§3.3 CROSS-SECTIONS

So far we have derived a procedure for calculating the transition probability amplitudes S_{fi} for any process in QED. Next we would like to relate S_{fi} to experimentally observable quantities in particular cross-sections. We have calculated transition amplitudes for several processes involving definite spin and polarization of the initial and final particles. In general this is not measured but rather the initial beams of particles are spin-averaged while we sum over all final polarizations. We will have to see how to do this also in our cross-section calculations.

To be definitive let's first consider the scattering process between two initial particles, either $e^{\pm}$ or γ, colliding, interacting and resulting in a final state with n final particles. The initial particles' energy-momentum is $p_1 = (E_1, \vec{p}_1)$, $p_2 = (E_2, \vec{p}_2)$ and is labelled p_i, for $i = 1, 2$. Similarly, the final particles' energy and momentum are labelled $p'_f = (E'_f, \vec{p}_f{}')$ for $f = 1, \ldots, n$. The initial and final particles are in definite spin and polarization states also. Recall that the S-matrix element always has the form

$$S_{fi} = \delta_{fi} + (2\pi)^4\, \delta^4(P_i - P_f)\, \mathcal{M}_{fi}, \tag{3.3.1}$$

where δ_{fi} represents the normalization of the free incoming and outgoing states. Since we are interested in the transition probability we consider $|S_{fi}|^2$ for $i \neq f$. Notice however that we will have $[\delta^4(P_i - P_f)]^2 = \delta^4(0)\delta^4(P_i - P_f)$, a meaningless expression. The reason is twofold; first, we should initially consider transition amplitudes occurring for a finite time $\pm T$, square it for the probability, and then let $T \longrightarrow \infty$, as described in the adiabatic hypothesis discussion. Hence

$$2\pi\delta(E_i - E_f) = \int_{-\infty}^{+\infty} e^{-i(E_i - E_f)t'}\, dt' \tag{3.3.2}$$

should be replaced by

$$\begin{aligned} 2\pi\delta(E_i - E_f) &= \lim_{T \to \infty} \int_{-T}^{+T} e^{-i(E_i - E_f)t'}\, dt' \\ &= \lim_{T \to \infty} \left\{ 2i \frac{\sin\left[(E_i - E_f)T\right]}{i(E_i - E_f)} \right\} \end{aligned} \tag{3.3.3}$$

to give the transition amplitude for time $2T$.

However, in any process, decay or scattering, we are not interested in the total transition probability but in the transition probability per unit time, or transition rate.

Since for finite T, $|S_{fi}|^2$ $(i \neq f)$ gives the transition probability during time $2T$ we have that the transition rate w_{fi} is, for $i \neq f$,

$$w_{fi} = \lim_{T \to \infty} \frac{|S_{fi}|^2}{2T}. \tag{3.3.4}$$

Now in calculating $|S_{fi}|^2$ the expression $|2\pi\delta(E_i - E_f)|^2$ is replaced, for finite time, by

$$I(E_i - E_f) \equiv 4\,\frac{\sin^2\left[(E_i - E_f)T\right]}{(E_i - E_f)^2}. \tag{3.3.5}$$

But recall that

$$\lim_{T \to \infty} \int \frac{I(E)}{2T} f(E)\,dE = \lim_{T \to \infty} \frac{1}{2T}\left[f(0)\int_{-\infty}^{+\infty} I(E)\,dE + \mathcal{O}(1)\right] \tag{3.3.6}$$

This last term, $\lim_{T\to\infty} \frac{1}{2T}\mathcal{O}(1)$, remains finite as $T \to \infty$, since $I(E)$ is sharply peaked about $E = 0$.

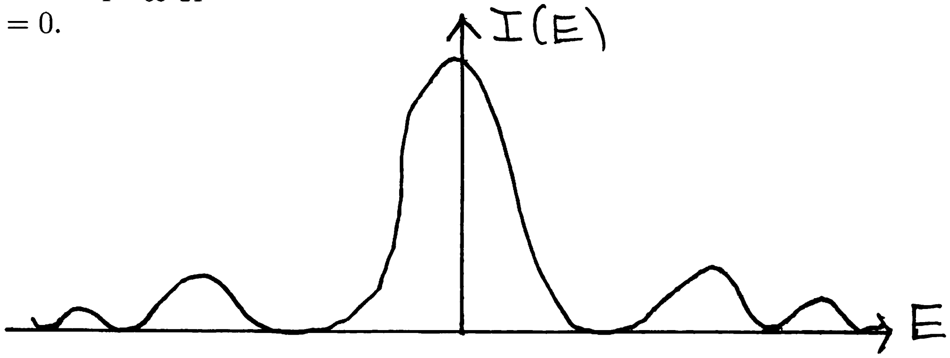

Figure 3.3.1

But

$$\int_{-\infty}^{+\infty} I(E)\,dE = 2\pi(2T), \tag{3.3.7}$$

hence

$$\lim_{T \to \infty} \int \frac{I(E)}{2T} f(E)\,dE = 2\pi f(0). \tag{3.3.8}$$

Thus

$$\frac{I(E)}{2T} = 2\pi\delta(E) \tag{3.3.9}$$

in the limiting sense. This yields

$$\begin{aligned} w_{fi} &= \lim_{T \to \infty} \frac{|S_{fi}|^2}{2T} \\ &= 2\pi\delta(E_i - E_f)\left|(2\pi)^3\delta^3(\vec{P}_i - \vec{P}_f)\right|^2 \left|\mathcal{M}_{fi}\right|^2. \end{aligned} \tag{3.3.10}$$

That is, more directly,

$$|2\pi\delta(E_i - E_f)|^2 = 2\pi\delta(0)(2\pi)\delta(E_i - E_f) \tag{3.3.11}$$

but

$$\begin{aligned} 2\pi\delta(0) &= \lim_{T \to \infty} \lim_{E_f \to E_i} \int_{-T}^{+T} e^{-i(E_i - E_f)t}\, dt \\ &= \lim_{T \to \infty} 2T. \end{aligned} \tag{3.3.12}$$

Since $2T$ is the time during which the transition has taken place, $|S_{fi}|^2$ divided by $2T$ is the transition rate.

In a similar manner we have the spatial momentum delta function squared, which is also nonsense. We should be using momentum wave packet states rather than plane wave states. Then at the end of the calculation, i.e. after squaring S_{fi} and calculating a density, we may take the plane wave limit. More directly we could place our particles in a finite volume or box ($T << V^{\frac{1}{3}}$ to avoid reflections). Then

$$\begin{aligned} \left|(2\pi)^3\, \delta^3(\vec{P}_i - \vec{P}_f)\right|^2 &= (2\pi)^3\, \delta^3(\vec{P}_i - \vec{P}_f) \lim_{V \to \infty} \int_V d^3x \left(e^{-i(\vec{P}_i - \vec{P}_f)\cdot\vec{x}}\right) \\ &= (2\pi)^3\, \delta^3(\vec{P}_i - \vec{P}_f) \lim_{V \to \infty} V. \end{aligned} \tag{3.3.13}$$

We can then define the transition rate per unit volume or transition probability per unit space-time

$$\overline{w}_{fi} = \lim_{V \to \infty} \frac{w_{fi}}{V} = (2\pi)^4\, \delta^4(P_i - P_f)\, |\mathcal{M}_{fi}|^2 \tag{3.3.14}$$

where $\overline{w}_{fi}$ is the transition rate density to one definite final state. To obtain the transition rate to states with momenta differentially close to $\vec{p}_f^{\,\prime}$ for $f = 1, \ldots n$, we must multiply $\overline{w}_{fi}$ by the number of states in the volume d^3p_f' around $\vec{p}_f^{\,\prime}$, $f = 1, \ldots, n$.

