§3.2 THE S-MATRIX FEYNMAN RULES

Of course we cannot evaluate $(f|S|i)$ in closed form, so we will perturb in the interaction. To be specific let's evaluate the S-matrix for the process $e^-\gamma \longrightarrow e^-\gamma$. This is called Compton scattering. The initial state is an electron of momentum $\vec{p}$, spin $\frac{(-1)^{s+1}}{2}$ and a photon of momentum $\vec{k}$, helicity λ. The final state consists of the same particles with momenta and spins that are changed due to the scattering. So the final state is an electron of momentum $\vec{p}\,'$, spin $\frac{(-1)^{s'+1}}{2}$ and a photon of momentum $\vec{k}'$, helicity λ'. So

$$\begin{aligned} |i) &= |(\vec{p}, \frac{(-1)^{s+1}}{2}, -), (\vec{k}, \lambda)\rangle \\ &= b_s^\dagger(\vec{p})\, a_{(\lambda)}^\dagger(\vec{k})\,|0\rangle, \end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned} |f) &= |(\vec{p}\,', \frac{(-1)^{s'+1}}{2}, -), (\vec{k}', \lambda')\rangle \\ &= b_{s'}^\dagger(\vec{p}\,')\, a_{(\lambda')}^\dagger(\vec{k}')\,|0\rangle, \end{aligned} \tag{3.2.2}$$

which becomes the bra-vector

$$(f| = \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,'). \tag{3.2.3}$$

The transition probability amplitude to go from the initial state (at $t \to -\infty$) to the final state (at $t \to +\infty$) is given by

$$\begin{aligned} S_{fi} &= (f|S|i) \\ &= \langle(\vec{p}\,', \frac{(-1)^{s'+1}}{2}, -), (\vec{k}', \lambda')|S|(\vec{p}, \frac{(-1)^{s+1}}{2}, -), (\vec{k}, \lambda)\rangle \\ &= \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')\mathrm{T}e^{-ie\int d^4x\, A_\mu^{IP}(x)\overline{\Psi}^{IP}(x)\gamma^\mu\Psi^{IP}(x)} b_s^\dagger(\vec{p})\, a_{(\lambda)}^\dagger(\vec{k})\,|0\rangle, \end{aligned} \tag{3.2.4}$$

where we recall the beauty of the IP is that A_μ^{IP}, Ψ^{IP}, and $\overline{\Psi}^{IP}$ are all free fields Fourier expanded with coefficients a, $a^\dagger$, b, $b^\dagger$, d, and $d^\dagger$. Hence we only need to apply Wick's Theorem to evaluate this product of free fields. Since we cannot evaluate this exactly we expand and view this as a perturbative series.

So let's work through second order in e, ignoring $\mathcal{O}(e^3)$ terms. Hence

$$\begin{aligned} S_{fi} = \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')\Big[1 - ie\int d^4x\, \mathrm{T}A^{IP}_{\mu}(x)\overline{\Psi}^{IP}(x)\gamma^{\mu}\Psi^{IP}(x) \\ + \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \mathrm{T}A^{IP}_{\mu_1}(x_1)\overline{\Psi}^{IP}(x_1)\gamma^{\mu_1}\Psi^{IP}(x_1) \\ A^{IP}_{\mu_2}(x_2)\overline{\Psi}^{IP}(x_2)\gamma^{\mu_2}\Psi^{IP}(x_2) \\ + \mathcal{O}(e^3)\Big] b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle. \end{aligned} \tag{3.2.5}$$

Thus

$$\begin{aligned} S_{fi} = \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle \\ - ie\int d^4x \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')\Big(\mathrm{T}A^{IP}_{\mu}(x)\overline{\Psi}^{IP}(x)\gamma^{\mu}\Psi^{IP}(x)\Big) b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle \\ + \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')\Big(\mathrm{T}A^{IP}_{\mu_1}(x_1)\overline{\Psi}^{IP}(x_1) \\ \times \gamma^{\mu_1}\Psi^{IP}(x_1)A^{IP}_{\mu_2}(x_2)\overline{\Psi}^{IP}(x_2)\gamma^{\mu_2}\Psi^{IP}(x_2)\Big) b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle. \\ + \mathcal{O}(e^3). \end{aligned} \tag{3.2.6}$$

Let's analyze this term by term. Starting with the first; since all fields here are free we can evaluate this by using the CAR and CCR

$$\begin{aligned} \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle. \\ = \langle 0|\Big[a_{(\lambda')}(\vec{k}'), b_{s'}(\vec{p}\,')\, b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\Big]\Big|0\rangle. \\ = \langle 0|b_{s'}(\vec{p}\,')\, b^{\dagger}_s(\vec{p})\Big[a_{(\lambda')}(\vec{k}'), a^{\dagger}_{(\lambda)}(\vec{k})\Big]\,|0\rangle, \end{aligned} \tag{3.2.7}$$

where the first step is allowed since $a_{(\lambda')}(\vec{k}')|0\rangle = 0$ and since photon operators commute with $\overline{e}$ operators. Using the CCR and CAR we find that our expression becomes

$$= \delta_{\lambda\lambda'}(2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')\langle 0|b_{s'}(\vec{p}\,')\, b^{\dagger}_s(\vec{p})|0\rangle. \tag{3.2.8}$$

(Recall that $\lambda, \lambda' = 1, 2$ only for transverse photons so $-g_{\lambda\lambda'} = +\delta_{\lambda\lambda'}$.) Proceeding similarly for the e^- using the CAR, we find

$$\begin{aligned} \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')b^{\dagger}_s(\vec{p})\, a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle \\ = \delta_{\lambda\lambda'}(2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')\delta_{ss'}(2\pi)^3 2\omega_p \delta^3(\vec{p} - \vec{p}\,'). \end{aligned} \tag{3.2.9}$$

The 1 in the S operator just yields the result that the initial state evolves freely without scattering and hence must be in a final state which is the same as the initial state. There has been no scattering. This is always the case in general, any set of particles can go through without scattering. We are not interested in such terms since there is no "physics" in it; it does not probe the interaction. Hence in what will follow let's consider the case where $\vec{k} \neq \vec{k}'$ and $\vec{p} \neq \vec{p}\,'$ so that both particles must scatter.

Proceeding to the second term in (3.2.6), here we must evaluate

$$\begin{aligned} \langle 0|a_{(\lambda')}(\vec{k}')\, b_{s'}(\vec{p}\,')\Big(\mathrm{T}A_{\mu}^{IP}(x)\overline{\Psi}^{IP}(x)\gamma^{\mu}\Psi^{IP}(x)\Big) \\ \times b_{s}^{\dagger}(\vec{p})\, a_{(\lambda)}^{\dagger}(\vec{k})\,|0\rangle \\ = 0. \end{aligned} \tag{3.2.10}$$

Again, by the utility of the IP, all the fields are free fields and we can evaluate this vacuum matrix element. Since there are an odd number of photon fields present we see that we will always be left over with one $a_{(\rho)}$ or $a_{(\rho)}^{\dagger}$. Since $a_{(\rho)}|0\rangle = 0$ and $< 0|a_{(\rho)}^{\dagger} = 0$ this second term vanishes, as indicated.

So we have our final term in (3.2.6) to evaluate. As we have seen in the first term, it is easiest to evaluate when we have the vacuum expectation value of a string of creation and annihilation operators. To re-write the T-product as such a string, we turn to the general form of Wick's Theorem for operators which relates time-ordered operators to normal- ordered operators. Let's state the result first:

Wick's Theorem

$$
\begin{aligned}
\mathrm{T}\phi(x_1)\cdots\phi(x_n) &= \mathrm{N}\big[\phi(x_1)\cdots\phi(x_n)\big] \\
&+ \sum_{\substack{(1\cdots n)\xrightarrow{P^1}(i_1 j_1)\frac{(1\cdots n)}{(i_1 j_1)} \\ i_1<j_1}} (-1)^{|P^1|}\langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle\,\mathrm{N}\left[\frac{\phi(x_1)\cdots\phi(x_n)}{\phi(x_{i_1})\phi(x_{j_1})}\right] \\
&+ \sum_{\substack{(1\cdots n)\xrightarrow{P^2}(i_1 j_1)(i_2 j_2)\frac{(1\cdots n)}{(i_1 j_1)(i_2 j_2)} \\ i_1<j_1 \\ i_2<j_2 \\ i_1<i_2}} (-1)^{|P^2|}\langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle \\
&\qquad \times\langle 0|\mathrm{T}\phi(x_{i_2})\phi(x_{j_2})|0\rangle\,\mathrm{N}\Big[\frac{\phi(x_1)\cdots\phi(x_n)}{\phi(x_{i_1})\phi(x_{i_2})\phi(x_{j_1})\phi(x_{j_2})}\Big] \\
&+\cdots\begin{cases}
+\sum_{(1\cdots n)\xrightarrow{P^{\frac{n}{2}}}(i_1 j_1)\cdots(i_{\frac{n}{2}} j_{\frac{n}{2}})} (-1)^{|P^{\frac{n}{2}}|}\langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle\cdots \\
\qquad\qquad \times\langle 0|\mathrm{T}\phi(x_{i_{\frac{n}{2}}})\phi(x_{j_{\frac{n}{2}}})|0\rangle \quad , \mathrm{n}=\text{even} \\
+\sum (-1)^{|P^{\frac{n-1}{2}}|}\langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle\cdots \\
\qquad\qquad \times\langle 0|\mathrm{T}\phi(x_{i_{\frac{n-1}{2}}})\phi(x_{j_{\frac{n-1}{2}}})|0\rangle \\
\qquad\qquad \times\mathrm{N}[\Phi(x_{i_{\frac{n+1}{2}}})] \quad , \mathrm{n}=\text{odd}.
\end{cases}
\end{aligned}
\tag{3.2.11}
$$

where $(-1)^{|P^i|}$ is the parity of the permutation that takes the order of the Fermi fields on the RHS and permutes to the original order on the LHS.

More simply written Wick's Theorem states

$$
\begin{aligned}
\mathrm{T}\phi(x_1)\cdots\phi(x_n) &= \mathrm{N}\big[\phi(x_1)\cdots\phi(x_n)\big] \\
&+ \sum_{\substack{\text{single} \\ \text{contractions}}} \langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle\,\mathrm{N}\Big[\frac{\phi(x_1)\cdots\phi(x_n)}{\phi(x_{i_1})\phi(x_{j_1})}\Big] \\
&+ \sum_{\substack{\text{double} \\ \text{contractions}}} \langle 0|\mathrm{T}\phi(x_{i_1})\phi(x_{j_1})|0\rangle\langle 0|\mathrm{T}\phi(x_{i_2})\phi(x_{j_2})|0\rangle \\
&\qquad \times\mathrm{N}\Big[\frac{\phi(x_1)\cdots\phi(x_n)}{\phi(x_{i_1})\phi(x_{j_1})\phi(x_{i_2})\phi(x_{j_2})}\Big] \\
&+\cdots.
\end{aligned}
\tag{3.2.12}
$$

This is proven by induction. First note that

$$\mathrm{T}\phi(x) = \mathrm{N}\big[\phi(x)\big] = \phi(x)$$

$$
\begin{aligned}
\mathrm{T}\phi(x)\phi(y) &= \theta(x^0-y^0)\phi(x)\phi(y) \pm \theta(y^0-x^0)\phi(y)\phi(x) \\
&= \theta(x^0-y^0)\big(\phi^+(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\phi^+(x)\phi^-(y)+\phi^-(x)\phi^+(y)\big) \\
&\quad \pm \theta(y^0-x^0)\big(\phi^+(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\phi^+(x)\phi^-(y)+\phi^-(x)\phi^+(y)\big) \\
&= \theta(x^0-y^0)\Big(\mathrm{N}\big[\phi(x)\phi(y)\big]+\big[\phi^+(x),\phi^-(y)\big]_\pm\Big) \\
&\quad \pm \theta(y^0-x^0)\Big(\mathrm{N}\big[\phi(y)\phi(x)\big]+\big[\phi^+(y),\phi^-(x)\big]_\pm\Big). \qquad (3.2.13)
\end{aligned}
$$

But $\mathrm{N}\big[\phi(x)\phi(y)\big] = \mathrm{N}\big[\phi(y)\phi(x)\big]$ by definition. So with $+$ for Fermi fields and $-$ for Bose fields we get

$$
\begin{aligned}
&\mathrm{T}\phi(x)\phi(y) \\
&= \mathrm{N}\big[\phi(x)\phi(y)\big] + \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle \pm \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle. \qquad (3.2.14)
\end{aligned}
$$

Thus we secure

$$
\mathrm{T}\phi(x)\phi(y) = \mathrm{N}\big[\phi(x)\phi(y)\big] + \langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle. \qquad (3.2.15)
$$

Now suppose we use this result for the next step multiply (3.2.15) by $\phi(z)$ on the right where $x^0, y^0 > z^0$. Then

$$
\Big(\mathrm{T}\phi(x)\phi(y)\Big)\phi(z) = \mathrm{T}\phi(x)\phi(y)\phi(z) \qquad \text{for } x^0, y^0 > z^0. \qquad (3.2.16)
$$

On the other hand

$$
\begin{aligned}
\Big(\mathrm{N}\big[\phi(x)\phi(y)\big]\Big)\phi(z) &= \mathrm{N}\big[\phi(x)\phi(y)\big]\phi^+(z) + \mathrm{N}\big[\phi(x)\phi(y)\big]\phi^-(z) \\
&= \mathrm{N}\big[\phi(x)\phi(y)\phi^+(z)\big] + \mathrm{N}\big[\phi(x)\phi(y)\big]\phi^-(z).
\end{aligned} \qquad (3.2.17)
$$

