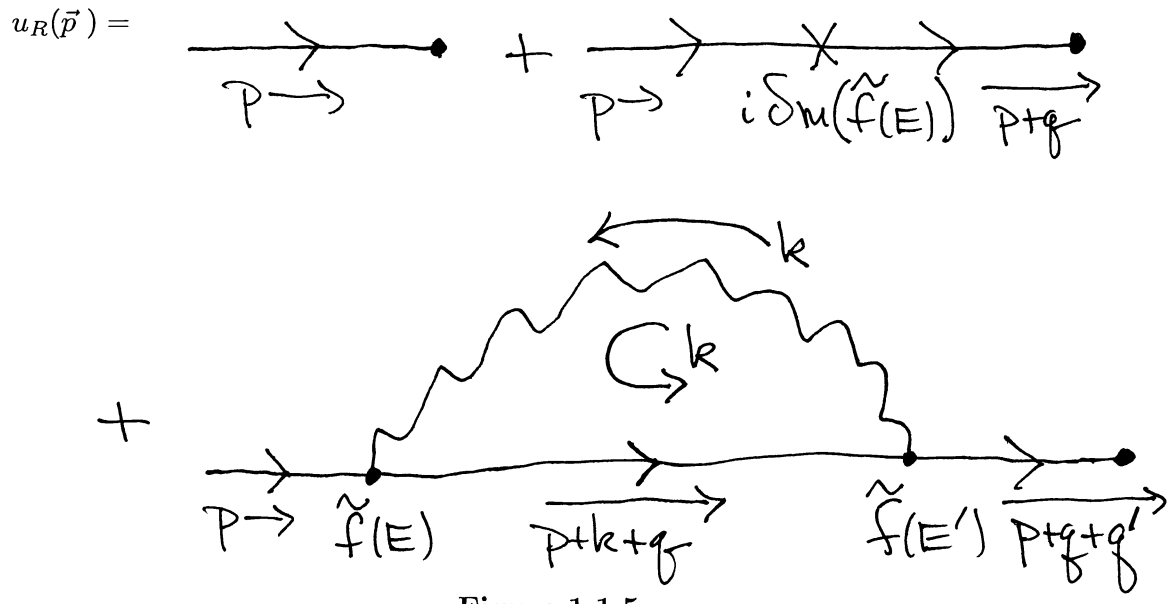


Figure 1.1.5

renormalization factors cancel between the internal and external lines and the definition of the physical electric charge. The S-matrix elements are finite expressions in terms of the physical mass m_e and physical electric charge e_R.

To repeat, according to the second method of renormalized perturbation theory by using the renormalized fields, mass, charge and their counter-term Lagrangian as well as the single particle wavefunctions (which are solutions of $(\not{p} - m_e)u(\vec{p}\,) = 0$ with m_e the physical mass), the S-matrix elements can be rendered finite and dependent only on the renormalized mass m_e and coupling constant e_R.

Before concluding the introductory remarks let us briefly recall that similar statements can be made about the photon self-energy and its wavefunction renormalization as well as those of the positron. In addition there is a generally independent charge renormalization due to the quantum radiative corrections to the vertex function defining the strength ofthe coupling constant given by the Feynman graphs in Figure 1.1.6, for example, in QED.