Since we normalized our states to the continuum, we have that the density of states is

$$\langle \vec{p}\, | \vec{q}\, \rangle = (2\pi)^3 2\omega_p\, \delta^3(\vec{p} - \vec{q}\,) \tag{3.3.15}$$

and so

$$\int \frac{d^3p}{(2\pi)^3 2\omega_p} |\vec{p}\,\rangle\langle\vec{p}\,| = 1. \tag{3.3.16}$$

Hence there are $\frac{d^3p'_f}{(2\pi)^3 2\omega_{p'_f}}$ states differeentially close to $\vec{p}_f{}'$, and the transition rate per unit volume into all states differentially close to $\vec{p}_f{}'$ is thus

$$\overline{w}_{fi} \prod_{f=1}^{n} \frac{d^3p'_f}{(2\pi)^3 2\omega_{p'_f}}. \tag{3.3.17}$$

Returning to our case of two incoming particles and n-outgoing particles, the differential cross-section $d\sigma_{fi}$ is defined in the laboratory frame as the transition rate density per target density per incident flux

$$d\sigma_{fi} = \frac{\overline{w}_{fi}}{n_t\, F_I} \prod_{f=1}^{n} \frac{d^3p'_f}{(2\pi)^3 2\omega_{p'_f}}, \tag{3.3.18}$$

where

$$\begin{aligned} n_t &\equiv \text{target density} \\ &= \text{number of scattering centers per volume} \\ \text{and} \quad F_I &\equiv \text{incident flux} \\ &= \text{rate of incoming particles per area.} \end{aligned}$$

Now if we take particle 2 as the target particle we have that, due to the momentum normalization,

$$\langle\vec{p}\,|\vec{q}\,\rangle = (2\pi)^3 2\omega_p \delta^3(\vec{p} - \vec{q}\,), \tag{3.3.19}$$

that the $\vec{x}$-space normalization is found from the $\vec{x}$-wavefunction

$$\langle\vec{x}\,|\vec{p}\,\rangle = [2\omega_p]^{\frac{1}{2}} e^{i\vec{p}\cdot\vec{x}}. \tag{3.3.20}$$

So the probabiltiy of finding a particle per unit volume is $[2\omega_p]$, and the target density is then

$$n_t = (2\omega_{p_2}). \tag{3.3.21}$$

Similarly the incident particle flux is just the incident particle density, also $2\omega_{p_1}$, times the relative velocity of the two particles in the lab frame $v_{\rm rel}$. Thus the incident flux is

$$F_I = 2\omega_{p_1} v_{\rm rel}. \tag{3.3.22}$$

Putting this together we find

$$
\begin{aligned}
d\sigma_{fi} &\equiv \frac{\overline{w}_{fi}}{n_t\, F_I} \prod_{f=1}^{n} \frac{d^3 p'_f}{(2\pi)^3 2\omega_{p'_f}} \\
&= \frac{1}{2\omega_{p_1} 2\omega_{p_2} v_{\mathrm{rel}}} \prod_{f=1}^{n} \frac{d^3 p'_f}{(2\pi)^3 2\omega_{p'_f}} (2\pi)^4 \delta^4(P_i - P_f)\, |\mathcal{M}_{fi}|^2
\end{aligned}
. \tag{3.3.23}
$$

We can write the expression $\omega_{p_1}\omega_{p_2}v_{\mathrm{rel}}$ in a more Lorentz invariant way. (Notice that all frames in which the particles are moving collinearly will yield the above expression.)

$$
\begin{aligned}
\omega_{p_1}\omega_{p_2}v_{\mathrm{rel}} &= \omega_{p_1}\omega_{p_2}\left|\frac{\vec{p}_1}{\omega_{p_1}} - \frac{\vec{p}_2}{\omega_{p_2}}\right| \\
&= \left[\omega_{p_1}^2\omega_{p_2}^2\left(\frac{\vec{p}_1^{\,2}}{\omega_{p_1}^2} + \frac{\vec{p}_2^{\,2}}{\omega_{p_2}^2} - \frac{2\vec{p}_1\cdot\vec{p}_2}{\omega_{p_1}\omega_{p_2}}\right)\right]^{\frac{1}{2}} \\
&= \left[\omega_{p_2}^2\vec{p}_1^{\,2} + \omega_{p_1}^2\vec{p}_2^{\,2} - 2\omega_{p_1}\vec{p}_1\cdot\omega_{p_2}\vec{p}_2\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.3.24}
$$

But recall

$$
\begin{aligned}
(p_1p_2)^2 &= (\omega_{p_1}\omega_{p_2} - \vec{p}_1\cdot\vec{p}_2)^2 \\
&= (\omega_{p_1}\omega_{p_2})^2 + (\vec{p}_1\cdot\vec{p}_2)^2 - 2\omega_{p_1}\vec{p}_1\cdot\omega_{p_2}\vec{p}_2.
\end{aligned} \tag{3.3.25}
$$

So we find

$$
\begin{aligned}
\omega_{p_1}\omega_{p_2}v_{\mathrm{rel}} &= \left[(p_1p_2)^2 - \omega_{p_1}^2\omega_{p_2}^2 - (\vec{p}_1\cdot\vec{p}_2)^2 + \omega_{p_2}^2\vec{p}_1^{\,2} + \omega_{p_1}^2\vec{p}_2^{\,2}\right]^{\frac{1}{2}} \\
&= \Big[(p_1p_2)^2 - m_1^2m_2^2 - \vec{p}_1^{\,2}\vec{p}_2^{\,2} - m_1^2\vec{p}_2^{\,2} - m_2^2\vec{p}_1^{\,2} \\
&\quad - (\vec{p}_1\cdot\vec{p}_2)^2 + \vec{p}_1^{\,2}\vec{p}_2^{\,2} + m_1^2\vec{p}_2^{\,2} + m_2^2\vec{p}_1^{\,2} + \vec{p}_1^{\,2}\vec{p}_2^{\,2}\Big]^{\frac{1}{2}} \\
&= \left[(p_1p_2)^2 - m_1^2m_2^2 + \vec{p}_1^{\,2}\vec{p}_2^{\,2} - (\vec{p}_1\cdot\vec{p}_2)^2\right]^{\frac{1}{2}}
\end{aligned} \tag{3.3.26}
$$

Now if the frame is such that the colliding particles momenta is co-linear or one particle is at rest we have

$$
\vec{p}_1 \times \vec{p}_2 = 0 \tag{3.3.27}
$$

and thus either $\vec{p}_2 = 0$ or

$$
\vec{p}_2 = \alpha\vec{p}_1, \tag{3.3.28}
$$

then the last two terms cancel to yield

$$\omega_{p_1}\omega_{p_2}v_{\rm rel} = \Big[(p_1p_2)^2 - m_1^2m_2^2\Big]^{\frac{1}{2}} \tag{3.3.29}$$

in all colinear frames.
So the lab frame cross-section takes the form

$$d\sigma_{fi} = \frac{(2\pi)^4\,\delta^4(p_1+p_2-\sum_f p'_f)}{4\Big[(p_1p_2)^2 - m_1^2m_2^2\Big]^{\frac{1}{2}}}\Big|\mathcal{M}_{fi}\Big|^2\prod_{f=1}^{n}\Big(\frac{d^3p'_f}{(2\pi)^3 2\omega_{p'_f}}\Big), \tag{3.3.30}$$

which is our final Lorentz invariant definition for the cross–section.

Two important colinear frames are:

1.) The laboratory frame; $\vec{p}_2 = 0$ (massive particle)

$$\omega_{p_1}\omega_{p_2}v_{\rm rel} = \omega_{p_1}m_2\frac{|\vec{p}_1|}{\omega_{p_1}} = m_2|\vec{p}_1| \qquad \text{(Lab)} \tag{3.3.31}$$

2.) The center of mass frame; $\vec{p}_1 = -\vec{p}_2$, $v_{\rm rel} = \frac{|\vec{p}_1|}{\omega_{p_1}} + \frac{|\vec{p}_2|}{\omega_{p_2}}$ that is $v_{\rm rel} = |\vec{p}_1|\frac{E_1+E_2}{E_1E_2} = |\vec{p}_1|\frac{\omega_{p_1}+\omega_{p_2}}{\omega_{p_1}\omega_{p_2}}$, so

$$\begin{aligned}\omega_{p_1}\omega_{p_2}v_{\rm rel} &= \omega_{p_1}\omega_{p_2}\Big\{\frac{|\vec{p}_1|}{\omega_{p_1}} + \frac{|\vec{p}_2|}{\omega_{p_2}}\Big\}\\ &= \omega_{p_1}\omega_{p_2}\Big\{|\vec{p}_1|\frac{E_1+E_2}{E_1E_2}\Big\}\\ &= |\vec{p}_1|(\omega_{p_1}+\omega_{p_2}) \qquad \text{(COM).}\end{aligned} \tag{3.3.32}$$

The differential cross-section $d\sigma_{fi}$ given above is for transitions from initial states with specified momenta and spins and polarizations to final states with specified spins and polarizations and with momenta differentially close to a specified value. In an experiment usually we will not polarize the spins of the incoming particles but just average over them, and likewise we in general do not detect the spin states the final particles are in but rather sum over all spin states. Of course some experiments do study polarization effects but by and large they do not.