Now we must bring the creation operator into the normal product,

$$
\begin{aligned}
\mathrm{N}\big[\phi(x)\phi(y)\big]\phi^-(z) &= \phi^+(x)\phi^+(y)\phi^-(z) + \phi^-(x)\phi^+(y)\phi^-(z) \\
&\quad \pm \phi^-(y)\phi^+(x)\phi^-(z) + \phi^-(x)\phi^-(y)\phi^-(z)
\end{aligned} \qquad (3.2.18)
$$

as we see the only difficulty is bringing $\phi^-(z)$ through the annihilator operators since the ϕ^+, ϕ^- do not commute. Thus we must just successively commute $\phi^-(z)$ through,

picking up the commutators,

$$\begin{aligned}
\mathrm{N}\big[\phi(x)\phi(y)\big]\phi^-(z) &= \mp\phi^-(z)\phi^+(x)\phi^+(y) + \Big[\phi^+(x)\phi^+(y),\phi^-(z)\Big]_\pm \\
&\quad \mp \phi^-(x)\phi^-(z)\phi^+(y) + \phi^-(x)\Big[\phi^+(y),\phi^-(z)\Big]_\pm \\
&\quad \mp \phi^-(y)\phi^-(z)\phi^+(x) + \phi^-(y)\Big[\phi^+(x),\phi^-(z)\Big]_\pm \\
&\quad + \phi^-(x)\phi^-(y)\phi^-(z) \\
&= \mathrm{N}\big[\phi(x)\phi(y)\phi^-(z)\big] + \Big[\phi^+(x)\phi^+(y),\phi^-(z)\Big]_\pm \\
&\quad + \phi^-(x)\Big[\phi^+(y),\phi^-(z)\Big]_\pm \mp \phi^-(y)\Big[\phi^+(x),\phi^-(z)\Big]_\pm \\
&= \mathrm{N}\big[\phi(x)\phi(y)\phi^-(z)\big] + \phi(x)\Big[\phi^+(y),\phi^-(z)\Big]_\pm \\
&\quad \mp \phi(y)\Big[\phi^+(x),\phi^-(z)\Big]_\pm .
\end{aligned} \tag{3.2.19}$$

Combining the expressions we find that, for $x^0, y^0 > z^0$,

$$\begin{aligned}
\mathrm{T}\phi(x)\phi(y)\phi(z) &= \mathrm{N}\big[\phi(x)\phi(y)\phi(z)\big] + \langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle\phi(z) \\
&\quad + \langle 0|\mathrm{T}\phi(y)\phi(z)|0\rangle\phi(x) \pm \langle 0|\mathrm{T}\phi(x)\phi(z)|0\rangle\phi(y)
\end{aligned} \tag{3.2.20}$$

where we used $\langle 0|\mathrm{T}\phi(x)\phi(z)|0\rangle = \langle 0|\phi(x)\phi(z)|0\rangle = \Big[\phi^+(x),\phi^-(y)\Big]_\pm$ for $x^0 > z^0$. Also, in the last term the "+" is for bosons and the "−" is for fermions.

Now the above expression was derived for $x^0, y^0 > z^0$ but the expressions on the LHS and $\mathrm{N}\big[\phi(x)\phi(y)\phi(z)\big]$ on the RHS and the sum of the last three terms on the RHS are invariant under permutation of x, y, and z. That is, we could have started with $\big(\mathrm{T}\phi(y)\phi(z)\big)\phi(x)$ for $y^0, z^0 > x^0$ and gone through the same steps and arrived at the same expression for the RHS. Hence the above formula is valid for all times (x^0, y^0, z^0).

The general form of Wick's Theorem is proved similarly. By starting with the expression for n-fields we derive it for $(n+1)$, recall the derivations in chapters 2.2, 2.3, 2.4.

So let's return to the S-matrix evaluation. We have

$$
\begin{aligned}
&\mathrm{T}A_{\mu_1}(x_1)\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)A_{\mu_2}(x_2)\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)\\
&\qquad = \mathrm{N}\Big[A_{\mu_1}(x_1)\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)A_{\mu_2}(x_2)\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)|0\rangle \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)|0\rangle \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)|0\rangle \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{b_1}(x_1)\Psi_{b_2}(x_2)|0\rangle \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle \mathrm{N}\Big[\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)|0\rangle\\
&\qquad\qquad \times \mathrm{N}\Big[\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle \langle 0|\mathrm{T}\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)|0\rangle\\
&\qquad\qquad \mathrm{N}\Big[\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)|0\rangle\\
&\qquad\qquad \times \mathrm{N}\Big[\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle \langle 0|\mathrm{T}\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)|0\rangle\\
&\qquad\qquad \times \mathrm{N}\Big[\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)|0\rangle \langle 0|\mathrm{T}\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)|0\rangle\\
&\qquad\qquad \times \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)|0\rangle \langle 0|\mathrm{T}\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)|0\rangle\\
&\qquad\qquad \times \mathrm{N}\Big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\Big]\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_1}(x_1)|0\rangle \langle 0|\mathrm{T}\overline{\Psi}_{a_2}(x_2)\Psi_{b_2}(x_2)|0\rangle\\
&\qquad\qquad \times \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle\\
&\qquad\quad + \langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\Psi_{b_2}(x_2)|0\rangle \langle 0|\mathrm{T}\Psi_{b_1}(x_1)\overline{\Psi}_{a_2}(x_2)|0\rangle\\
&\qquad\qquad \times \langle 0|\mathrm{T}A_{\mu_1}(x_1)A_{\mu_2}(x_2)|0\rangle
\end{aligned}
\tag{3.2.21}
$$

Now we can apply this to the process at hand, the scattering of $e^-\,\gamma$. Recall we must sandwich these terms between creation and annihilation operators

$$\langle 0|a_{(\lambda')}(\vec{k}')\,b_{s'}(\vec{p}\,')\,(\mathrm{T}\cdots)b_s^\dagger(\vec{p})\,a_{(\lambda)}^\dagger(\vec{k})|0\rangle \tag{3.2.22}$$

If $a_{(\lambda')}(\vec{k}')$ contracts with $a_{(\lambda)}^\dagger(\vec{k})$ in our evaluation we get a factor of $\delta^3(\vec{k}-\vec{k}')$. Since we desire $\vec{k}\neq\vec{k}'$ scattering, these terms will be zero. Hence we first consider only that normal product in the sum which has two photon operators since they will eventually commute with the initial and final state operators yielding $\vec{k}\neq\vec{k}'$.

Similarly we must have $\vec{p}\neq\vec{p}\,'$ so that term of interest in $(\mathrm{T}\cdots)$ must have in the normal product another $b^\dagger$ for $b_{s'}(\vec{p}\,')$ to anti-commute with and another b for $b_s^\dagger(\vec{p})$ to anti-commute with – that is, we need a $\Psi\overline{\Psi}$ in the normal product. Altogether then we must have at least two A's and Ψ, $\overline{\Psi}$ in the $(\mathrm{N}\cdots)$ terms of interest. This restricts the number of terms considerably to just the first five listed on the previous page. However we are not finished. The first term does not contribute since we will always be left over with two operators after the initial and final states' operators are commuted through. But these leftover operators are normal-ordered; hence their vacuum expectation value (VEV) is zero. Cryptically,

$$\begin{aligned}\langle 0|a(\vec{k}')\,b(\vec{p}\,')\mathrm{N}\big[AA\overline{\Psi}\Psi\overline{\Psi}\Psi\big]b^\dagger(\vec{p})\,a^\dagger(\vec{k})|0\rangle&\\ \sim\langle 0|b(\vec{p}\,')\mathrm{N}\big[\overline{\Psi}\Psi\overline{\Psi}\Psi\big]b^\dagger(\vec{p})|0\rangle&\\ \sim\langle 0|\mathrm{N}\big[\overline{\Psi}\Psi\big]|0\rangle=0.&\end{aligned} \tag{3.2.23}$$

Further the second and third terms are non-sensical for two reasons. First, physically, when we, for instance, contract $a_{(\lambda)}(\vec{k})$ with $\int d^4x\,A_\mu(x)$ we see that we get a $\delta^3(\vec{k})$, the incoming photon (or, in the other case, the out-going photon) must have zero momentum. Such photons we don't have as initial or final states so these terms do not contribute. Secondly, for consistency of the theory we eliminate these terms by renormalization. That is, note that they involve the propagator

$$\begin{aligned}\langle 0|\mathrm{T}\Psi(x)\,\overline{\Psi}(x)|0\rangle&=\lim_{x\to y}\int\frac{d^4l}{(2\pi)^4}\,e^{-il(x-y)}\frac{i(\not{l}+m)}{l^2-m^2+i\epsilon}\\&=\int\frac{d^4l}{(2\pi)^4}\,\frac{i(\not{l}+m)}{l^2-m^2+i\epsilon}\end{aligned} \tag{3.2.24}$$

which for large integration variable values diverges

$$\sim\int^\Lambda\frac{d^4l\,l}{l^2}\sim\Lambda^3\longrightarrow\infty\quad\text{as }\Lambda\to\infty, \tag{3.2.25}$$

and hence leads to a meaningless infinite expression for S_{fi}. The source of the difficulty again is that we must be careful how we define the Hamiltonian. Since we want $H^{IP}|0\rangle_{IP} = 0$ for our system at all times we must normal order the free IP fields in $\mathcal{H}_I^{IP}$ as well as $\mathcal{H}_o^{IP}$!

Thus we really have

$$\mathcal{H}_I^{IP} = e\mathrm{N}\big[A_\mu^{IP}\overline{\Psi}^{IP}\gamma^\mu\Psi^{IP}\big] \tag{3.2.26}$$

and

$$S = \mathrm{T}e^{-ie\int d^4x\,\mathrm{N}\left[A_\mu^{IP}(x)\overline{\Psi}^{IP}(x)\gamma^\mu\Psi^{IP}(x)\right]}. \tag{3.2.27}$$

Note S is <u>not</u> normal ordered; that is

$$\mathrm{T}\Big(e^{\int d^4x\,\mathrm{N}[\mathcal{O}]}\Big) \neq \mathrm{N}\Big(\mathrm{T}e^{\int d^4x\,\mathcal{O}}\Big). \tag{3.2.28}$$

Now, Wick's Theorem becomes slightly modified when we have the time-ordered product of normal-ordered products of fields. The form of the modification can be found by approaching the desired result as a limit of point-split products. That is, we are now interested in

$$\mathrm{T}\Big[\mathrm{N}\big[\phi_{11}(x_1)\cdots\phi_{1a_1}(x_1)\big]\mathrm{N}\big[\phi_{21}(x_2)\cdots\phi_{2a_2}(x_2)\big]\cdots\mathrm{N}\big[\phi_{n1}(x_n)\cdots\phi_{na_n}(x_n)\big]\Big]. \tag{3.2.29}$$

In each case the normal products can be written as a limit with the substitution of $x^0+\epsilon$ in the argument of the creation operators and of $x^0-\epsilon$ in the argument of the annihilation operators. So we define $\hat{\phi}(x) \equiv \phi^+(x^0-\epsilon,\vec{x}) + \phi^-(x^0+\epsilon,\vec{x})$. Then

$$\begin{aligned}\mathrm{T}\Big[\mathrm{N}\big[\phi_{11}(x_1)\cdots\phi_{1a_1}(x_1)\big]\cdots\mathrm{N}\big[\phi_{n1}(x_n)\cdots\phi_{na_n}(x_n)\big]\Big] \\ = \lim_{\epsilon\to 0^+}\mathrm{T}\Big[\hat{\phi}_{11}(x_1)\cdots\hat{\phi}_{1a_1}(x_1)\hat{\phi}_{21}(x_2)\cdots\hat{\phi}_{na_n}(x_n)\Big].\end{aligned} \tag{3.2.30}$$

So the normal ordering of the operators within each equal time group is the same as the time ordering before the $\epsilon\to 0^+$ limit, and hence results in the time-ordered product of normal products.

We may now use Wick's Theorem to expand the time-split time-ordered product. Whenever there is a contraction between fields split about the same time we find zero since they are normal-ordered by the splitting, i.e.

$$\begin{aligned}\langle 0|\mathrm{T}\phi^+(x-\epsilon)\,\phi^-(x+\epsilon)|0\rangle &= \langle 0|\phi^-(x+\epsilon)\,\phi^+(x-\epsilon)|0\rangle \\ &= 0.\end{aligned} \tag{3.2.31}$$

Hence the time-ordered product of normal products is just the same as the time-ordered product of all the fields with the contractions of the equal-time fields omitted, i.e.

$$\begin{aligned} &\mathrm{T}\Big[\mathrm{N}\big[\phi_{11}(x_1)\cdots\phi_{1a_1}(x_1)\big]\cdots\mathrm{N}\big[\phi_{n1}(x_n)\cdots\phi_{na_n}(x_n)\big]\Big] \\ &\qquad = \mathrm{T}\Big[\hat{\phi}_{11}(x_1)\cdots\hat{\phi}_{na_n}(x_n)\Big]\Big|_{\substack{\text{no equal-time}\\ \text{contractions}}} . \end{aligned} \tag{3.2.32}$$

(Equal-time contractions are often called self-contractions.)