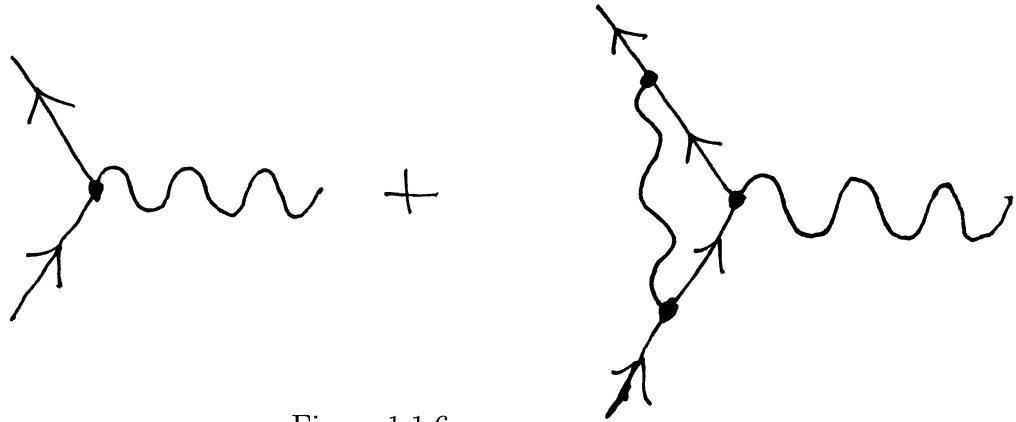


Figure 1.1.6

Then the finite S-matrix elements are obtained from a renormalized perturbation expansion in which we use the renormalized fields (letting $Z = Z_2$ here)

$$\Psi_R = Z_2^{-\frac{1}{2}} \Psi$$

$$A_R^\mu = Z_3^{-\frac{1}{2}} A^\mu, \tag{1.1.58}$$

the renormalized physical mass and charge

$$m_e = m + \delta m$$

$$e_R = Z_1^{-1} Z_2 Z_3^{\frac{1}{2}} e. \tag{1.1.59}$$

(At the same time if the bare field theory approach is used it must be checked that the initial and final states' full single particle wavefunctions are the renormalized ones

$$u_R(\vec{p}\,) = Z_2^{\frac{1}{2}} u(\vec{p}\,) \quad , \quad \bar{u}_R(\vec{p}\,) = Z_2^{\frac{1}{2}} \bar{u}(\vec{p}\,)$$

$$v_R(\vec{p}\,) = Z_2^{\frac{1}{2}} v(\vec{p}\,) \quad , \quad \bar{v}_R(\vec{p}\,) = Z_2^{\frac{1}{2}} \bar{v}(\vec{p}\,)$$

$$\epsilon_R^\mu(k,\lambda) = Z_3^{\frac{1}{2}} \epsilon^\mu(k,\lambda). \tag{1.1.60}$$

In general we will use the second counter-term method.) The QED Lagrangian then contains additional interaction terms due to this renormalization

$$\mathcal{L} = -\frac{Z_3}{4} F_{R\mu\nu} F_R^{\mu\nu} - \frac{Z_3}{2\alpha} (\partial_\lambda A_R^\lambda)^2 + Z_2 \bar{\Psi}_R (i\partial\!\!\!/ - m_e) \Psi_R - Z_1 e_R \bar{\Psi}_R \gamma_\mu \Psi_R A_R^\mu + Z_2 \delta m \bar{\Psi}_R \Psi_R. \tag{1.1.61}$$

The unperturbed Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4} F_{R\mu\nu} F_R^{\mu\nu} - \frac{1}{2\alpha} (\partial_\lambda A^\lambda)^2 + \bar{\Psi}_R (i\partial\!\!\!/ - m_e) \Psi_R, \tag{1.1.62}$$

while the interaction Lagrangian is just

$$\begin{aligned} \mathcal{L}_I &= \mathcal{L} - \mathcal{L}_0 \\ &= -\frac{(Z_3 - 1)}{4} F_{R\mu\nu} F_R^{\mu\nu} - \frac{(Z_3 - 1)}{2\alpha} (\partial_\lambda A^\lambda)^2 + (Z_2 - 1) \bar{\Psi}_R (i\partial\!\!\!/ - m_e) \Psi_R \\ &\quad + Z_2 \delta m \bar{\Psi}_R \Psi_R - Z_1 e_R \bar{\Psi}_R \gamma_\mu \Psi_R A_R^\mu. \end{aligned} \tag{1.1.63}$$

The mass shift δm is defined so that the full propagator has a pole at the physical mass m_e,

$$S_F'^{-1}(p)\big|_{p\!\!\!/ = m_e} = 0. \tag{1.1.64}$$

The wavefunction renormalization factors are defined so that the full propagators' residues are i,

$$(p\!\!\!/ - m_e) S_F'(p)\big|_{p\!\!\!/ = m_e} = i$$

$$(p^2 g^{\mu\nu} - p^\mu p^\nu)\Delta'_{F\ \nu\mu}(p)|_{p^2=0} = -3i. \tag{1.1.65}$$

In a similar manner, the charge renormalization factor Z_1 is defined so that the physical charge is e_R. These results have been motivated by studing the second order electron self-energy. That the procedure works in a consistent manner to arbitrary order in perturbation theory is the subject of renormalization theory.