To make clear all of the above discussion let's apply our definitions to calculate the cross-sections in Compton scattering.

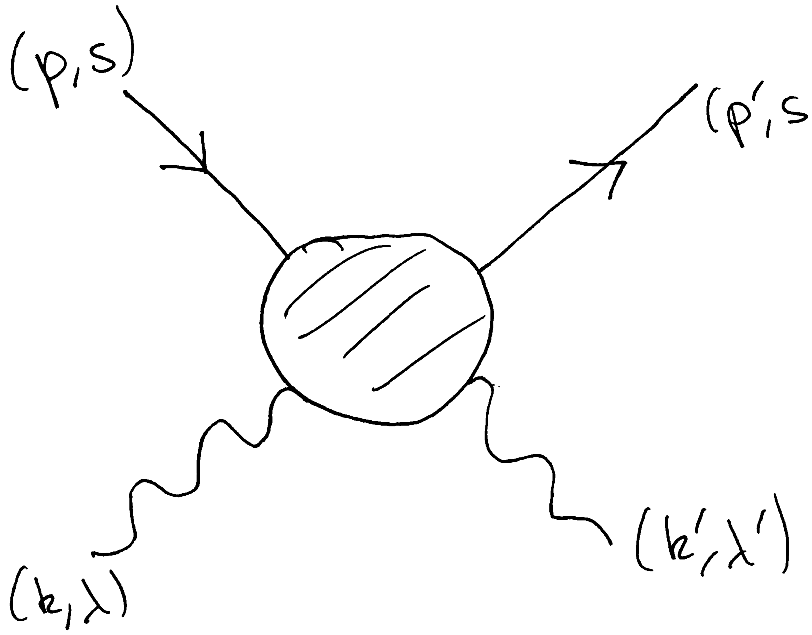

Figure 3.3.2

First for totally specified initial states, an initial photon of momentum $\vec{k}$, polarization $\epsilon_\mu(k,\lambda)$, and an electron with momentum $\vec{p}$, spin $\frac{(-1)^{s+1}}{2}$ scatter to produce final electrons with spin $\frac{(-1)^{s'+1}}{2}$ and photons with polarization $\epsilon_\mu(k',\lambda')$ with momenta differentially close to $\vec{p}'$ for the e^- and $\vec{k}'$ for the γ.

The differential cross-section for scattering is

$$d\sigma(e^-\gamma \to e^-\gamma) = \frac{(2\pi)^4\,\delta^4(p+k-p'-k')}{4p\cdot k}\left|\mathcal{M}_{fi}\right|^2 \frac{d^3p'}{(2\pi)^3\,2\omega_{p'}}\,\frac{d^3k'}{(2\pi)^3\,2\omega_{k'}} \tag{3.3.33}$$

where $\mathcal{M}_{fi}$ was calculated previously in second order perturbation theory according to our Feynman rules and is given by equation (3.2.50)

$$\begin{aligned}&\mathcal{M}(e^-\gamma \to e^-\gamma)\\ &\quad= (-ie)^2\Big[\overline{u}^{(s')}(\vec{p}')\,\not\epsilon(k',\lambda')\,\frac{i}{(\not p+\not k)-m}\,\not\epsilon(k,\lambda)\,u^{(s)}(\vec{p})\\ &\qquad+\overline{u}^{(s')}(\vec{p}')\,\not\epsilon(k,\lambda)\,\frac{i}{(\not p-\not k')-m}\,\not\epsilon(k',\lambda')\,u^{(s)}(\vec{p})\Big].\end{aligned} \tag{3.3.34}$$

Since we have overall energy-momentum conservation, all the final momenta are not independent. (If we take $\vec{k}$ as our polar axis then (k',θ,φ) are the spherical coordinates of $\vec{k}\,'$ with $\vec{k}$ as the polar axis.)

Since there are four delta functions and six integrals, we can do all but two integrals. We will choose these two integrals to be the (θ,φ) polar angles of the scattered photon relative to the incident photon.

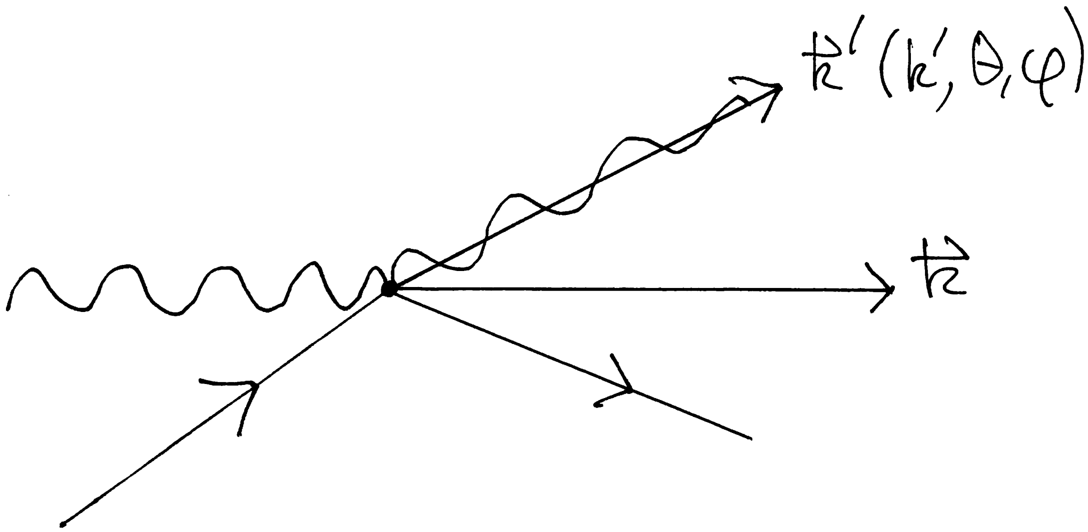

Figure 3.3.3

Thus we will find the cross-section for scattering into electron states of all momenta and photon states of all magnitudes of momenta that are in the solid angle $d\Omega$ centered about (θ, φ).
That is,

$$d^3k' = k'^2\,dk'\,\sin\theta\,d\theta\,d\varphi = k'^2\,dk'\,d\Omega, \tag{3.3.35}$$

so that the cross-section becomes

$$\frac{d\sigma\,(e^-\gamma \to e^-\gamma)}{d\Omega}$$

$$\equiv \int_{-\infty}^{+\infty} \frac{d^3p'}{(2\pi)^3\,2\omega_{p'}} \int_0^{+\infty} \frac{k'^2\,dk'}{(2\pi)^3\,2\omega_{k'}} \frac{(2\pi)^4\,\delta^4(p+k-p'-k')}{4p\cdot k} \left|\mathcal{M}_{fi}\right|^2. \tag{3.3.36}$$