Thus we define the S-matrix to be the time-ordered product of the exponentiated normal-ordered interaction Hamiltonian. This renormalization eliminates the infinities arising from the self-contractions. Thus we find only the two terms left contributing in second order to the Compton amplitude. So we find, gathering all terms so far, for $\vec{k}' \neq \vec{k}$ and $\vec{p}\,' \neq \vec{p}$,

$$\begin{aligned} S_{fi} = (\frac{-ie}{2!})^2 \int d^4x_1\, d^4x_2 \Big\{ &\langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)|0\rangle \\ &\times \langle 0|a_{(\lambda')}(\vec{k}')\,b_{s'}(\vec{p}\,')\,\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2) \\ &\quad \times \Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)\big]\gamma^{\mu_1}_{a_1b_1}\gamma^{\mu_2}_{a_2b_2}b^\dagger_s(\vec{p})\,a^\dagger_{(\lambda)}(\vec{k})|0\rangle \\ + \langle 0|\mathrm{T}&\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle \\ &\times \langle 0|a_{(\lambda')}(\vec{k}')\,b_{s'}(\vec{p}\,')\,\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2) \\ &\quad \times \overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)\big]\gamma^{\mu_1}_{a_1b_1}\gamma^{\mu_2}_{a_2b_2}b^\dagger_s(\vec{p})\,a^\dagger_{(\lambda)}(\vec{k})|0\rangle \Big\}. \end{aligned} \tag{3.2.33}$$

The first thing we note is that the second term is the same as the first. To see this just re-label the dummy indices $x_1 \leftrightarrow x_2$, $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $\mu_1 \leftrightarrow \mu_2$ and recall that

$$\langle 0|\mathrm{T}\Psi_{b_2}(x_2)\,\overline{\Psi}_{a_1}(x_1)|0\rangle = -\langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)|0\rangle \tag{3.2.34}$$

while

$$\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)\,\overline{\Psi}_{a_2}(x_2)\,\Psi_{b_1}(x_1)\big] = -\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)\,\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)\big]. \tag{3.2.35}$$

So we finally find (since $2 \times \frac{1}{2!} = 1$) that for $\vec{k}' \neq \vec{k}$ and $\vec{p}\,' \neq \vec{p}$,

$$S_{fi} = (-ie)^2 \int d^4x_1\, d^4x_2\, \langle 0|\mathrm{T}\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle\gamma^{\mu_1}_{a_1b_1}\gamma^{\mu_2}_{a_2b_2}$$

$$\times\langle 0|a_{(\lambda')}(\vec{k}')\,b_{s'}(\vec{p}\,')\,\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)\,\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)\big]\,b_s^\dagger(\vec{p})\,a_{(\lambda)}^\dagger(\vec{k})|0\rangle. \tag{3.2.36}$$

Now the annihilation operators from the normal product commute with the $b^\dagger$ and $a^\dagger$ of the initial state, and the creation operators commute with the a and b of the final state so that we are left with just

$$\begin{aligned}
&\langle 0|a_{(\lambda')}(\vec{k}')\,b_{s'}(\vec{p}\,')\,\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)\,\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)\big]\,b_s^\dagger(\vec{p})\,a_{(\lambda)}^\dagger(\vec{k})|0\rangle \\
&\quad= \langle 0|a_{(\lambda')}(\vec{k}')\big[A^-_{\mu_1}(x_1)\,A^+_{\mu_2}(x_2)+A^-_{\mu_2}(x_2)\,A^+_{\mu_1}(x_1)\big]a_{(\lambda)}^\dagger(\vec{k})|0\rangle \\
&\qquad\langle 0|b_{s'}(\vec{p}\,')\big[\overline{\Psi}^\dagger_{a_1}(x_1)\,\Psi^\dagger_{b_2}(x_2)\big]\,b_s^\dagger(\vec{p})|0\rangle \\
&\quad= \Big\{\langle 0|a_{(\lambda')}(\vec{k}')\,A^-_{\mu_1}(x_1)|0\rangle\langle 0|A^+_{\mu_2}(x_2)\,a_{(\lambda)}^\dagger(\vec{k})|0\rangle \\
&\qquad\langle 0|a_{(\lambda')}(\vec{k}')\,A^-_{\mu_2}(x_2)|0\rangle\langle 0|A^+_{\mu_1}(x_1)\,a_{(\lambda)}^\dagger(\vec{k})|0\rangle\Big\} \\
&\qquad\times\langle 0|b_{s'}(\vec{p}\,')\,\overline{\Psi}^-_{a_1}(x_1)|0\rangle\langle 0|\Psi^+_{b_2}(x_2)\,b_s^\dagger(\vec{p})|0\rangle.
\end{aligned} \tag{3.2.37}$$

Now we can evaluate these vacuum expectation values (VEV) simply using

$$\begin{aligned}
A^+_\mu(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{\rho=0}^{3}\epsilon_\mu(l,\rho)\,a_{(\rho)}(\vec{l})\,e^{-ilx} \\
A^-_\mu(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{\rho=0}^{3}\epsilon_\mu(l,\rho)\,a^\dagger_{(\rho)}(\vec{l})\,e^{+ilx} \\
\Psi^+_a(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{r=1}^{2}b_r(\vec{l})\,u_a^{(r)}(\vec{l})\,e^{-ilx} \\
\Psi^-_a(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{r=1}^{2}d_r^\dagger(\vec{l})\,v_a^{(r)}(\vec{l})\,e^{+ilx} \\
\overline{\Psi}^+_a(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{r=1}^{2}d_r(\vec{l})\,\overline{v}_a^{(r)}(\vec{l})\,e^{-ilx} \\
\overline{\Psi}^-_a(x) &= \int\frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{r=1}^{2}b_r^\dagger(\vec{l})\,\overline{u}_a^{(r)}(\vec{l})\,e^{+ilx}
\end{aligned} \tag{3.2.38}$$

Thus

$$\begin{aligned}\langle 0|a_{(\lambda)}(\vec{k})\,A_\mu^-(x)|0\rangle &= \int \frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{\rho=0}^{3}\epsilon_\mu(l,\rho)\,e^{+ilx}\langle 0|a_{(\lambda)}(\vec{k})\,a^\dagger_{(\rho)}(\vec{l})|0\rangle \\ &= \int \frac{d^3l}{(2\pi)^3 2\omega_l}\sum_{\rho=0}^{3}\epsilon_\mu(l,\rho)\,e^{+ilx}(-g_{\lambda\rho})(2\pi)^3 2\omega_k\,\delta^3(\vec{k}-\vec{l}) \\ &= \epsilon_\mu(k,\lambda)\,e^{+ikx}\end{aligned} \tag{3.2.39}$$

Similarly,

$$\langle 0|A_\mu^+(x)\,a^\dagger_{(\lambda)}(\vec{k})|0\Big\rangle = \epsilon_\mu(k,\lambda)\,e^{-ikx}. \tag{3.2.40}$$

And likewise for the fermions

$$\begin{aligned}\langle 0|b_s(\vec{p})\,\overline{\Psi}_a^-(x)|0\rangle &= \overline{u}_a^{(s)}(\vec{p})\,e^{+ipx} \\ \langle 0|\Psi_b^+(x)\,b_s^\dagger(\vec{p})|0\rangle &= u_b^{(s)}(\vec{p})\,e^{-ipx}.\end{aligned} \tag{3.2.41}$$

Also, for the anti-particles

$$\begin{aligned}\langle 0|d_s(\vec{p})\,\Psi_b^-(x)|0\rangle &= v_b^{(s)}(\vec{p})\,e^{+ipx} \\ \langle 0|\overline{\Psi}_a^+(x)\,d_s^\dagger(\vec{p})|0\rangle &= \overline{v}_a^{(s)}(\vec{p})\,e^{-ipx}.\end{aligned} \tag{3.2.42}$$

So we have then

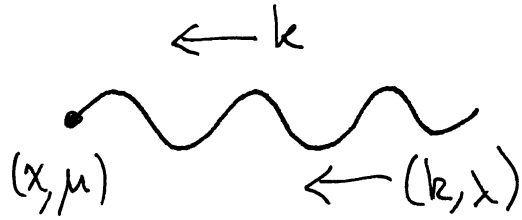
$$\begin{aligned}S_{fi} &= (-ie)^2\int d^4x_1\,d^4x_2\,\langle 0|T\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle \\ &\quad\times\Big\{\epsilon_{\mu_1}(k',\lambda')\,e^{+ik'x_1}\,\epsilon_{\mu_2}(k,\lambda)\,e^{-ikx_2}+\epsilon_{\mu_2}(k',\lambda')\,e^{+ik'x_2}\,\epsilon_{\mu_1}(k,\lambda)\,e^{-ikx_1}\Big\} \\ &\quad\times\overline{u}^{(s')}_{a_1}(\vec{p}\,')\,e^{+ip'x_1}\,u^{(s)}_{b_2}(\vec{p})\,e^{-ipx_2}\gamma^{\mu_1}_{a_1b_1}\gamma^{\mu_2}_{a_2b_2} \\ &= (-ie)^2\int d^4x_1\,d^4x_2\Big[\epsilon_{\mu_1}(k',\lambda')\,e^{+ik'x_1}\Big]\Big[\overline{u}^{(s')}(\vec{p}\,')\,e^{+ip'x_1}\Big] \\ &\quad\times\gamma^{\mu_1}\,S_F(x_1-x_2)\,\gamma^{\mu_2}\Big[u^{(s)}(\vec{p})\,e^{-ipx_2}\Big]\Big[\epsilon_{\mu_2}(k,\lambda)\,e^{-ikx_2}\Big] \\ &\quad+(-ie)^2\int d^4x_1\,d^4x_2\Big[\epsilon_{\mu_2}(k',\lambda')\,e^{+ik'x_2}\Big]\Big[\overline{u}^{(s')}(\vec{p}\,')\,e^{+ip'x_1}\Big] \\ &\quad\times\gamma^{\mu_1}\,S_F(x_1-x_2)\,\gamma^{\mu_2}\Big[u^{(s)}(\vec{p})\,e^{-ipx_2}\Big]\Big[\epsilon_{\mu_1}(k,\lambda)\,e^{-ikx_1}\Big].\end{aligned} \tag{3.2.43}$$

In order to visualize what is happening in this scattering process we can pictorially represent this mathematical expression by diagrams called Feynman graphs or diagrams in coordinate space. We see that our incoming particles are represented by their plane wave wavefunctions, and similarly for outgoing particles. So we represent

Incoming particles :

photon

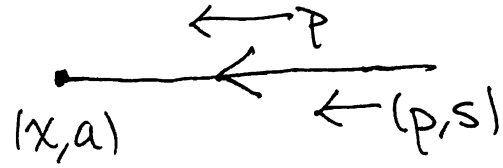
$$\epsilon_\mu(k,\lambda)\, e^{-ikx} \quad \longleftrightarrow$$



electron

(arrow follows charge of particle)

$$u_a^{(s)}(\vec{p})\, e^{-ipx} \quad \longleftrightarrow$$



Outgoing particles :

Photon

$$\epsilon_\mu(k,\lambda)\, e^{+ikx} \quad \longleftrightarrow$$

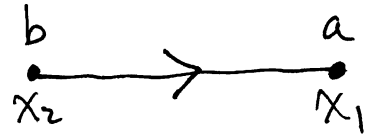
Electron

$$\overline{u}_a^{(s)}(\vec{p})\, e^{+ipx} \quad \longleftrightarrow$$

The electrons and photons interact at a space-time point x with strength $-ie\gamma^\mu$, and we sum over all interaction points $\int d^4x$. The electron then can propagate through space-time to the next interaction point via the propagator factor $S_F(x_1 - x_2)$ which we represent pictorially by a line joining the two vertices at x_1 and x_2.

Propagator

$$S_{Fab}(x_1 - x_2) \quad \longleftrightarrow$$



Vertices are where photons and particles meet. They are represented by

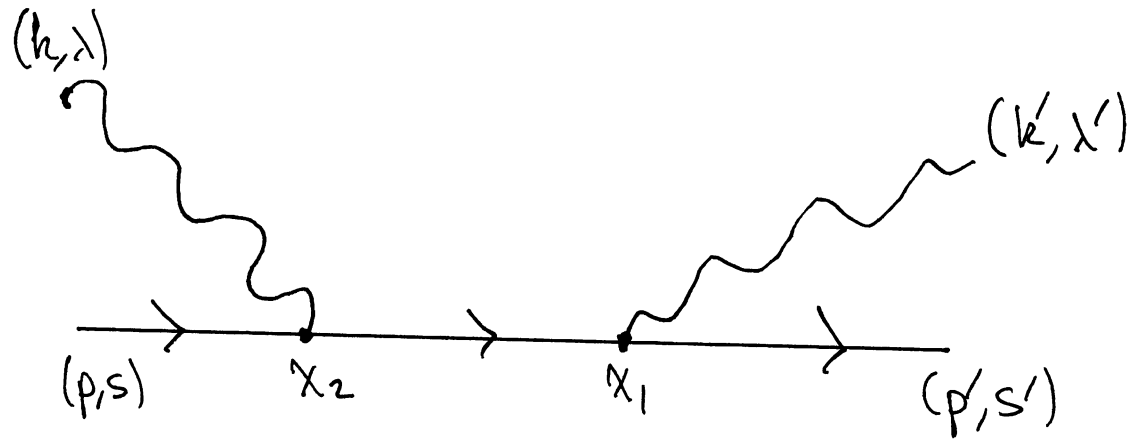
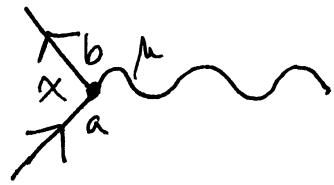


Figure 3.2.1

$$-ie \int d^4x \, \gamma^{\mu}_{ba} \quad \longleftrightarrow$$



Thus we can represent the first term in Compton Scattering pictorially as and the second term as given in Figure 3.2.2.