The point of all this is that the above presentation is quite complex and it is unclear exactly what are the physical assumptions of the field theory and what are the necessities of the calculational scheme in which we are working. We would like to separated these issues to see how far we might be able to go without appealing to an approximation scheme. Although in the final analysis the non-linear field equations can only be solved by means of approximation techniques, it is just a question as to when to apply them. When we want a phenomenological prediction, that is a number, we will resort to some sort of calculational approximation.

To conclude the introduction we point out that pivotal in our analysis of the S-matrix has been the use of the dynamics of the field operators. Since at best this is only analyzed using approximation techniques it will be useful to determine how far we might be able to study a field theory without resorting to the details of the dynamics. Hence a more abstract approach will be needed to persue this line of investigation. Since the separation of the Hamiltonian into free part and a perturbation is arbitrary and, as we have seen,needs to be renormalized, it first of all seems necessary for our analysis to take place in the Heisenberg or Schrödinger pictureswhere the dynamics is governed by the whole Hamiltonian. The field operators depend on space, hence in order to maintain manifest Lorentz invariance more easily the Heisenberg picture will be chosen. The field operators then depend on space-time. Moreover, it has been shown that for systems with an infinite number of degrees of freedom the interaction representation does not have the properties we have demanded of it. That is Haag's Theorem states that the only fields unitarily related to free field operators are other free fields. Thus the operator $U(t, t_0)$ used to relate the interaction picture to the Heisenberg picture is not a proper unitary operator. In addition it has been shown by Van Hove in the framework of a specific model that the eigenstates of the Hamiltonian H are orthogonal to the eigenstates of the unperturbed Hamiltonian H_0. Since our perturbation theory analysis relies on expanding one set of eigenstates in terms of another, the procedure is on shaky grounds. Van Hove went on to show that in fact the eigenstates for a particular value of the coupling constant were orthogonal to the eigenstates for any other value of the coupling constant, no matter how close. Hence, the whole idea of the adiabatic switching on and off of the interaction throws the system into (a continuum of)completely orthogonal vector spaces. Hence $U(t, t_0)$ relates operators in orthogonal spaces, it is an improper unitary operator, its matrix elementsin any one space are zero. Fortunately the physical predictions of the perturbative calculations to avoid probing these foundation questions.

Even if these formal difficulties were not present, we have a limited number of systems we can analyze with our methodology. Utilizing the adiabatic hypothesis to construct the initial and final states of the system excludes Hamiltonians that have bound states in their spectrum. For if we imagine that the interactions all vanish at early and late times, there is nothing to bind the particles together. So a more general formalism that can include

bound states is needed. Along these same lines the perturbative analysis excludes strongly interacting systems since the higher order terms in the series are as important as the low order terms. As well there are forces so strong that the fundamental fields cannot give rise to asymptotic particle states directly but only as indivisible composite particles. The quark and gluon fields of quantum chromodynamics (QCD) form colorless bound states that are observable in nature as protons, neutrons, pions and the like. In our present formulation it is not even clear what quantities to calculate to observe such a particle spectrum.

The abstract approach to field theory shows us how to extract matrix elements of observables from the vacuum expectation values of time ordered products of fields without a knowledge of the dynamics of the field theory. We will then independently develop general calculational techniques for these time ordered functions based on field dynamics. These more general functional methods will encompase the perturbative results outlined above as well as provide a means to begin to answer the more complex dynamical problems of strongly interacting theories. Throughout the principles upon which quantum field theory are based will be clearly separated from the assumptions of particular calculational schemes. Towards this goal we will first re-express our formal perturbative scattering theory results in terms of Heisenberg picture fields and states.