The integral over $\vec{p}'$ can be done immediately by just setting
$\vec{p}' = \vec{k} + \vec{p} - \vec{k}'$ everywhere.
So

$$\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega} = \int_0^{+\infty} \frac{k'^2\,dk'}{(2\pi)^3\,2\omega_{k'}} \frac{(2\pi)\;\delta\;(\omega_p + \omega_k - \omega_{p'} - \omega_{k'})}{2\omega_{p'}\,4p\cdot k} \left|\mathcal{M}_{fi}\right|^2 \tag{3.3.37}$$

where $\vec{p}' = \vec{k} + \vec{p} - \vec{k}'$ is understood.
Now,

$$\omega_{k'} = \sqrt{(\vec{k}\ ')^2 + m_\gamma^2} = k' \tag{3.3.38}$$

since $m_\gamma = 0$.
Unfortunately

$$\omega_{p'} = \sqrt{(\vec{p}')^2 + m^2} = \left((\vec{k} + \vec{p} - \vec{k}')^2 + m^2\right)^{\frac{1}{2}} \tag{3.3.39}$$

depends on k' so we cannot do the integral simply. Using

$$\int_{-\infty}^{+\infty} d^4k'\, \delta\big(f(k')\big)\, g(k') = \frac{g(k'_0)}{\left|\frac{\partial f}{\partial k'_0}\right|}\Bigg|_{f(k'_0)=0}, \tag{3.3.40}$$

we find

$$\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega} = \frac{|\vec{k}'|^2\, |\mathcal{M}_{fi}|^2}{(2\pi)^2\, 2\omega_{k'}\, 2\omega_{p'}\, 4p\cdot k}\left[\frac{\partial(\omega_{p'} + \omega_{k'})}{\partial|\vec{k}'|}\right]^{-1} \tag{3.3.41}$$

where $\vec{p}' = \vec{k} + \vec{p} - \vec{k}'$ and $\omega_{k'} = \omega_p + \omega_k - \omega_{p'}$. That is, $p_\mu{}' = k_\mu + p_\mu - k_\mu{}'$ and $\frac{\partial}{\partial|\vec{k}'|}$ is taken with θ and φ held fixed. More specifically, most experiments are performed in a frame where the electrons are at rest. Thus the lab frame has $p = (m, 0, 0, 0)$ and hence $\vec{p}' = \vec{k} - \vec{k}'$. So in this frame

$$\begin{aligned}\omega_{p'} &= \Big[m^2 + (\vec{k} - \vec{k}\ ')^2\Big]^{\frac{1}{2}} \\ &= \Big[m^2 + (\vec{k})^2 + (\vec{k}\ ')^2 - 2\vec{k}\cdot\vec{k}'\Big]^{\frac{1}{2}}.\end{aligned} \tag{3.3.42}$$

But

$$\vec{k}\cdot\vec{k}' = |\vec{k}|\,|\vec{k}'|\,\cos\theta = \omega_k\omega_{k'}\cos\theta, \tag{3.3.43}$$

so

$$\omega_{p'} = [m^2 + \omega_k{}^2 + \omega_{k'}{}^2 - 2\omega_k\omega_{k'}\cos\theta]^{\frac{1}{2}} \tag{3.3.44}$$

where we recall that $\omega_k = |\vec{k}|$ and $\omega_{k'} = |\vec{k}'|$. So we find

$$\begin{aligned}\frac{\partial(\omega_{p'} + \omega_{k'})}{\partial\omega_{k'}}\Bigg|_{(\theta,\varphi)} &= 1 + \frac{1}{2}\,\frac{1}{\omega_{p'}}\,(2\omega_{k'} - 2\omega_k\cos\theta) \\ &= 1 + \frac{(\omega_{k'} - \omega_k\cos\theta)}{\omega_{p'}} \\ &= \frac{\omega_{p'} + \omega_{k'} - \omega_k\cos\theta}{\omega_{p'}}.\end{aligned} \tag{3.3.45}$$

But $\omega_{p'} + \omega_{k'} = \omega_p + \omega_k = m + \omega_k$, hence

$$\left.\frac{\partial(\omega_{p'} + \omega_{k'})}{\partial \omega_{k'}}\right|_{(\theta,\varphi)} = \frac{m + \omega_k(1 - \cos\theta)}{\omega_{p'}}. \tag{3.3.46}$$

Now we can further use energy-momentum conservation to find $\omega_{k'}$ as a function of θ and ω_k.

$$\begin{aligned} {\omega_{p'}}^2 &= (\omega_p + \omega_k - \omega_{k'})^2 \\ &= (m + \omega_k - \omega_{k'})^2 \\ &= m^2 + {\omega_k}^2 + {\omega_{k'}}^2 + 2m\omega_k - 2m\omega_{k'} - 2\omega_k\omega_{k'}. \end{aligned} \tag{3.3.47}$$

From momentum conservation we had

$${\omega_{p'}}^2 = m^2 + {\omega_k}^2 + {\omega_{k'}}^2 - 2\omega_k\omega_{k'}\cos\theta. \tag{3.3.48}$$

Equating these last two expressions we arrive at the Compton condition:

$$\omega_{k'} = \frac{m\omega_k}{[m + \omega_k(1 - \cos\theta)]}. \tag{3.3.49}$$

So

$$\left.\frac{\partial(\omega_{p'} + \omega_{k'})}{\partial \omega_{k'}}\right|_{(\theta,\varphi)} = \frac{m\omega_k}{\omega_{p'}\omega_{k'}}. \tag{3.3.50}$$

Hence with these results we have that the differential cross-section in the laboratory frame

$$\left(\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}} = \frac{{\omega_{k'}}^2 \left|\mathcal{M}_{fi}\right|^2}{16\,(2\pi)^2\,\omega_{k'}\,\omega_{p'}\,m\omega_k}\left[\frac{\partial(\omega_{p'} + \omega_{k'})}{\partial|\vec{k}'|}\right]^{-1} \tag{3.3.51}$$

becomes

$$\left(\frac{d\sigma\,(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}} = \frac{1}{(8\pi)^2}\left(\frac{\omega_{k'}}{m\omega_k}\right)^2\left|\mathcal{M}_{fi}\right|^2 \tag{3.3.52}$$

with $\omega_{k'} = \frac{m\omega_k}{[m+\omega_k(1-\cos\theta)]}$.

With the kinematics out of the way we must now face the evaluation of $|\mathcal{M}_{fi}|^2$. We consider scattering with unpolarized photons and electrons. Thus we must average over the initial spin and polarization states of the e^- and the γ, and then sum over the final e^- spin and the final γ polarization. Thus the spin and polarization averaged-and-summed cross-section in the lab frame is

$$\left(\frac{d\sigma^{\text{unpol.}}(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}} = \frac{1}{2}\sum_{s=1}^{2}\frac{1}{2}\sum_{\lambda=1}^{2}\sum_{s'=1}^{2}\sum_{\lambda'=1}^{2}\left(\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}}. \quad (3.3.53)$$

This is called the unpolarized cross-section. In it,
$\frac{1}{2}\sum_{s=1}^{2}$ represents the average over the initial electron spin,
$\frac{1}{2}\sum_{\lambda=1}^{2}$ represents the average over the initial photon polarization, and
$\sum_{s'=1}^{2}\sum_{\lambda'=1}^{2}$ is the sum over the final spin and polarization states.

First let's consider the electron spin sums. Note that $\mathcal{M}_{fi}$ has the form

$$\mathcal{M}_{fi} = \overline{u}_a^{(s')}(\vec{p}')\,\Gamma_{ab}u_b^{(s)}(\vec{p}). \quad (3.3.54)$$

Thus

$$|\mathcal{M}_{fi}|^2 = \left(\overline{u}_a^{(s')^*}(\vec{p}')\,\Gamma_{ab}{}^* u_b^{(s)^*}(\vec{p})\right)\left(\overline{u}_c^{(s')}(\vec{p}')\,\Gamma_{cd}u_d^{(s)}(\vec{p})\right). \quad (3.3.55)$$

Defining

$$\tilde{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0 \quad (3.3.56)$$

we have

$$|\mathcal{M}_{fi}|^2 = \left(\overline{u}_d^{(s')}(\vec{p}')\,\Gamma_{da}u_a^{(s)}(\vec{p})\right)\left(\overline{u}_b^{(s)}(\vec{p})\,\tilde{\Gamma}_{bc}u_c^{(s')}(\vec{p}')\right). \quad (3.3.57)$$

Now, for the sum over s and s', recall that

$$\sum_{s=1}^{2} u_a^{(s)}(\vec{p})\,\overline{u}_b^{(s)}(\vec{p}) = (\not{p}+m)_{ab}, \quad (3.3.58)$$

so

$$\begin{aligned}\sum_{s,s'=1}^{2}|\mathcal{M}_{fi}|^2 &= \sum_{s=1}^{2}\sum_{s'=1}^{2} u_c^{(s')}(\vec{p}')\,\overline{u}_d^{(s')}(\vec{p}')\,u_a^{(s)}(\vec{p})\,\overline{u}_b^{(s)}(\vec{p})\,\Gamma_{da}\tilde{\Gamma}_{bc}\\ &= (\not{p}'+m)_{cd}(\not{p}+m)_{ab}\Gamma_{da}\tilde{\Gamma}_{bc}\\ &= (\not{p}'+m)_{cd}\Gamma_{da}(\not{p}+m)_{ab}\tilde{\Gamma}_{bc}\\ &= \text{Tr}\left[(\not{p}'+m)\Gamma(\not{p}+m)\tilde{\Gamma}\right].\end{aligned} \quad (3.3.59)$$

Hence we just have to evaluate the γ-matrix traces when we spin-average.