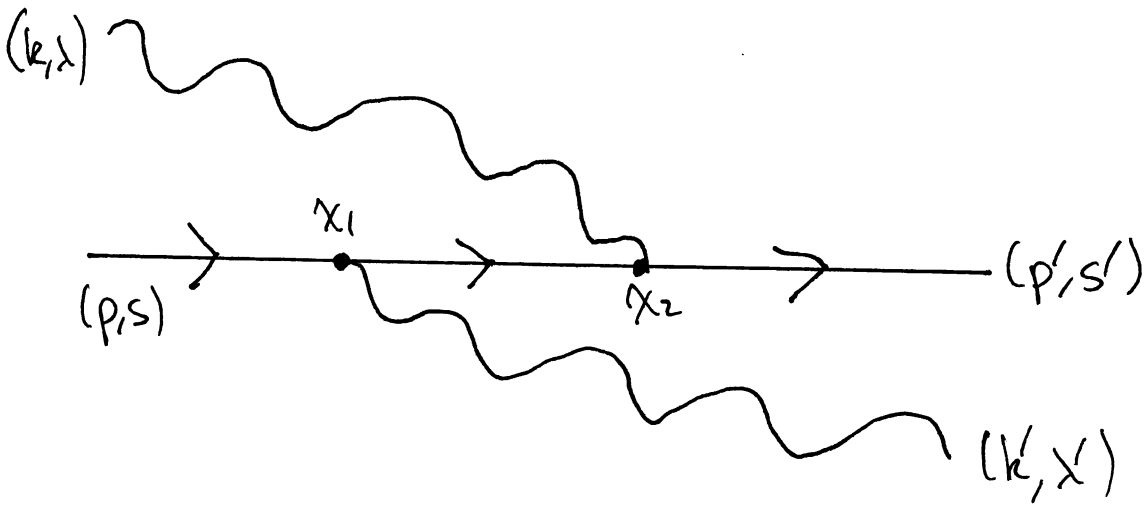
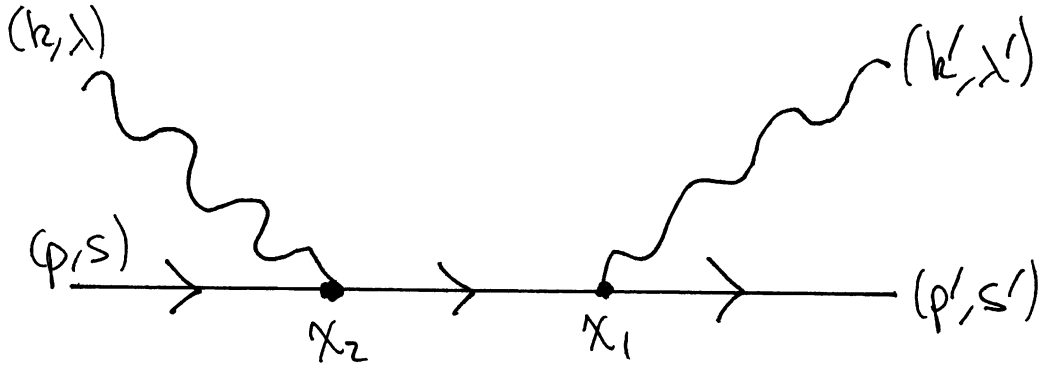


Figure 3.2.2

So to calculate S_{fi} we follow these rules. In coordinate space write down all distinct graphs made with incoming and outgoing lines for the initial and final states desired. Then join them by internal lines at vertices. The mathematical expression for S_{fi} is obtained by writing the associated factors and integrals. For example, the graph

below contributes to S_{fi} the term $S_{fi}^{(1)}$:

$$S_{fi}^{(1)} = (-ie)^2 \int d^4x_1\, d^4x_2\, \overline{u}^{(s')}(\vec{p}\,')\, e^{+ip'x_1}\, \not{\epsilon}(k',\lambda')\, e^{+ik'x_1} \times S_F(x_1 - x_2)\, \not{\epsilon}(k,\lambda)\, e^{-ikx_2}\, u^{(s)}(\vec{p})\, e^{-ipx_2} \tag{3.2.44}$$



as we found initially.

As we see it is more convenient to work in momentum space since we are starting and ending with momentum eigenstates. Thus we perform the x_1 and x_2 integrals in S_{fi}, using $S_F(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{i}{\not{q} - m}$.

$$\begin{aligned} S_{fi} = (-ie)^2 & \int d^4x_1\, d^4x_2\, e^{i(k'+p')x_1}\, e^{-i(k+p)x_2} \\ & \times \int \frac{d^4q}{(2\pi)^4} e^{-iq(x_1-x_2)}\, \overline{u}^{(s')}(\vec{p}\,')\, \not{\epsilon}(k',\lambda')\, \frac{i}{\not{q} - m}\, \not{\epsilon}(k,\lambda)\, u^{(s)}(\vec{p}) \\ & + (-ie)^2 \int d^4x_1\, d^4x_2\, e^{i(k'-p)x_2}\, e^{i(p'-k)x_1} \\ & \times \int \frac{d^4q}{(2\pi)^4} e^{-iq(x_1-x_2)}\, \overline{u}^{(s')}(\vec{p}\,')\, \not{\epsilon}(k,\lambda)\, \frac{i}{\not{q} - m}\, \not{\epsilon}(k',\lambda')\, u^{(s)}(\vec{p}). \end{aligned} \tag{3.2.45}$$

The x_1 and x_2 integrals in the first term yield

$$(2\pi)^4\, \delta^4(q - k' - p')\, (2\pi)^4\, \delta^4(k + p - q) \tag{3.2.46}$$

which equals

$$(2\pi)^4\, \delta^4(k + p - k' - p')\, (2\pi)^4\, \delta^4(k + p - q). \tag{3.2.47}$$

The x_1 and x_2 integrals in the second term yield

$$(2\pi)^4\,\delta^4(p'-k-q)\,(2\pi)^4\,\delta^4(k'-p+q) \tag{3.2.48}$$

which equals

$$(2\pi)^4\,\delta^4(p+k-p'-k')\,(2\pi)^4\,\delta^4(p-k'-q). \tag{3.2.49}$$

Hence we can perform the q-integration to obtain

$$\begin{aligned} S_{fi} = {} & (-ie)^2\,(2\pi)^4\,\delta^4(p+k-p'-k') \\ & \times \Big[\overline{u}^{(s')}(\vec{p}\,')\,\not{\epsilon}(k',\lambda')\,\frac{i}{(\not{p}+\not{k})-m}\,\not{\epsilon}(k,\lambda)\,u^{(s)}(\vec{p}) \\ & + \overline{u}^{(s')}(\vec{p}\,')\,\not{\epsilon}(k,\lambda)\,\frac{i}{(\not{p}-\not{k}')-m}\,\not{\epsilon}(k',\lambda')\,u^{(s)}(\vec{p})\Big]. \end{aligned} \tag{3.2.50}$$

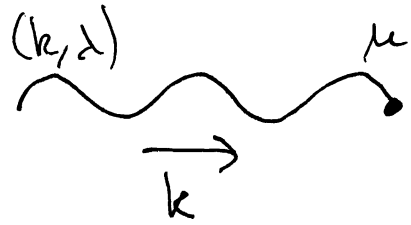
Again we can represent this mathematical expression graphically by Feynmann diagrams in momentum space.

Each incoming or outgoing particle is represented by

Incoming particles :

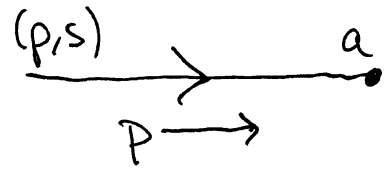
photon

$\epsilon_\mu(k,\lambda) \longleftrightarrow$



electron

$u_a^{(s)}(\vec{p}) \longleftrightarrow$



Outgoing particles :

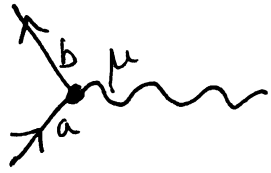
Photon

$\epsilon_\mu(k,\lambda) \longleftrightarrow$

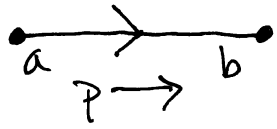
Electron

$\overline{u}_a^{(s)}(\vec{p}) \longleftrightarrow$ 

These interact at a vertex where the electron absorbs the photon and propagates as a virtual (its momentum $q^2 \neq m^2$) particle with the momentum $p+k$ of the $e^- + \gamma$. The interaction is represented by a vertex

$-ie\gamma^\mu_{ba} \longleftrightarrow$ 

and the propagating electron by a line joining the first vertex with the second where the virtual e^- emits the final state photon and becomes a real particle again ($p'^2 = m^2$), the final e^-. This is called an internal line – it joins two vertices,

$\left(\frac{i}{\not{p} - m}\right)_{ba} \longleftrightarrow$ 

The final vertex is exactly as the first.

Finally we always leave a $(2\pi)^4\,\delta^4(P_i - P_f)$ where P_i is the sum of the graph's initial energy-momenta and P_f is the sum of the graph's final particles' energy-momenta.

Hence we graphically represent S_{fi} as

$S_{fi} =$ Fig. 3.23

where we always label the momentum flow through the graph.

Using our graphical rules we can write down the mathematical expression for S_{fi} quickly to obtain the above momentum space result.

Suppose instead of initial and final state electrons we had positrons. The positron Compton scattering amplitude is found the same way. We have that for different initial and final momenta

$$\begin{aligned} S(e^+\gamma \to e^+\gamma) &= \langle (\vec{p}\,'), \frac{(-1)^{s'}}{2}, +), (\vec{k}', \lambda')|S|(\vec{p}), \frac{(-1)^s}{2}, +), (\vec{k}, \lambda)\rangle \\ &= \langle 0|a_{(\lambda')}(\vec{k}')\, d_{s'}(\vec{p}\,')\left(\mathrm{T}e^{-ie\int d^4x\, \mathrm{N}\left[A_\mu \overline{\Psi}\gamma^\mu \Psi\right](x)}\right) d_s^\dagger(\vec{p})\, a_{(\lambda)}(\vec{k})|0\rangle. \end{aligned} \tag{3.2.51}$$

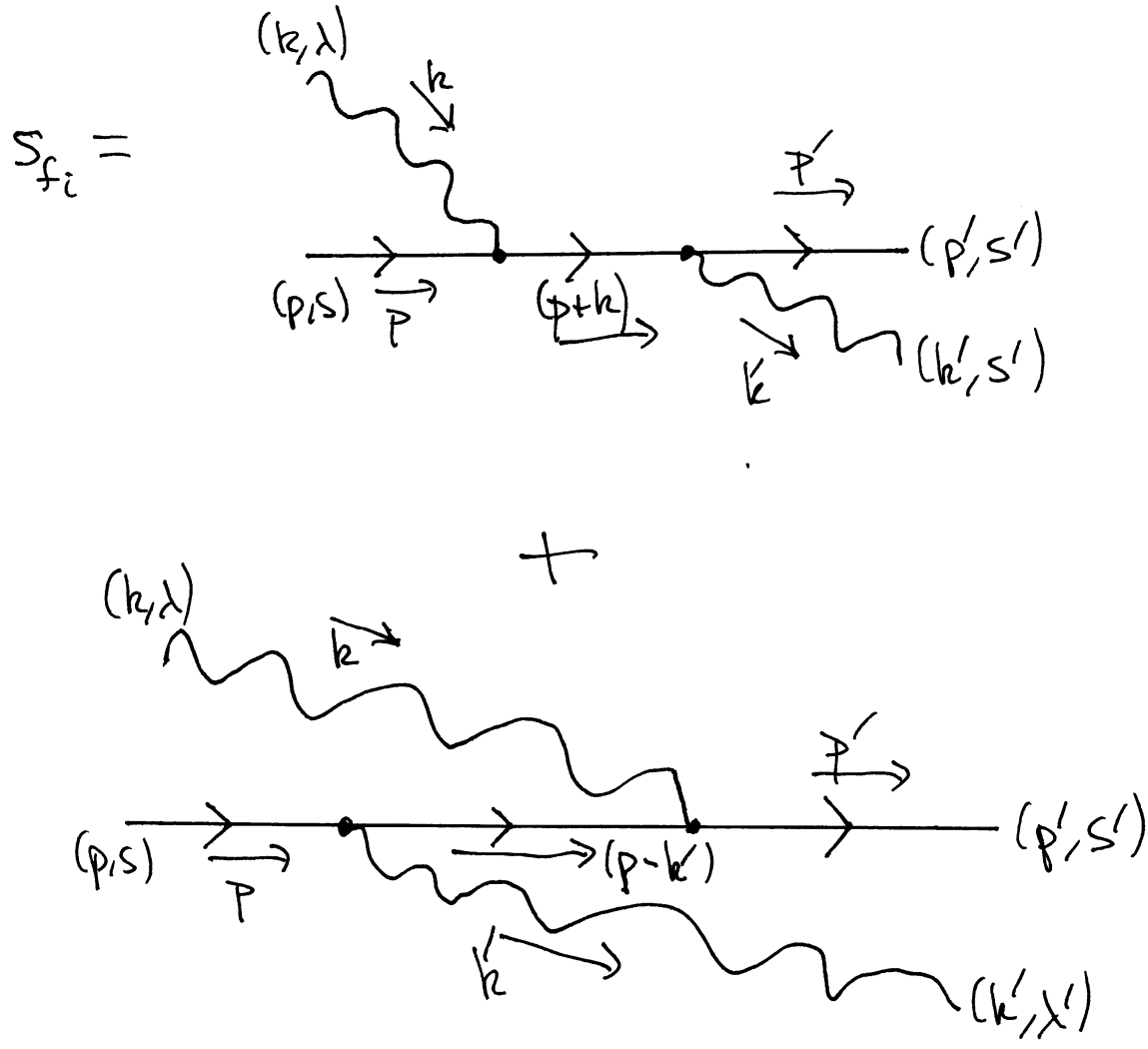


Figure 3.2.3

Using Wick's Theorem again we find the only terms contributing to scattering are the fourth and fifth terms in equation (3.2.21) which are equal and give a factor of 2 to cancel the $\frac{1}{2!}$. Thus

$$
\begin{aligned}
S(e^+\gamma \to e^+\gamma) = (-ie)^2 \int d^4x_1\, d^4x_2\, \langle 0|\mathrm{T}\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle \\
\times \langle 0|a_{(\lambda')}(\vec{k}')\, d_{s'}(\vec{p}\,')\,\mathrm{N}\big[A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)\,\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)\big] \\
\times d_s^\dagger(\vec{p})\, a_{(\lambda)}^\dagger(\vec{k})|0\rangle \gamma^{\mu_1}_{a_1 b_1} \gamma^{\mu_2}_{a_2 b_2}
\end{aligned}
\tag{3.2.52}
$$