Further consider the form of $\mathcal{M}_{fi}$'s polarization dependence.

$$\mathcal{M}_{fi} \equiv \epsilon^\mu(k,\lambda)\,\epsilon^\nu(k',\lambda')\,\mathcal{M}_{\mu\nu}(k,k') \quad (3.3.60)$$

So a polarization sum implies

$$\sum_{\lambda,\lambda'=1}^{2} |\mathcal{M}_{fi}|^2 = \sum_{\lambda,\lambda'=1}^{2} \epsilon^{\mu}(k,\lambda)\,\epsilon^{\rho}(k,\lambda)\,\epsilon^{\nu}(k',\lambda')\,\epsilon^{\kappa}(k',\lambda')\,\mathcal{M}_{\mu\nu}\mathcal{M}^{*}_{\rho\kappa}. \tag{3.3.61}$$

Recall the polarization vector completeness property

$$\sum_{\lambda=1}^{2} \epsilon^{\mu}(k,\lambda)\,\epsilon^{\rho}(k,\lambda) = -g^{\mu\rho} - \frac{k^{\mu}k^{\rho}}{(n\cdot k)^2 - k^2} - \frac{n^{\mu}n^{\rho}k^2}{(n\cdot k)^2 - k^2} + \frac{(n\cdot k)(k^{\mu}n^{\rho} + k^{\rho}n^{\mu})}{(n\cdot k)^2 - k^2}. \tag{3.3.62}$$

Now for photons $k^2 = 0$ and $n\cdot k = \omega_k$, so

$$\sum_{\lambda=1}^{2} \epsilon^{\mu}(k,\lambda)\,\epsilon^{\rho}(k,\lambda) = -g^{\mu\rho} - \frac{k^{\mu}k^{\rho}}{{\omega_k}^2} + \frac{(k^{\mu}g^{0\rho} + k^{\rho}g^{0\mu})}{{\omega_k}^2}. \tag{3.3.63}$$

Now we can further simplify this by using the gauge invariance property of the theory.

Since the theory is gauge invariant we should obtain the same physical results whether we use the fields $A_\mu(x)$ and $\Psi(x)$, or the fields $A_\mu + \partial_\mu\Lambda(x)$ and $e^{-ie\Lambda(x)}\Psi(x)$. In particular, all observables are gauge invariant if their operators commute with gauge transformations $U(\Lambda)$. So S-matrix elements are gauge invariant if

$$\subset f|U^{-1}(\Lambda)\,S\,U(\Lambda)|i\supset = \subset f|S|i\supset . \tag{3.3.64}$$

That is, if $|i'\supset = U(\Lambda)|i\supset$ and $|f'\supset = U(\Lambda)|f\supset$ then since $US = SU$ we have

$$\begin{aligned} {S_{fi}}' &= \subset f'|S|i'\supset = \subset f|U^{-1}(\Lambda)\,S\,U(\Lambda)|i\supset \\ &= \subset f|S|i\supset = S_{fi}. \end{aligned} \tag{3.3.65}$$

Now

$$U^{-1}(\Lambda)\,S\,U(\Lambda) = \frac{\mathrm{T}U^{-1}(\Lambda)\,e^{-i\int d^4x\, N[\mathcal{H}_I^{IP}](x)}\,U(\Lambda)}{\langle 0|\mathrm{T}e^{-i\int d^4x\, N[\mathcal{H}_I^{IP}](x)}|0\rangle}. \tag{3.3.66}$$

But

$$\mathcal{H}_I^{IP} = eA_\mu^{IP}\overline{\Psi}^{IP}\gamma^{\mu}\Psi^{IP}, \tag{3.3.67}$$

so

$$U^{-1}(\Lambda)\,\mathcal{H}_I^{IP}\,U(\Lambda) = e\Big[\big(A_\mu + \partial_\mu\Lambda\big)\overline{\Psi}\gamma^{\mu}\Psi\Big](x). \tag{3.3.68}$$

Thus

$$S_{fi}{}' = \frac{\subset f|\mathrm{T}e^{-i\int d^4x\, N[\mathcal{H}_I^{IP}](x)}\, e^{-i\int d^4x\, \partial_\mu \Lambda N[\overline{\Psi}\gamma^\mu\Psi]}|i\supset}{\langle 0|\mathrm{T}e^{-i\int d^4x\, N[\mathcal{H}_I^{IP}](x)}|0\rangle}. \tag{3.3.69}$$

For infinitesimal Λ we find

$$S_{fi}{}' = S_{fi} - ie\int d^4x\, \partial_\mu \Lambda(x)\, \frac{\subset f|\mathrm{T}N[\overline{\Psi}\gamma^\mu\Psi(x)]\, e^{-i\int d^4y\, N[\mathcal{H}_I^{IP}](y)}\,|i\supset}{\langle 0|\mathrm{T}e^{-i\int d^4x\, N[\mathcal{H}_I^{IP}](x)}|0\rangle}. \tag{3.3.70}$$

Thus if

$$\int d^4x\, \partial_\mu \Lambda(x)\, \subset f|\mathrm{T}N[\overline{\Psi}\gamma^\mu\Psi(x)]\, e^{-i\int d^4y\, N[\mathcal{H}_I^{IP}](y)}\,|i\supset = 0, \tag{3.3.71}$$

the S-matrix is gauge invariant. Or, stated otherwise, since the S-matrix is gauge invariant we have that the above expression is zero. Now since $\Lambda(x)$ is arbitrary, this implies that

$$\partial_\mu^x\, \subset f|\mathrm{T}J^\mu(x)\, e^{-i\int d^4y\, \mathrm{N}\left[\mathcal{H}_I^{IP}(y)\right]}\,|i\supset = 0 \tag{3.3.72}$$

where

$$J^\mu(x) = \mathrm{N}\left[\overline{\Psi}\gamma^\mu\Psi\right](x). \tag{3.3.73}$$

Now when we have a matrix element involving a photon in the state $|i,(k,\lambda)\supset$, for scattering this photon always attaches to a vertex

$$\begin{aligned} \subset f|S|i,(k,\lambda)\supset &= (2\pi)^4\, \delta^4(P_i + k - P_f)\, \mathcal{M}_{f\ i+(k,\lambda)} \\ &= -ie\epsilon^\mu(k,\lambda)\, \subset f|\mathrm{T}\tilde{J}^\mu(k)\, e^{-i\int d^4y\, N[\mathcal{H}_I^{IP}](y)}\,|i\supset . \end{aligned} \tag{3.3.74}$$

Now gauge invariance implies that

$$k_\mu \subset f|\mathrm{T}\tilde{J}^\mu(x)\, e^{-i\int d^4y\, N[\mathcal{H}_I^{IP}](y)}\,|i\supset = 0. \tag{3.3.75}$$

Thus if we change the classical photon wave function

$$A^\mu(x) = \mathrm{const.}\, \epsilon^\mu(k,\lambda)e^{-ikx} \tag{3.3.76}$$

by a gauge transformation $A^\mu \longrightarrow A^\mu + \partial^\mu\Lambda$ where $\Lambda(x) = \tilde{\Lambda}(k)\, e^{-ikx}$ we have that

$$\epsilon^\mu(k,\lambda) \longrightarrow \epsilon^\mu(k,\lambda) - ik^\mu\tilde{\Lambda}(k). \tag{3.3.77}$$

The S-matrix is invariant since $k_\mu \subset f|\mathrm{T}\tilde{J}^\mu(x)\, e^{-i\int d^4y\, N[\mathcal{H}_I^{IP}](y)}\,|i\supset= 0$. This means that for any process involving photons,

$$\mathcal{M}_{fi} = \epsilon^\mu(k,\lambda)\,\mathcal{M}_\mu(k), \tag{3.3.78}$$

we have by gauge invariance

$$k^\mu \mathcal{M}_\mu(k) = 0. \tag{3.3.79}$$

Now let's check explicitly that the Compton amplitude is gauge invariant. Pulling $\epsilon^\mu(k,\lambda)\,\epsilon^\nu(k',\lambda')$ out of $\mathcal{M}(e^-\gamma \to e^-\gamma)$ we have

$$\begin{aligned}\mathcal{M}_{\mu\nu}(k,k') = (-ie)^2 \Big[&\overline{u}^{(s')}(\vec{p}')\,\gamma_\nu\,\frac{i}{(\not{p}+\not{k})-m}\,\gamma_\mu u^{(s)}(\vec{p})\\ &+\overline{u}^{(s')}(\vec{p}')\,\gamma_\mu\,\frac{i}{(\not{p}-\not{k}')-m}\,\gamma_\nu u^{(s)}(\vec{p})\Big].\end{aligned} \tag{3.3.80}$$

So if S_{fi} is to be gauge invariant then

$$k^\mu \mathcal{M}_{\mu\nu} = 0 = k'^\nu \mathcal{M}_{\mu\nu}. \tag{3.3.81}$$

First

$$\begin{aligned}k^\mu \mathcal{M}_{\mu\nu} = (-ie)^2 \Big[&\overline{u}^{(s')}(\vec{p}')\,\gamma_\nu\,\frac{i}{(\not{p}+\not{k})-m}\,\not{k}\, u^{(s)}(\vec{p})\\ &+\overline{u}^{(s')}(\vec{p}')\,\not{k}\,\frac{i}{(\not{p}-\not{k}')-m}\,\gamma_\nu u^{(s)}(\vec{p})\Big],\end{aligned} \tag{3.3.82}$$

but by energy-momentum conservation $k = p' + k' - p$, so

$$\begin{aligned}\not{k}\, u^{(s)}(\vec{p}) &= (\not{p}' + \not{k}' - \not{p})u^{(s)}(\vec{p})\\ &= (\not{p}+\not{k})u^{(s)}(\vec{p}) - \not{p}\, u^{(s)}(\vec{p}).\end{aligned} \tag{3.3.83}$$

Similarly

$$\overline{u}^{(s')}(\vec{p}')\,\not{k} = \overline{u}^{(s')}(\vec{p}')\,(\not{p}' + \not{k}' - \not{p}) \tag{3.3.84}$$

but, from the Dirac equation,

$$\not{p}\, u^{(s)}(\vec{p}) = m\, u^{(s)}(\vec{p}) \qquad \text{and} \qquad \overline{u}^{(s')}(\vec{p}')\,\not{p}' = m\,\overline{u}^{(s')}(\vec{p}'). \tag{3.3.85}$$

Thus

$$
\begin{aligned}
k^\mu \mathcal{M}_{\mu\nu} &= (-ie)^2 \Big[\overline{u}^{(s')}(\vec{p}')\,\gamma_\nu\,\frac{i}{(\not{p}+\not{k})-m}\,\big(\not{p}+\not{k}-m\big)\,u^{(s)}(\vec{p}) \\
&\qquad + \overline{u}^{(s')}(\vec{p}')\,\big(-(\not{p}-\not{k}'-m)\big)\,\frac{i}{(\not{p}-\not{k}')-m}\,\gamma_\nu u^{(s)}(\vec{p})\Big] \\
&= (-ie)^2 \Big[\overline{u}^{(s')}(\vec{p}')\,i\,\gamma_\nu\,u^{(s)}(\vec{p}) - \overline{u}^{(s')}(\vec{p}')\,i\,\gamma_\nu\,u^{(s)}(\vec{p})\Big] \\
&= 0.
\end{aligned}
\tag{3.3.86}
$$

Note that each graph separately is not gauge invariant, only the sum of all the graphs in that order of perturbation theory is gauge invariant.

Likewise one shows that

$$k'^{\nu}\mathcal{M}_{\mu\nu} = 0. \tag{3.3.87}$$

The importance of this is that in our polarization sum we have

$$\sum_{\lambda\lambda'=1}^{2} |\mathcal{M}_{fi}|^2 = \sum_{\lambda\lambda'=1}^{2} \big(\epsilon^\mu(k,\lambda)\,\epsilon^\rho(k,\lambda)\big)\big(\epsilon^\nu(k',\lambda')\,\epsilon^\kappa(k',\lambda')\big)\mathcal{M}_{\mu\nu}(k,k')\,\mathcal{M}_{\rho\kappa}{}^*(k,k'). \tag{3.3.88}$$

Now if $k^\mu\mathcal{M}_{\mu\nu} = 0 = k'^{\nu}\mathcal{M}_{\mu\nu}$ we have that the polarization vectors for $\lambda, \lambda' = 0, 3$ cancel each other; i.e. the $\left(\frac{k^\mu k^\rho}{\omega_k{}^2} - \frac{k^\mu g^{0\rho} + k^\rho g^{0\mu}}{\omega_k}\right)$ terms vanish. Thus

$$
\begin{aligned}
\sum_{\lambda\lambda'=1}^{2} |\mathcal{M}_{fi}|^2 &= (-g^{\mu\rho})(-g^{\nu\kappa})\mathcal{M}_{\mu\nu}(k,k')\,\mathcal{M}_{\rho\kappa}{}^*(k,k') \\
&= \mathcal{M}_{\mu\nu}(k,k')\,\mathcal{M}^{\mu\nu\,*}(k,k') \\
&= \mathrm{T}\Big[\mathcal{M}(k,k')\,\mathcal{M}^\dagger(k,k')\Big]
\end{aligned}
\tag{3.3.89}
$$

where $\Big(\mathcal{M}^\dagger(k,k')\Big)^{\mu\nu} = \Big(\mathcal{M}^*(k,k')\Big)^{\nu\mu}$.

So first calculating the spin average cross-section

$$
\begin{aligned}
\left(\frac{d\sigma^{\overline{\text{spin}}}(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}} &\equiv \frac{1}{2}\sum_{s=1}^{2}\sum_{s'=1}^{2}\left(\frac{d\sigma(e^-\gamma\to e^-\gamma)}{d\Omega}\right)_{\text{lab}} \\
&= \frac{1}{2}\,\frac{1}{(8\pi)^2}\left(\frac{\omega_{k'}}{m\omega_k}\right)^2 \mathrm{Tr}\Big[(\not{p}'+m)\Gamma(\not{p}+m)\tilde{\Gamma}\Big]
\end{aligned}
\tag{3.3.90}
$$

where

$$\Gamma = (-ie)^2 \not\epsilon(k',\lambda') \frac{i}{(\not p + \not k) - m} \not\epsilon(k,\lambda) + (-ie)^2 \not\epsilon(k,\lambda) \frac{i}{(\not p - \not k') - m} \not\epsilon(k',\lambda') \quad (3.3.91)$$

or

$$\Gamma = -ie^2\, \epsilon^\mu(k,\lambda)\, \epsilon^\nu(k',\lambda')\, \Gamma_{\mu\nu} \quad (3.3.92)$$

with

$$\Gamma_{\mu\nu} = \gamma_\nu \frac{1}{(\not p + \not k) - m} \gamma_\mu + \gamma_\mu \frac{1}{(\not p - \not k') - m} \gamma_\nu. \quad (3.3.93)$$