The VEV of the creation and annihilation operators yields two terms again

$$\begin{aligned}
&\langle 0|a_{(\lambda')}(\vec{k}')\, d_{s'}(\vec{p}\,')\, \mathrm{N}\big[A_{\mu_1}(x_1)\, A_{\mu_2}(x_2)\, \overline{\Psi}_{a_1}(x_1)\, \Psi_{b_2}(x_2)\big]\, d_s^\dagger(\vec{p})\, a_{(\lambda)}^\dagger(\vec{k})|0\rangle \\
&\quad = \Big\{ \big(\epsilon_{\mu_1}(\vec{k}',\lambda')\, e^{+ik'x_1}\big)\, \big(\epsilon_{\mu_2}(\vec{k},\lambda)\, e^{-ikx_2}\big) \\
&\qquad + \big(\epsilon_{\mu_2}(\vec{k}',\lambda')\, e^{+ik'x_2}\big)\, \big(\epsilon_{\mu_1}(\vec{k},\lambda)\, e^{-ikx_1}\big) \Big\} \\
&\qquad \times (-1)\, \langle 0|d_{s'}(\vec{p}\,')\, \Psi_{b_2}^-(x_2)|0\rangle \langle 0|\overline{\Psi}_{a_1}^+(x_1)\, d_s^\dagger(\vec{p})|0\rangle .
\end{aligned} \tag{3.2.53}$$

(Note the factor of (-1) in the last line arises since we had to interchange $\overline{\Psi}_{a_1}$ and Ψ_{b_2}.) Thus

$$\begin{aligned}
&\langle 0|a_{(\lambda')}(\vec{k}')\, d_{s'}(\vec{p}\,')\, \mathrm{N}\big[A_{\mu_1}(x_1)\, A_{\mu_2}(x_2)\, \overline{\Psi}_{a_1}(x_1)\, \Psi_{b_2}(x_2)\big]\, d_s^\dagger(\vec{p})\, a_{(\lambda)}(\vec{k})|0\rangle \\
&\quad = \Big\{ \big(\epsilon_{\mu_1}(\vec{k}',\lambda')\, e^{+ik'x_1}\big)\, \big(\epsilon_{\mu_2}(\vec{k},\lambda)\, e^{-ikx_2}\big) \\
&\qquad + \big(\epsilon_{\mu_2}(\vec{k}',\lambda')\, e^{+ik'x_2}\big)\, \big(\epsilon_{\mu_1}(\vec{k},\lambda)\, e^{-ikx_1}\big) \Big\} \\
&\qquad \times (-1) v_{b_2}^{(s')}(\vec{p}\,')\, e^{+ip'x_2} \overline{v}_{a_1}^{(s)}(\vec{p})\, e^{-ipx_1} .
\end{aligned} \tag{3.2.54}$$

And thus

$$\begin{aligned}
&S(e^+\gamma \to e^+\gamma) \\
&\quad = (-ie)^2 \int d^4x_1\, d^4x_2\, (-1) \Big\{ \big(\overline{v}^{(s)}(\vec{p})\, e^{-ipx_1}\big)\, \big(\not{\epsilon}(\vec{k}',\lambda')\, e^{+ik'x_1}\big) \\
&\qquad \times S_F(x_1 - x_2)\, \big(\not{\epsilon}(\vec{k},\lambda)\, e^{-ikx_2}\big)\, \big(v^{(s)}(\vec{p}\,')\, e^{+ip'x_2}\big) \\
&\qquad + \big(\overline{v}^{(s)}(\vec{p})\, e^{-ipx_1}\big)\, \big(\not{\epsilon}(\vec{k},\lambda)\, e^{-ikx_2}\big)\, S_F(x_1 - x_2) \\
&\qquad \times \big(\not{\epsilon}(\vec{k}',\lambda')\, e^{+ik'x_1}\big)\, \big(v^{(s')}(\vec{p}\,')\, e^{+ip'x_2}\big) \Big\}
\end{aligned} \tag{3.2.55}$$

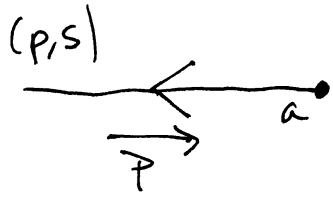
performing the Fourier transform as before we find

$$\begin{aligned}
S(e^+\gamma \to e^+\gamma) &= (-ie)^2\, (2\pi)^4\, \delta^4(p + k - p' - k') \\
&\quad \times \Big[\overline{v}^{(s)}(\vec{p})\, \not{\epsilon}(\vec{k},\lambda)\, \frac{i}{(-\not{p} - \not{k}) - m}\, \not{\epsilon}(\vec{k}',\lambda')\, v^{(s')}(\vec{p}\,') \\
&\qquad + \overline{v}^{(s)}(\vec{p})\, \not{\epsilon}(\vec{k}',\lambda')\, \frac{i}{(-\not{p} + \not{k}') - m}\, \not{\epsilon}(\vec{k},\lambda)\, \big(v^{(s)}(\vec{p}\,')\big) \Big] .
\end{aligned} \tag{3.2.56}$$

Thus we see that the Feynman rules for external positron lines are

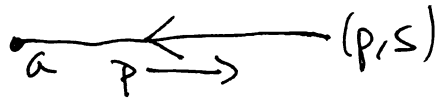
Incoming positron

$\overline{v}_a^{(s)}(\vec{p}) \longleftrightarrow$



Outgoing positron

$v_a^{(s)}(\vec{p}) \longleftrightarrow$



The extra overall (-1) results from the interchange of fermion operators in the normal ordering operator. That is, $\overline{v}^{(s)}(\vec{p})\, v^{(s)}(\vec{p}\,')$ is in the order opposite to the original state creation operators $d(\vec{p}\,')\, d^\dagger(\vec{p})$; thus a (-1) interchange factor is obtained.

The corresponding Feynman graphs for positron Compton scattering are shown in Figure 3.2.4.

Note if $(p+k)$ flows opposite the direction of the arrow on the propagator $-(p+k)$ flows in the direction of the arrow. Hence the propagator factor is $\frac{i}{(-\not{p}-\not{k})-m}$ in the first graph and $\frac{i}{(-\not{p}+\not{k}')-m}$ in the second graph.

In order to complete the discussion on Feynman rules for QED let's consider the second-order contribution to the vacuum-to-vacuum transition amplitude. That is let's evaluate the phase difference between initial and final vacuum states as discussed in the previous section

$$\begin{aligned}\langle 0|S|0\rangle &= \langle 0|\mathrm{T}e^{-ie\int d^4x\,\mathrm{N}\left[A_\mu \overline{\Psi}\gamma^\mu \Psi\right](x)}|0\rangle \\ &= 1+\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\,\mathrm{T}\Big[\mathrm{N}\left[A_{\mu_1}(x_1)\,\overline{\Psi}(x_1)\,\gamma^{\mu_1}\,\Psi(x_1)\right] \\ &\qquad \times \mathrm{N}\left[A_{\mu_2}(x_2)\,\overline{\Psi}(x_2)\,\gamma^{\mu_2}\,\Psi(x_2)\right]\Big]|0\rangle \\ &\quad + \mathcal{O}(e^3).\end{aligned} \tag{3.2.57}$$

According to Wick's theorem this is just given by

$$\begin{aligned}\langle 0|S|0\rangle = 1+\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\,\langle 0|\mathrm{T}A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)|0\rangle\gamma^{\mu_1}_{a_1b_1}\gamma^{\mu_2}_{a_2b_2} \\ \times\langle 0|\mathrm{T}\overline{\Psi}_{a_1}(x_1)\,\Psi_{b_2}(x_2)|0\rangle\langle 0|\mathrm{T}\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle.\end{aligned} \tag{3.2.58}$$

Notice that we will have a (-1) factor due to the interchange of $\overline{\Psi}_{a_1}$ and Ψ_{b_2}, so

$$\begin{aligned}&\langle 0|S|0\rangle \\ &= 1+\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\,\langle 0|\mathrm{T}A_{\mu_1}(x_1)\,A_{\mu_2}(x_2)|0\rangle(-\gamma^{\mu_1}_{a_1b_1}) \\ &\quad\times\langle 0|\mathrm{T}\Psi_{b_1}(x_1)\,\overline{\Psi}_{a_2}(x_2)|0\rangle\gamma^{\mu_2}_{a_2b_2}\langle 0|\mathrm{T}\Psi_{b_2}(x_2)\,\overline{\Psi}_{a_1}(x_1)|0\rangle.\end{aligned} \tag{3.2.59}$$

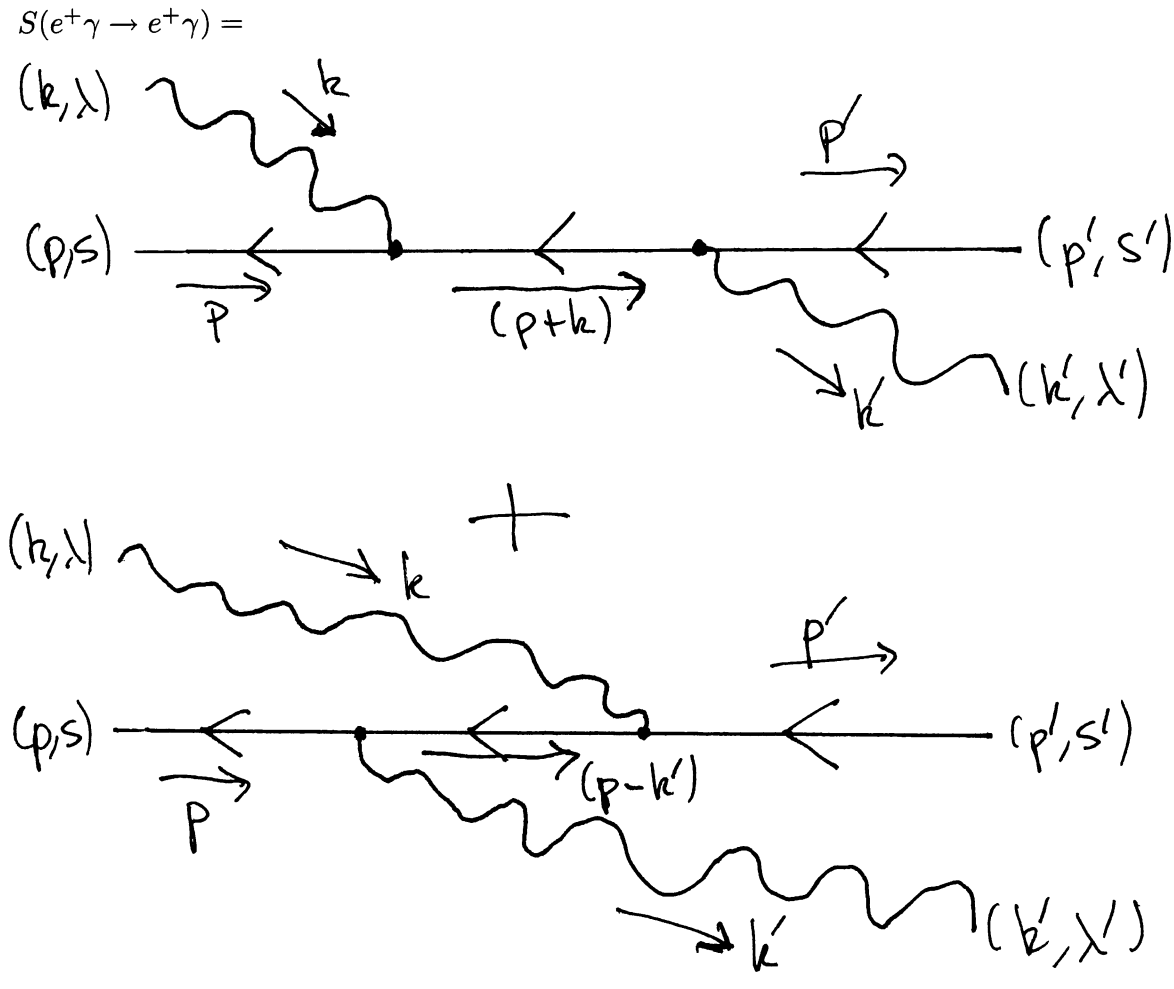


Figure 3.2.4

Now let's recall that the IP fields are free fields with the Feynman propagators given by

$$\begin{aligned}\Delta_{F\mu\nu}(x-y) &= \langle 0|\mathrm{T}A_\mu(x)\,A_\nu(y)|0\rangle \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\,\tilde{\Delta}_{F\mu\nu}(q) \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\left[\frac{-i}{k^2+i\epsilon}g_{\mu\nu} - \frac{i(\alpha-1)}{k^2+i\epsilon}\frac{k_\mu k_\nu}{(k^2+i\epsilon)}\right]\end{aligned} \tag{3.2.60}$$

in arbitrary Stueckelberg gauge (since S is gauge invariant we could calculate in any

gauge and still obtain the same answer). Of course we will work with $\alpha = 1$, the Feynman gauge.
And as we have seen

$$\begin{aligned} S_F(x-y) &= \langle 0|\mathrm{T}\Psi(x)\,\overline{\Psi}(y)|0\rangle \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{i}{\not{k}-m}. \end{aligned} \tag{3.2.61}$$

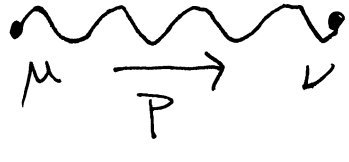
Substituting in the above we find

$$\begin{aligned} \langle 0|S|0\rangle &= 1 - \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4} \\ &\quad \times e^{-ik(x_1-x_2)}\, e^{-ip(x_1-x_2)}\, e^{-iq(x_2-x_1)} \\ &\quad \times \frac{-i}{k^2+i\epsilon} g_{\mu_1\mu_2}\gamma^{\mu_1}_{a_1b_1}\Big(\frac{i}{\not{p}-m}\Big)_{b_1a_2}\gamma^{\mu_2}_{a_2b_2}\Big(\frac{i}{\not{q}-m}\Big)_{b_2a_1} \\ &= 1 - \frac{(-ie)^2}{2!}\int \frac{d^4k}{(2\pi)^4}\,\frac{d^4p}{(2\pi)^4}\,\frac{d^4q}{(2\pi)^4}\,(2\pi)^4\,\delta^4(k+p-q) \\ &\quad \times (2\pi)^4\,\delta^4(k+p-q)\,\frac{-i}{k^2+i\epsilon}\mathrm{Tr}\Big[\gamma^\mu \frac{i}{\not{p}-m}\gamma_\mu \frac{i}{(\not{p}+\not{k})-m}\Big] \\ &= 1 - \frac{(-ie)^2}{2!}(2\pi)^4\,\delta^4(0)\int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\,\frac{-i}{(k_1-k_2)^2+i\epsilon} \\ &\qquad \mathrm{Tr}\Big[\gamma^\mu \frac{i}{\not{k}_1-m}\gamma_\mu \frac{i}{\not{k}_2-m}\Big]. \end{aligned} \tag{3.2.62}$$

where, in the last step, we have changed integration variables $k = k_1 - k_2$, $p = k_2$, and thus $p + k = k_1$.