Now

$$\begin{aligned}
\tilde\Gamma &= \gamma^0 \Gamma^\dagger \gamma^0 \\
&= +ie^2 \epsilon^\mu(k,\lambda)\, \epsilon^\nu(k',\lambda') \left\{ \gamma^0 \Big[\gamma_\mu^\dagger \left(\frac{+1}{(\not p + \not k) - m} \right)^\dagger \gamma_\nu^\dagger \right. \\
&\quad \left. + \gamma_\nu^\dagger \left(\frac{1}{(\not p - \not k') - m} \right)^\dagger \gamma_\mu^\dagger \Big] \gamma^0 \right\} \\
&= +ie^2 \epsilon^\mu(k,\lambda)\, \epsilon^\nu(k',\lambda') \left[\gamma_\mu \frac{1}{(\not p + \not k) - m} \gamma_\nu + \gamma_\nu \frac{1}{(\not p - \not k') - m} \gamma_\mu \right] \\
&= +ie^2 \epsilon^\mu(k,\lambda)\, \epsilon^\nu(k',\lambda')\, \Gamma_{\nu\mu}.
\end{aligned} \quad (3.3.94)$$

So the spin averaged cross-section becomes

$$\begin{aligned}
&\left(\frac{d\sigma^{\overline{\text{spin}}}(e^-\gamma \to e^-\gamma)}{d\Omega} \right)_{\text{lab}} \\
&\quad = \frac{1}{2} \frac{e^4}{(8\pi)^2} \left(\frac{\omega_{k'}}{m\omega_k} \right)^2 \epsilon^\mu(k,\lambda)\, \epsilon^\nu(k',\lambda')\, \epsilon^\rho(k,\lambda)\, \epsilon^\kappa(k',\lambda') \\
&\quad \times \operatorname{Tr}\Big[(\not p' + m) \Big(\gamma_\nu \frac{1}{(\not p + \not k) - m} \gamma_\mu + \gamma_\mu \frac{1}{(\not p - \not k') - m} \gamma_\nu \Big) \\
&\quad \times (\not p + m) \Big(\gamma_\rho \frac{1}{(\not p + \not k) - m} \gamma_\kappa + \gamma_\kappa \frac{1}{(\not p - \not k') - m} \gamma_\rho \Big) \Big].
\end{aligned} \quad (3.3.95)$$

Now we can work out the trace! Notice that

$$\begin{aligned}
\frac{1}{\not p + \not k - m} &= \frac{\not p + \not k + m}{(p+k)^2 - m^2} = \frac{\not p + \not k + m}{p^2 - m^2 + k^2 + 2p\cdot k} \\
&= \frac{\not p + \not k + m}{2p\cdot k}
\end{aligned} \quad (3.3.96)$$

and similarly,

$$\begin{aligned}\frac{1}{\not p-\not k'-m} &= \frac{\not p-\not k'+m}{(p-k')^2-m^2} = \frac{\not p-\not k'+m}{p^2-m^2+k'^2-2p\cdot k'} \\ &= -\frac{\not p-\not k'+m}{2p\cdot k'}.\end{aligned} \tag{3.3.97}$$

So we have

$$\begin{aligned}\mathrm{Tr}\Big[(\not p'+m)\Big(\not\epsilon'\,\frac{1}{(\not p+\not k)-m}\,\not\epsilon+\not\epsilon\,\frac{1}{(\not p-\not k')-m}\,\not\epsilon'\Big)(\not p+m) \\ \times\Big(\not\epsilon\,\frac{1}{(\not p+\not k)-m}\,\not\epsilon'+\not\epsilon'\,\frac{1}{(\not p-\not k')-m}\,\not\epsilon\Big)\Big] \\ =\frac{1}{4}\Big[\frac{\mathrm{X}_{aa}}{(p\cdot k)^2}+\frac{\mathrm{X}_{bb}}{(p\cdot k')^2}-\frac{\mathrm{X}_{ab}+\mathrm{X}_{ba}}{(p\cdot k)\,(p\cdot k')}\Big]\end{aligned} \tag{3.3.98}$$

where

$$\begin{aligned}\not\epsilon &\equiv \gamma_\mu\epsilon^\mu(k,\lambda) \\ \not\epsilon' &\equiv \gamma_\mu\epsilon^\mu(k',\lambda') \\ \mathrm{X}_{aa} &\equiv \mathrm{Tr}\Big[(\not p'+m)\not\epsilon'(\not p+\not k+m)\not\epsilon(\not p+m)\not\epsilon(\not p+\not k+m)\not\epsilon'\Big] \\ \mathrm{X}_{bb} &\equiv \mathrm{Tr}\Big[(\not p'+m)\not\epsilon(\not p-\not k'+m)\not\epsilon'(\not p+m)\not\epsilon'(\not p-\not k'+m)\not\epsilon\Big] \\ \mathrm{X}_{ab} &\equiv \mathrm{Tr}\Big[(\not p'+m)\not\epsilon'(\not p+\not k+m)\not\epsilon(\not p+m)\not\epsilon'(\not p-\not k'+m)\not\epsilon\Big] \\ \mathrm{X}_{ba} &\equiv \mathrm{Tr}\Big[(\not p'+m)\not\epsilon(\not p-\not k'+m)\not\epsilon'(\not p+m)\not\epsilon(\not p+\not k+m)\not\epsilon'\Big].\end{aligned} \tag{3.3.99}$$

Notice that if $k \leftrightarrow -k'$ and $\epsilon \leftrightarrow \epsilon'$ in X_{aa}, then we get X_{bb}. So only X_{aa} need be calculated. Also, the interchange $k \leftrightarrow -k'$ and $\epsilon \leftrightarrow \epsilon'$ in X_{ab} yields X_{ba}. So only X_{ab} need be found.

Still these are very messy expressions involving the trace of up to eight γ-matrices. We can simplify things by first making use of the lab frame and the fact that we can always work in a gauge in this frame where $\epsilon^\mu(k,\lambda)=(0,\vec\epsilon_\lambda(\vec k))$ for $\lambda=1,2$.
Then

$$p_\mu\epsilon^\mu(k,\lambda)=p_\mu\epsilon^\mu(k',\lambda')=0 \tag{3.3.100}$$

and

$$k_\mu\epsilon^\mu(k,\lambda)={k_\mu}'\epsilon^\mu(k',\lambda')=0. \tag{3.3.101}$$

Now

$$(\not{p} + m)\not{\epsilon} = \not{\epsilon}(m - \not{p}) \tag{3.3.102}$$

so

$$\begin{aligned}(\not{p} + m)\not{\epsilon}(\not{p} + \not{k} + m) &= -\not{\epsilon}(\not{p} - m)(\not{p} + m + \not{k}) \\ &= -\not{\epsilon}(\not{p}\not{p} + m\not{p} + \not{p}\not{k} - m\not{p} - m^2 - m\not{k}) \\ &= -\not{\epsilon}(\not{p} - m)\not{k}.\end{aligned} \tag{3.3.103}$$

Then

$$\begin{aligned}(\not{p} + \not{k} + m)\not{\epsilon}(\not{p} + m)\not{\epsilon}(\not{p} + \not{k} + m) &= -(\not{p} + \not{k} + m)\not{\epsilon}\not{\epsilon}(\not{p} - m)\not{k} \\ &= +(\not{p} + \not{k} + m)(\not{p} - m)\not{k} \\ &= \not{k}(\not{p} - m)\not{k}.\end{aligned} \tag{3.3.104}$$

So

$$\begin{aligned}\mathrm{X}_{aa} &= \mathrm{Tr}\Big[(\not{p}' + m)\not{\epsilon}'\not{k}(\not{p} - m)\not{k}\not{\epsilon}'\Big] \\ &= \mathrm{Tr}\Big[\not{p}'\not{\epsilon}'\not{k}\not{p}\not{k}\not{\epsilon}'\Big] - m^2\mathrm{Tr}\Big[\not{\epsilon}'\not{k}\not{k}\not{\epsilon}'\Big].\end{aligned} \tag{3.3.105}$$