Thus we obtain four new rules from this calculation.

1). The e^- and e^+ meet at a vertex and annihilate giving rise to a photon which propagates from this point to the next where it pair produces the e^- - e^+. Hence where we have <u>internal</u> photon lines connecting two vertices, we represent this with a line



with momentum p flowing through it and it corresponds to the momentum-space

propagator

$$\tilde{\Delta}_{F\mu\nu}(p) = \frac{-i}{p^2 + i\epsilon} g_{\mu\nu} \qquad \text{(in Feynman gauge).}$$

2). For each line we had a Fourier transform momentum integral to perform; at each vertex a space-time integral to perform. They result in momentum-conserving delta functions at every vertex – that is, they result in overall conservation and conserving delta functions at every vertex minus one $(V-1)$. The energy-momentum flowing into a vertex equals the energy-momentum flowing out of a vertex. Since there are $V-1$ delta functions of energy-momentum variables (V = the number of vertices in the diagram) and L energy-momentum integrals (L = the number of internal lines), there are left $4m$ energy-momentum integrations to do where

$$m = L - V + 1, \qquad \text{the number of loops in a graph.}$$

(This is a topological relation that for any diagram with L internal lines and V vertices there are $m = L - V + 1$ loops.)

In the case above we have three internal lines and two vertices, so $m = 3-2+1 = 2$. As we found there are $4m = 8$ integrations left – called loop integrals or internal integrations. Hence, associated with each loop of the diagram is an independent momentum flow through the lines of the loop which obeys energy-momentum conservation at the vertices and which we integrate over $\int \frac{d^4k}{(2\pi)^4}$.

3). For each fermion loop we have a trace over the Dirac indices of the fermion propagators and Dirac gamma matrices of the vertices in the closed loop. Further the trace will always involve an odd number of Fermi field interchanges among the normal products of the interaction Hamiltonian involved in the trace, i.e.

$$\begin{aligned}
&\langle 0|\mathrm{T}\Big[\mathrm{N}\big[\overline{\Psi}(1)\,\Psi(1)\big]\cdots\mathrm{N}\big[\overline{\Psi}(n)\,\Psi(n)\big]\Big]|0\rangle \\
&\quad = \sum_P (-1)^P \langle 0|\mathrm{T}\overline{\Psi}(1)\,\Psi(n)|0\rangle\langle 0|\mathrm{T}\Psi(1)\,\overline{\Psi}(2)|0\rangle \\
&\qquad\qquad \langle 0|\mathrm{T}\Psi(2)\,\overline{\Psi}(3)|0\rangle\cdots\langle 0|\mathrm{T}\Psi(n-1)\,\overline{\Psi}(n)|0\rangle \\
&\quad = -\sum_P (-1)^P \langle 0|\mathrm{T}\Psi(n)\,\overline{\Psi}(1)|0\rangle\langle 0|\mathrm{T}\Psi(1)\,\overline{\Psi}(2)|0\rangle \\
&\qquad \times \langle 0|\mathrm{T}\Psi(2)\,\overline{\Psi}(3)|0\rangle\cdots\langle 0|\mathrm{T}\Psi(n-1)\,\overline{\Psi}(n)|0\rangle.
\end{aligned} \tag{3.2.63}$$

Thus for each closed fermion loop there is an extra factor of (-1).

4). Notice that $\langle 0|S|0\rangle \neq 0$. Further it equals infinity due not only to the $\delta^4(0)$ for energy- momentum conservation but also to the fact that the loop integrals are divergent for large integration momenta. Hence, for consistency, we would like to see that this infinity is renormalized away also. Physically we are calculating the stability of the vacuum state; i.e. the transition probability amplitude for the vacuum to stay the vacuum is $\langle 0|S|0\rangle$.

Note that due to energy-momentum conservation $S|0\rangle$ must be proportioned to $|0\rangle$ again,

$$\mathcal{P}^\mu\big(S|0\rangle\big) = \big[\mathcal{P}^\mu, S\big]|0\rangle = 0. \tag{3.2.64}$$

Thus $S|0\rangle$ is the zero eigenstate of $\mathcal{H}$ also, implying

$$S|0\rangle = \omega|0\rangle \tag{3.2.65}$$

The norm of this state is

$$\langle 0|S^\dagger S|0\rangle = |\omega|^2\langle 0|0\rangle. \tag{3.2.66}$$

But since S is unitary

$$S^\dagger S = 1 \quad \Longrightarrow \quad \langle 0|0\rangle = |\omega|^2\langle 0|0\rangle \tag{3.2.67}$$

$$\Longrightarrow \quad |\omega|^2 = 1. \tag{3.2.68}$$

So ω is a phase only

$$\omega = e^{i\varphi} = \langle 0|S|0\rangle. \tag{3.2.69}$$

Now, since the final states can differ from the initial states by this phase, when we calculate we should normalize by this transition amplitude. Hence we may define the S operator to leave the vacuum invariant : $S|0\rangle \equiv |0\rangle$. Thus we divide our old definition by ω as discussed in section 3.1,

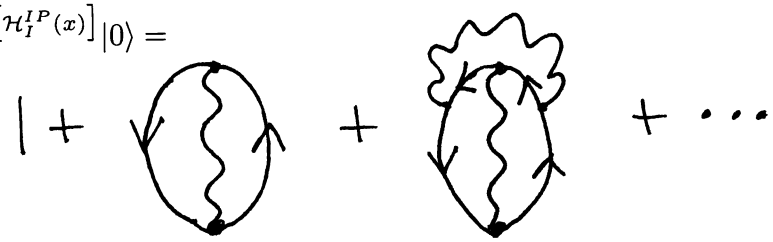
$$S = \frac{\mathrm{T}e^{-i\int d^4x\, \mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}}{\langle 0|\mathrm{T}e^{-i\int d^4x\, \mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|0\rangle}. \tag{3.2.70}$$

So $S|0\rangle = |0\rangle$ and the S matrix elements are

$$S_{fi} = \frac{\subset f|\mathrm{T}e^{-i\int d^4x\, \mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|i \supset}{\langle 0|\mathrm{T}e^{-i\int d^4x\, \mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|0\rangle}. \tag{3.2.71}$$

We can show that this phase factors out of the numerator also so that it will cancel the denominator. Graphically this is easy to see since the Feynman graphs contributing to

$\langle 0|\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|0\rangle =$



have no external lines on them, that is, no incoming or outgoing particles. These are called vacuum fluctuations or vacuum bubbles. The graphs that contribute to any $\subset f|T\exp(\cdots)|i\supset$ matrix element have the structure of being written as disjoint subgraphs of two types. One type will have at least one external line on it while the other type will be a vacuum bubble, having no external lines. We have already seen such possibilities in studying Compton scattering. In the cases we ignored due to energy-momentum conservation were graphs like

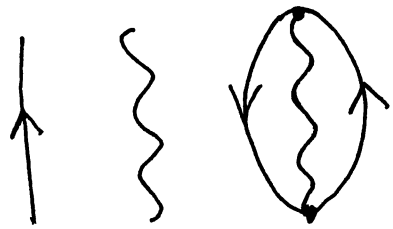


Figure 3.2.5

If we went on to higher order we would find an expansion like that seen in Figure 3.2.6.

We see that the vacuum bubbles always factor out of the diagrammatic sum and cancel the denominator as shown in Figure 3.2.7. Thus

$\subset f|\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|i\supset =$ Fig. 3.2.7.

for every fixed set of subgraphs with external lines we can sum over all possible (including 1) vacuum bubbles. Hence the sum over all $i\to f$ Feynman graphs Γ_{fi} is a

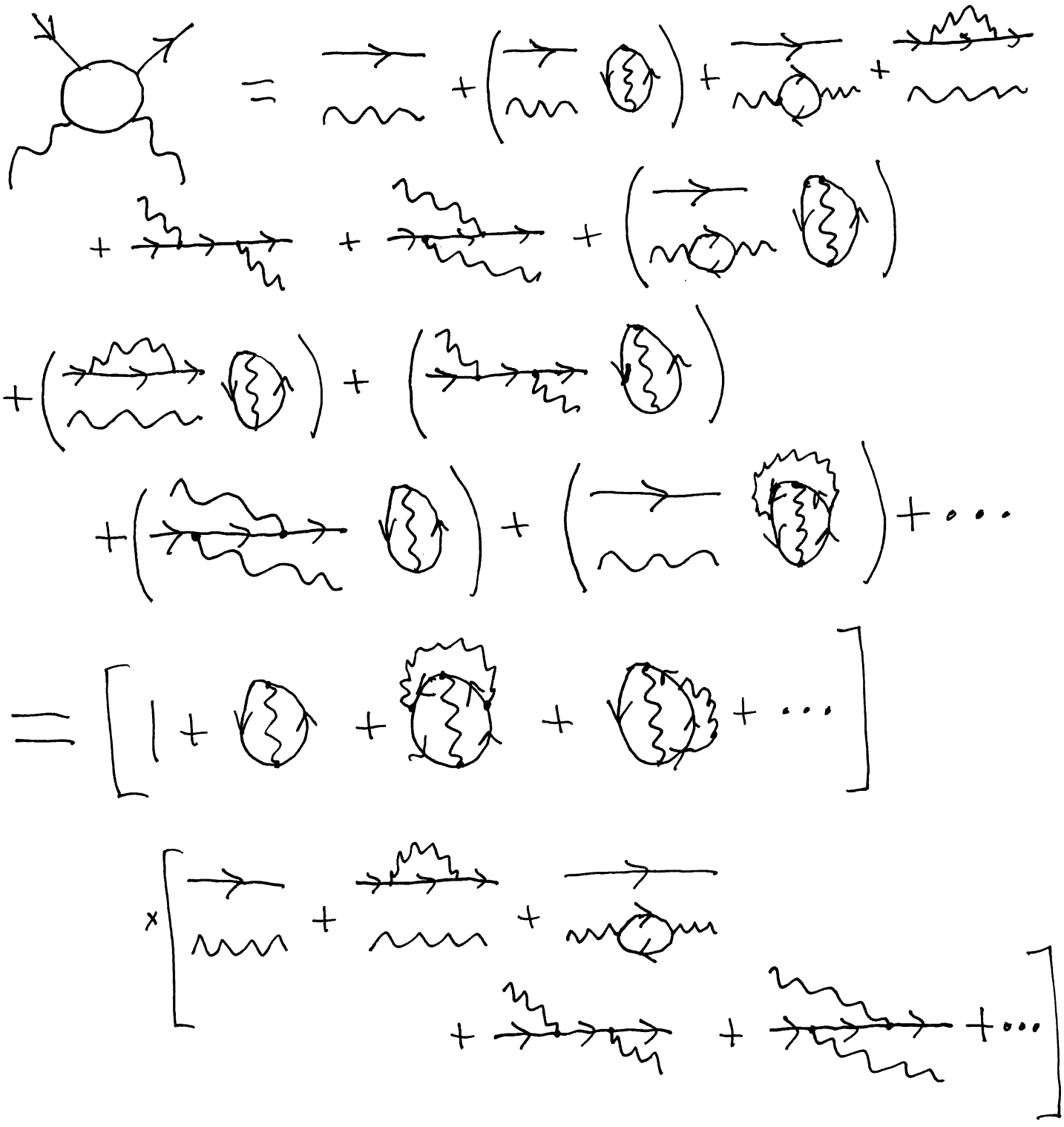


Figure 3.2.6

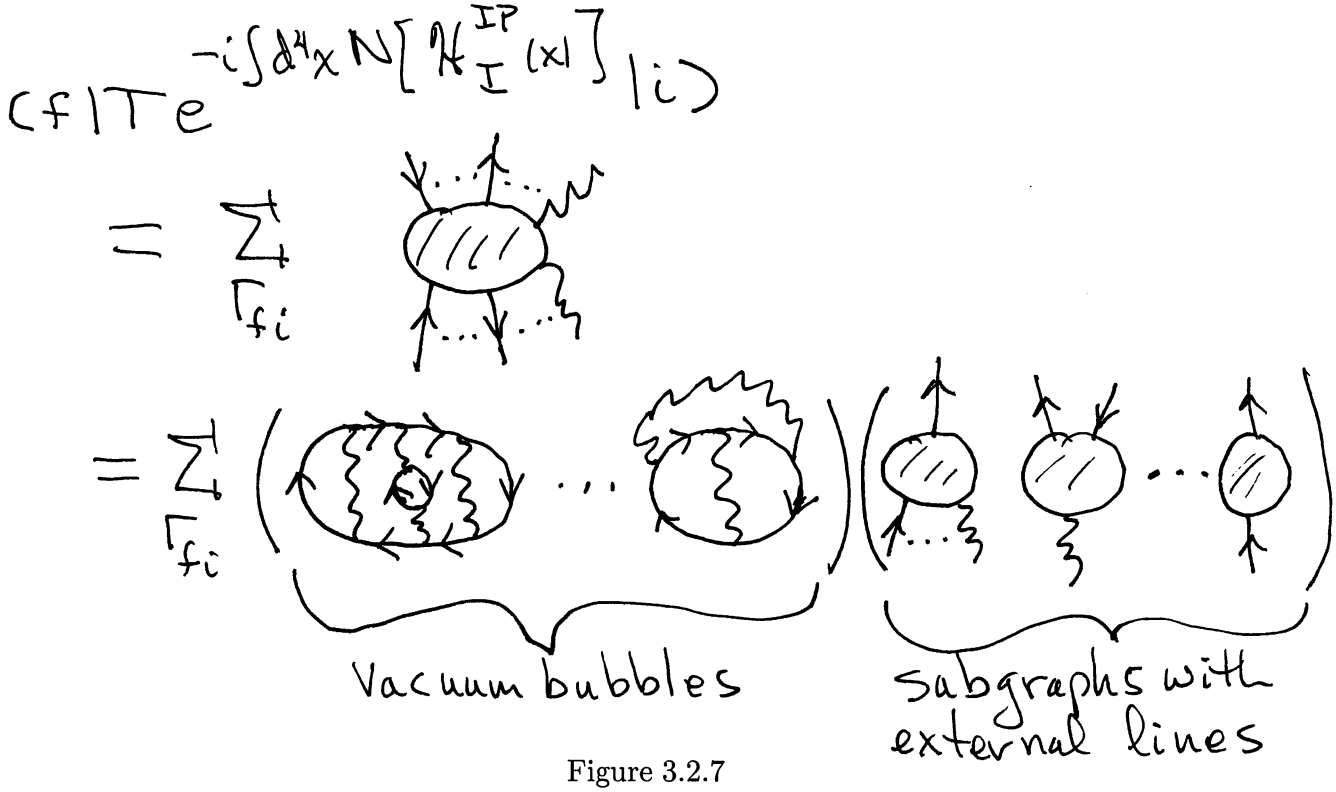


Figure 3.2.7

sum over all vacuum bubbles (including 1) times a sum over $i \to f$ Feynman diagrams with external lines on each subdiagram. Thus we have the structure shown in Figure 3.2.8.

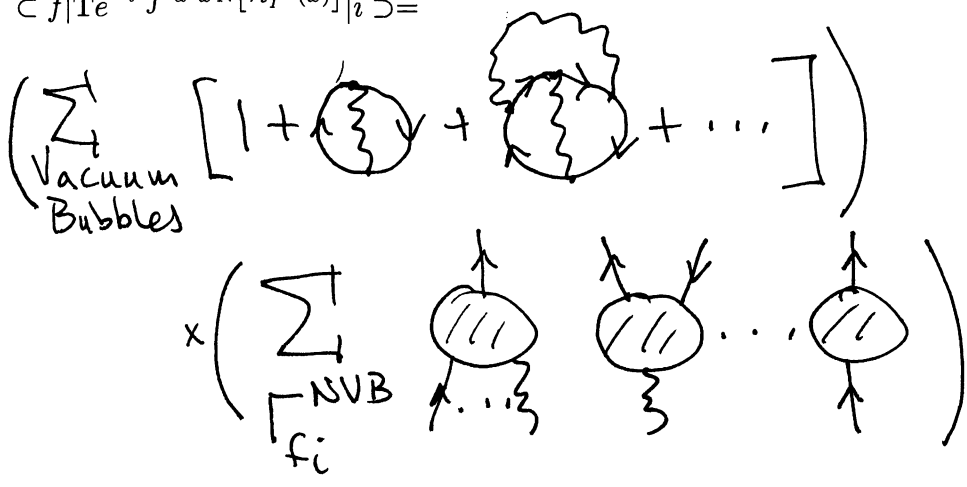


Figure 3.2.8

where $\sum_{\Gamma_{fi}^{\mathrm{NVB}}}$ is the sum over all Feynman diagrams with i-incoming and f–

outgoing lines and no vacuum bubbles (NVB).

Hence

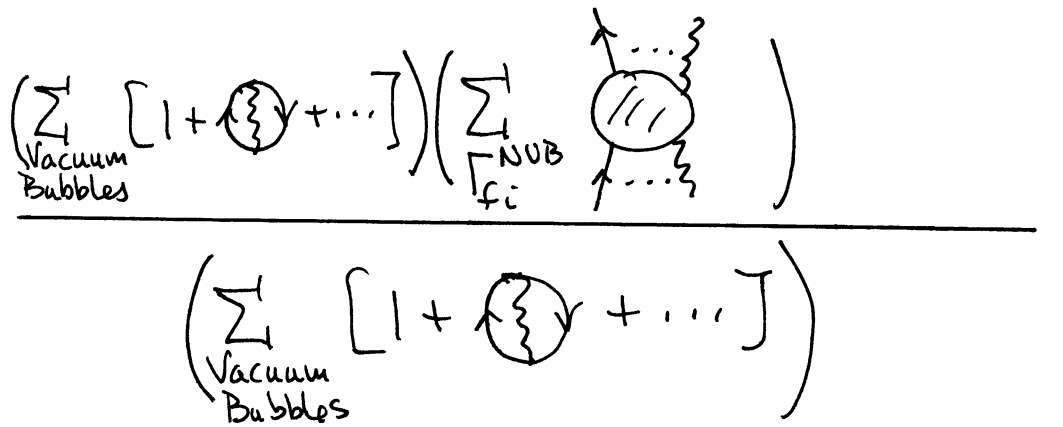
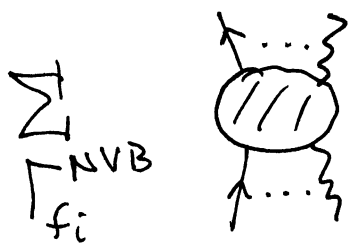
$S_{fi} =$ 

Figure 3.2.9

The vacuum bubble contribution cancels so that

$S_{fi} =$ 

$$= \subset f|\mathrm{T}_{\mathrm{NVB}} e^{-i\int d^4x\, \mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|i\supset \tag{3.2.72}$$

where "NVB" indicates that no vacuum bubble graphs contribute.

Analytically we can arrive at the same result by introducing time-ordered functions and operators that have the vacuum bubbles subtracted out of themselves. Hence, separating out the vacuum bubble contributions to
$\mathrm{T}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$, we define the no-vacuum-bubble time-ordered product of operators

$\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$ recursively

$$
\begin{aligned}
&\mathrm{T}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\\
&\qquad=\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\\
&\qquad\quad+\sum_{i=1}^{n}(-1)^P\,\langle 0|\mathrm{T}\mathcal{O}_i(x_i)|0\rangle\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_1(x_1)\cdots\cancel{\mathcal{O}_i(x_i)}\cdots\mathcal{O}_n(x_n)\\
&\qquad\quad+\sum_{\substack{\text{pairs}\\ i<j}}(-1)^P\,\langle 0|\mathrm{T}\mathcal{O}_i(x_i)\,\mathcal{O}_j(x_j)|0\rangle\\
&\qquad\qquad\times\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_1(x_1)\cdots\cancel{\mathcal{O}_i(x_i)}\cdots\cancel{\mathcal{O}_j(x_j)}\cdots\mathcal{O}_n(x_n)\\
&\qquad\quad+\cdots+\langle 0|\mathrm{T}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle
\end{aligned}
\tag{3.2.73}
$$

$$
\begin{aligned}
&=\sum_{\substack{(1,\ldots,n)\to(i_1,\ldots,i_l)(i_{l+1},\ldots,i_n)\\ i_1<i_2\cdots<i_l\\ i_{l+1}<i_{i+2}\cdots<i_n}}(-1)^P\,\langle 0|\mathrm{T}\mathcal{O}_{i_1}(x_{i_1})\cdots\mathcal{O}_{i_l}(x_{i_l})|0\rangle\\
&\qquad\qquad\times\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_{i_{l+1}}(x_{i_{l+1}})\cdots\mathcal{O}_{i_n}(x_{i_n}).
\end{aligned}
$$

Note that $\langle 0|\mathrm{T}_{\mathrm{NVB}}\mathcal{O}_1(x_1)\cdots\mathcal{O}_l(x_l)|0\rangle=0$; $\mathrm{T}_{\mathrm{NVB}}(\cdots)$ has no vacuum bubbles left in it!

Now for identical operators that are integrated over, this sum becomes (in the bose case)

$$
\begin{aligned}
&\int d^4x_1\cdots d^4x_n\,\mathrm{T}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\\
&\qquad=\int d^4x_1\cdots d^4x_n\sum_{l=0}^{n}\frac{n(n-1)\cdots(n-l+1)}{l!}\\
&\qquad\qquad\times\langle 0|\mathrm{T}\mathcal{O}(x_1)\cdots\mathcal{O}(x_l)|0\rangle\mathrm{T}_{\mathrm{NVB}}\mathcal{O}(x_{l+1})\cdots\mathcal{O}(x_n)\\
&\qquad=\sum_{l=0}^{n}\frac{n!}{l!\,(n-l)!}\int d^4x_1\cdots d^4x_l\,\langle 0|\mathrm{T}\mathcal{O}(x_1)\cdots\mathcal{O}(x_l)|0\rangle\\
&\qquad\qquad\times\int d^4x_{l+1}\cdots d^4x_n\,\mathrm{T}_{\mathrm{NVB}}\mathcal{O}(x_{l+1})\cdots\mathcal{O}(x_n)
\end{aligned}
\tag{3.2.74}
$$

The factor $\frac{n(n-1)\cdots(n-l+1)}{l!}=\frac{n!}{l!\,(n-l)!}$ is exactly the number of ways to pick l things out of n when the things are indistinguishable.

So applying this result we find

$$\begin{aligned}
&\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]} \\
&\quad=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l}{l!}\int d^4x_1\cdots d^4x_l\,\langle 0|\mathrm{TN}\left[\mathcal{H}_I^{IP}(x_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(x_l)\right]|0\rangle \\
&\qquad\times\frac{(-i)^{n-l}}{(n-l)!}\int d^4y_1\cdots d^4y_{n-l}\,\mathrm{T}_{\mathrm{NVB}}\Big[\mathrm{TN}\left[\mathcal{H}_I^{IP}(y_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(y_{n-l})\right]\Big].
\end{aligned} \tag{3.2.75}$$

Now cryptically calling the first terms

$$V_l\equiv\frac{(-1)^l}{l!}\int d^4x_1\cdots d^4x_l\,\langle 0|\mathrm{TN}\left[\mathcal{H}_I^{IP}(x_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(x_l)\right]|0\rangle \tag{3.2.76}$$

and the second set of terms

$$\mathrm{T}_{n-l}\equiv\frac{(-i)^{n-l}}{(n-l)!}\int d^4y_1\cdots d^4y_{n-l}\,\mathrm{T}_{\mathrm{NVB}}\Big[\mathrm{TN}\left[\mathcal{H}_I^{IP}(y_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(y_{n-l})\right]\Big], \tag{3.2.77}$$

we see that our sum becomes

$$\begin{aligned}
\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]} &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}V_lT_{n-l} \\
&= V_0T_0 \\
&+ V_0T_1+V_1T_0 \\
&+ V_0T_2+V_1T_1+V_2T_0 \\
&+\cdots \\
&+ V_0T_{n-1}+V_1T_{n-2}+V_2T_{n-3}\cdots+V_{n-1}T_0 \\
&+ V_0T_n+V_1T_{n-1}+V_2T_{n-2}\cdots+V_{n-1}T_1+V_nT_0 \\
&+\cdots.
\end{aligned} \tag{3.2.78}$$

So instead of summing in rows as we have let's re-sum in columns. Then we have

$$\begin{aligned}
&\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]} \\
&\quad=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} V_l T_m \\
&\quad=\Big[\sum_{l=0}^{\infty}\frac{(-i)^l}{l!}\int d^4x_1\cdots d^4x_l\,\langle 0|\mathrm{TN}\left[\mathcal{H}_I^{IP}(x_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(x_l)\right]|0\rangle\Big] \\
&\qquad\times\Big[\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int d^4y_1\cdots d^4y_m\,\mathrm{T}_{\mathrm{NVB}}\Big[\mathrm{TN}\left[\mathcal{H}_I^{IP}(y_1)\right]\cdots\mathrm{N}\left[\mathcal{H}_I^{IP}(y_m)\right]\Big]
\end{aligned} \tag{3.2.79}$$

and we find the desired result

$$\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}=\langle 0|\mathrm{T}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|0\rangle\,\mathrm{T}_{\mathrm{NVB}}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}. \tag{3.2.80}$$

The phase ω factors out, hence

$$S=\mathrm{T}_{\mathrm{NVB}}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}. \tag{3.2.81}$$

And we find

$$S_{fi}=\subset f|\mathrm{T}_{\mathrm{NVB}}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|i\supset \tag{3.2.82}$$

has no vacuum bubbles contributing.

Thus we are finally in a position to summarize the rules for calculating S-matrix elements as sums over Feynman graphs. Even though we have gleaned our rules from studying a few examples, the same procedure goes through in different processes with the same correspondence between mathematical expressions and the components of the Feynman graphs. Hence we can list the rules for calculating any $\subset f|S|i\supset$ for QED. The transition amplitude for $|i\supset\rightarrow|f\supset$ will be written as a sum of Feynman diagrams

$$\begin{aligned}
S_{fi}&=\frac{\subset f|\mathrm{T}_{\mathrm{NVB}}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|i\supset}{\langle 0|\mathrm{T}_{\mathrm{NVB}}e^{-i\int d^4x\,\mathrm{N}\left[\mathcal{H}_I^{IP}(x)\right]}|0\rangle} \\
&=\sum_{\Gamma\in G_{fi}} S_{fi}^{\Gamma}
\end{aligned} \tag{3.2.83}$$

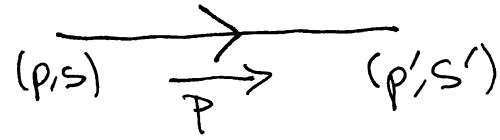
where the sum is over all Feynman diagrams Γ which are elements of the set of all topologically distinct Feynman diagrams (with no self-contractions) G_{fi} made from the lines and vertices listed below which have the incoming lines pertaining to the initial state $|i\supset$ and the external lines pertaining to the final state $|f\supset$. These states are specified by the type of particle, its momentum, its spin or helicity (polarization), mass and charge. Further there are no vacuum bubbles in Γ; that is, every largest subgraph of Γ has external lines (initial or final lines) attached to it. The contribution to S_{fi} from diagram Γ, S_{fi}^{Γ}, is made from the Feynman rules.