But $\not{k}\not{k} = k^2 = 0$ and $\not{k}\not{p}\not{k} = 2p \cdot k\not{k}$, thus

$$\begin{aligned}\mathrm{X}_{aa} &= 2\mathrm{Tr}\Big[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}'\Big]p \cdot k \\ &= 2\mathrm{Tr}\Big[\not{p}'\not{k} + \not{p}'\not{\epsilon}'2k \cdot \epsilon'\Big]p \cdot k \\ &= 8(p \cdot k)\Big[p' \cdot k + 2(k \cdot \epsilon')(p' \cdot \epsilon')\Big].\end{aligned} \tag{3.3.106}$$

But $p' - k = p - k'$, so $\epsilon' \cdot p' = \epsilon' \cdot k$ and $p' \cdot k = p \cdot k'$.
So

$$\mathrm{X}_{aa} = 8(p \cdot k)\Big[2(\epsilon' \cdot k)^2 + (p \cdot k')\Big] \tag{3.3.107}$$

and thus

$$\mathrm{X}_{bb} = -8(p \cdot k')\Big[2(\epsilon \cdot k')^2 - (p \cdot k)\Big]. \tag{3.3.108}$$

Now we have X_{ab} to evaluate. Using

$$\begin{aligned}(\not{p} + \not{k} + m)\not{\epsilon}(\not{p} + m) &= \not{k}\not{\epsilon}(\not{p} + m) + \not{\epsilon}(m - \not{p})(\not{p} + m) \\ &= \not{k}\not{\epsilon}(\not{p} + m)\end{aligned} \tag{3.3.109}$$

and

$$\begin{aligned}(\not{p} + m)\not{\epsilon}'(\not{p} - \not{k}' + m) &= -(\not{p} + m)\not{\epsilon}'\not{k}' + (\not{p} + m)\not{\epsilon}'(\not{p} + m) \\ &= -(\not{p} + m)\not{\epsilon}'\not{k}'.\end{aligned} \tag{3.3.110}$$

So X_{ab} becomes

$$\begin{aligned}\mathrm{X}_{ab} &= -\mathrm{Tr}\Big[(\not{p}' + m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}\Big] \\ &= -\mathrm{Tr}\Big[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\Big] - m^2\mathrm{Tr}\Big[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}'\not{k}'\not{\epsilon}\Big].\end{aligned} \tag{3.3.111}$$

Using $p' = p + k - k'$

$$\begin{aligned}\mathrm{X}_{ab} = &-\mathrm{Tr}\Big[\not{p}\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\Big] - \mathrm{Tr}\Big[\not{k}\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\Big] \\ &+ \mathrm{Tr}\Big[\not{k}'\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\Big] - m^2\mathrm{Tr}\Big[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}'\not{k}'\not{\epsilon}\Big].\end{aligned} \tag{3.3.112}$$

The evaluation of X_{ab} is messier than that of X_{aa}. It leads to

$$\mathrm{X}_{ab} = -8(p\cdot k)(p\cdot k')\Big[2(\epsilon\cdot\epsilon')^2 - 1\Big] - 8(k\cdot\epsilon')^2(p\cdot k') + 8(k'\cdot\epsilon)^2(p\cdot k) \tag{3.3.113}$$

so that, in fact,

$$\mathrm{X}_{ab} = \mathrm{X}_{ba}. \tag{3.3.114}$$

Recall that

$$\left(\frac{d\overline{\sigma^{\mathrm{spin}}}(e^-\gamma\to e^-\gamma)}{d\Omega}\right)_{\mathrm{lab}} = \frac{e^4}{8\,(8\pi)^2}\left(\frac{\omega_{k'}}{m\omega_k}\right)^2\left[\frac{\mathrm{X}_{aa}}{(p\cdot k)^2} + \frac{\mathrm{X}_{bb}}{(p\cdot k')^2} - \frac{\mathrm{X}_{ab}+\mathrm{X}_{ba}}{(p\cdot k)\,(p\cdot k')}\right] \tag{3.3.115}$$

with the electromagnetic fine structure constant given by $\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137.04}$.

Finally putting all this together we obtain the initial and final electron spin averaged and summed Compton scattering differential cross-section in the laboratory frame for definite initial and final photon polarizations. This is known as the Klein-Nishina formula

$$\begin{aligned}\left(\frac{d\overline{\sigma^{\mathrm{Klein-Nishina}}}(e^-\gamma\to e^-\gamma)}{d\Omega}\right)_{\mathrm{lab}} &= \left(\frac{d\overline{\sigma^{\mathrm{spin}}}(e^-\gamma\to e^-\gamma)}{d\Omega}\right)_{\mathrm{lab}} \\ &= \frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left\{\frac{\omega_k}{\omega_{k'}} + \frac{\omega_{k'}}{\omega_k} + 4\left(\epsilon^\mu(k,\lambda)\,\epsilon_\mu(k',\lambda')\right)^2 - 2\right\}.\end{aligned} \tag{3.3.116}$$

Since $\epsilon^\mu(k,\lambda)\,\epsilon_\mu(k',\lambda') = -\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}_{\lambda'}(\vec{k}')$, we can simply perform the average and sum over photon polarization to find the unpolarized Compton scattering differential cross-section, hence we must evaluate

$$\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2}\left(\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}_{\lambda'}(\vec{k}')\right)^2. \tag{3.3.117}$$

Since $(\vec{\epsilon}_1, \vec{\epsilon}_2, \frac{\vec{k}}{|\vec{k}|})$ forms an orthonormal basis we have

$$\vec{\epsilon}_{\lambda'}(\vec{k}') = \big(\hat{k} \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\big)\hat{k} + \sum_{\lambda=1}^{2} \big(\vec{\epsilon}_{\lambda}(\vec{k}) \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\big)\vec{\epsilon}_{\lambda}(\vec{k}) \tag{3.3.118}$$

and thus

$$\sum_{\lambda=1}^{2} \Big(\vec{\epsilon}_{\lambda}(\vec{k}) \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\Big)^2 = 1 - \left(\frac{\vec{k}}{|\vec{k}|} \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\right)^2. \tag{3.3.119}$$

Similarly $(\vec{\epsilon}_1(\vec{k}'), \vec{\epsilon}_2(\vec{k}'), \frac{\vec{k}'}{|\vec{k}'|})$ is an orthonormal basis so

$$\sum_{\lambda'=1}^{2} \Big(\frac{\vec{k}}{|\vec{k}|} \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\Big)^2 = 1 - \left(\frac{\vec{k}}{|\vec{k}|} \cdot \frac{\vec{k}'}{|\vec{k}'|}\right)^2 \tag{3.3.120}$$

but $\vec{k} \cdot \vec{k}' = |\vec{k}|\,|\vec{k}'| \cos\theta$ so

$$\begin{aligned}\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2} \Big(\vec{\epsilon}_{\lambda}(\vec{k}) \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')\Big)^2 &= 2 - (1 - \cos^2\theta) \\ &= 2 - \sin^2\theta \\ &= \cos^2\theta + 1.\end{aligned} \tag{3.3.121}$$

So the unpolarized cross-section becomes

$$\left(\frac{d\sigma^{\text{unpol.}}(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{\text{lab}} = \frac{\alpha^2}{2m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2 \left\{\frac{\omega_k}{\omega_{k'}} + \frac{\omega_{k'}}{\omega_k} - \sin^2\theta\right\} \tag{3.3.122}$$

where the Compton relation gives

$$\omega_{k'} = \frac{m\omega_k}{m + \omega_k(1 - \cos\theta)}. \tag{3.3.123}$$

In the low energy limit $\omega_k << m$ and $\omega_{k'} \sim \omega_k$, the recoil electron's kinetic energy is negligible. Thus our expression reduces to the Thompson scattering cross-section

$$\left(\frac{d\sigma_{\text{Thompson}}}{d\Omega}\right)_{\text{lab}} = \frac{\alpha^2}{2m^2}\big(1 + \cos^2\theta\big). \tag{3.3.124}$$