The Feynman diagrams are made from the following graphical elements:

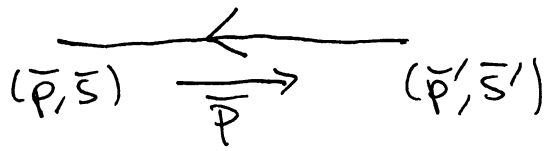
I.) External Lines

1.) Incoming and outgoing lines that do not interact but go straight through; they are not attached to any vertex.

a.) Initial electron to final electron



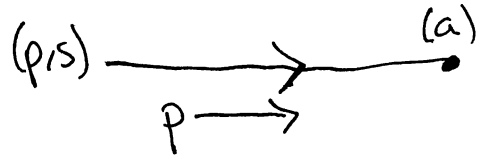
b.) Initial positron to final positron



c.) Initial photon to final photon

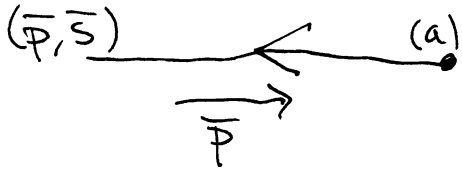
2.) Initial and final lines that terminate (enter) or begin (leave) on a vertex of the diagram

a.) Initial (incoming) electron

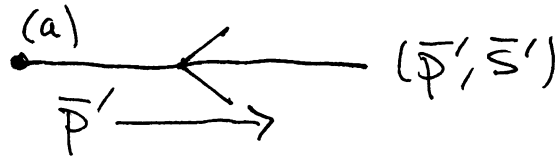


b.) Final (outgoing) electron

c.) Initial (incoming) positron



d.) Final (outgoing) positron

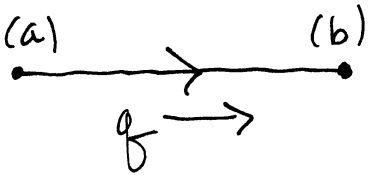


e.) Initial (incoming) photon

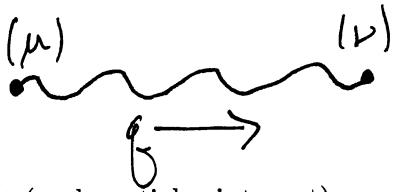
f.) Final (outgoing) photon

II.) Internal lines or propagators connecting vertices

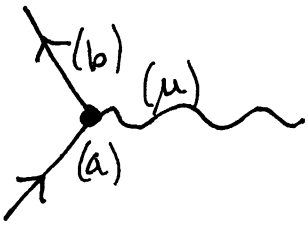
1.) Fermion line



2.) Photon line



III.) Vertices where lines meet (and particles interact)



IV.) Once the Feynman diagram is drawn the 4-momentum is routed through the lines of the graph with energy-momentum conservation at each vertex. Each incoming line brings its own momentum into the graph at its vertex, and each outgoing line takes its momentum out of the graph at its vertex. The external momenta are thus independently routed through the graph with conservation at each vertex being the only constraint. For each independent closed loop of internal lines in the graph there is an interval momentum circulating around the loop. The internal momenta are also routed so that energy-momentum conservation occurs independently for the loop momenta at each vertex. The sum of incoming, outgoing and internal loop momenta at each vertex

is conserved since they are separately. Note that internal lines carry external as well as internal momentum.

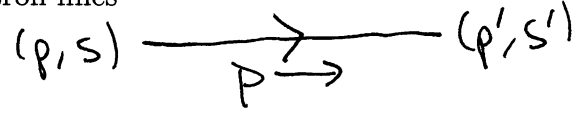
For each Feynman diagram, Γ, there is a corresponding contribution to S_{fi}, S_{fi}^{Γ}, made according to the Feynman rules:

I.) For each largest connected subgraph of Γ there is a factor of $(2\pi)^4\,\delta^4(\sum p_i - \sum p_f)$, an energy-momentum conserving delta function of the sum of the initial energy-momentum minus the sum of the final energy-momentum flowing into and out of that subgraph.
(Note: there is always a $(2\pi)^4\,\delta^4(P_i - P_f)$ where P_i is the sum of all the incoming momenta of $|i\supset$ and P_f is the sum of all the outgoing momenta of $|f\supset$.)

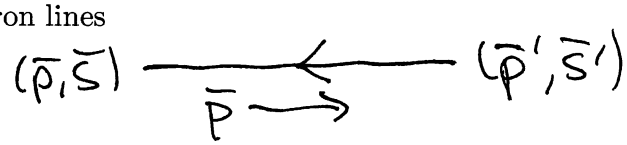
II.) For each non-internal line a mathematical factor is present

1.) For each non-interacting line there is a factor

a.) $(2\pi)^3\,2\omega_p\,\delta_{ss'}\delta^3(\vec{p}-\vec{p}\,')$ for electron lines



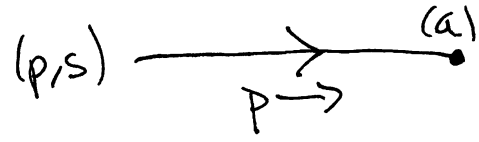
b.) $(2\pi)^3\,2\omega_{\overline{p}}\,\delta_{\overline{s}\overline{s}'}\delta^3(\vec{\overline{p}}-\vec{\overline{p}}\,')$ for positron lines



c.) $(2\pi)^3\,2\omega_k\,\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')$ for photon lines

2.) For each external line write the factor

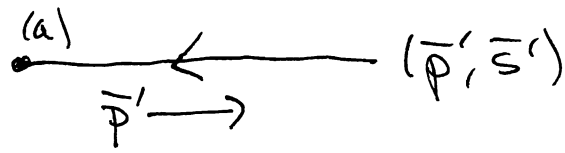
a.) $u_a^{(s)}(\vec{p})$ for incoming electrons



b.) $\overline{u}_a^{(s')}(\vec{p}')$ for outgoing electrons

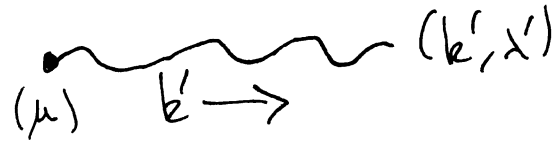
c.) $\overline{v}_a^{(\overline{s})}(\vec{\overline{p}})$ for incoming positrons

d.) $v_a^{(\bar{s}')}(\vec{\bar{p}}')$ for outgoing positrons



e.) $\epsilon_\mu(k, \lambda)$ for incoming photons

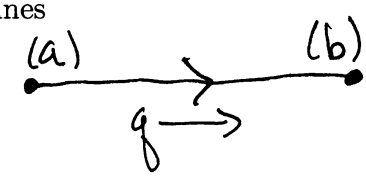
f.) $\epsilon_\mu^*(k', \lambda')$ for outgoing photons



(Note that for linearly polarized photons, $\epsilon_\mu^* = \epsilon_\mu$ is real.)

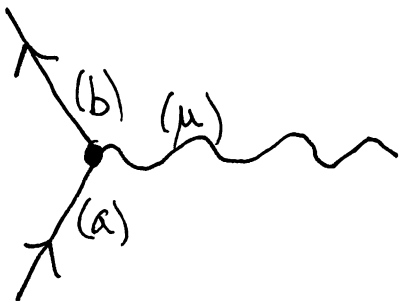
III.) For each internal line write a propagator factor

1.) $\tilde{S}_{F_{ba}}(q) = \left[\frac{i}{\not{q}-m+i\epsilon}\right]_{ba}$ for fermion internal lines



2.) $\tilde{D}_F^{\mu\nu}(q) = \left[\frac{-i}{q^2+i\epsilon}\right] g^{\mu\nu}$ for photon internal lines

IV.) Write a factor $-ie\gamma_{ba}^\mu$ for each vertex



V.) In writing the factors for lines and vertices (γ^μ, $\tilde{S}_F$, ϵ_μ, etc.), they are ordered

so that as you follow the fermion lines in the opposite direction to their arrow you write the factors from left to right on your page. For example, see Figure 3.3.10.

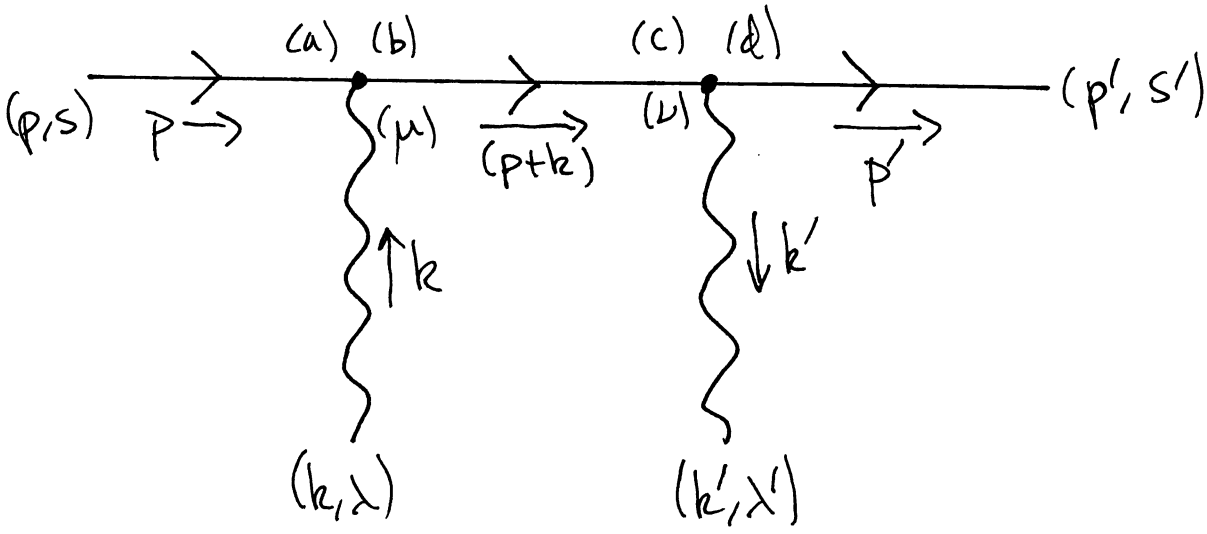


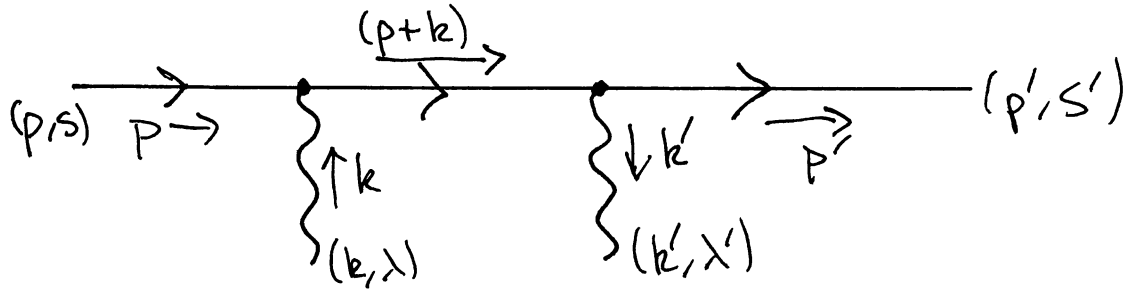
Figure 3.2.10

Start at the right of the diagram and write from the left of the page

$$\overline{u}_d^{(s')}(\vec{p}\,')\,(-ie\gamma^{\nu}_{dc})\,\Big(\frac{i}{(\not{p}+\not{k})-m}\Big)_{cb}\,(-ie\gamma^{\mu}_{ba})\,u_d^{(s)}(\vec{p})$$

all times $\epsilon_\nu(\vec{k}',\lambda')\,\epsilon_\mu(\vec{k},\lambda)$.

So following this rule it is unnecessary to always specify the (a), (μ) dummy indices; i.e.



$$= \overline{u}^{(s')}(\vec{p}\,')\,(-ie\not{\epsilon}(\vec{k}',\lambda'))\,(\frac{i}{(\not{p}+\not{k})-m})\,(-ie\not{\epsilon}(\vec{k},\lambda))\,u^{(s)}(\vec{p})$$

a spinor and matrix product.

VI.) For each independent closed loop, integrate over its internal loop momentum

$$\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}$$

if Γ has $m(\Gamma)$ loops. (This is an integration over each momentum not fixed by conservation from each original vertex space-time integral.)

V.) For each closed fermion loop, multiply by (-1) and take the trace over its spinor product indices.

VI.) For open fermion lines, multiply by a factor $(-1)^P$ where $P = 0$ if the permutation of the external lines is even in order to bring them into the same order as given in the initial and final states and where $P = 1$ if the permutation is odd. More specifically, each initial and final line will be connected to each other line singly; directly or via internal fermion lines. This association of lines in pairs defines a permutation of the pairing of initial and final lines relative to a standard pairing order. For the purposes of permutation counting, label the initial state electrons by $1,\ldots,m$ and the initial positrons by $\overline{1},\ldots\overline{m}$ so $|i\supset = |1,\ldots,m,\overline{1},\ldots,\overline{m}\rangle$; and label the final state electrons by $1',\ldots,n'$ and the final state positrons by $\overline{1}',\ldots,\overline{n}'$ so $|f\supset = |1',\ldots,n',\overline{1}',\ldots,\overline{n}'\rangle$ or $\subset f| = \langle\overline{n}',\ldots,\overline{1}',n',\ldots,1'|$

Since $|i\supset$ is formed by creation operators we have (suppressing the superscripts and all other labels)

$$|i\supset = b_1^\dagger\cdots b_m^\dagger d_{\overline{1}}^\dagger\cdots d_{\overline{m}}^\dagger|0\rangle \tag{3.2.84}$$

and

$$\subset f| = \langle 0|d_{\overline{n}'}\cdots d_{\overline{1}'}b_{n'}\cdots b_{1'}. \tag{3.2.85}$$

Recall that the number of $(b^\dagger+d)$'s must equal the number of $(d^\dagger+b)$'s ; that is, the charge of the final state must equal the charge of the initial state. So $\overline{m}-m = \overline{n}'-n'$ or $m+\overline{n}' = \overline{m}+n'$.

Now each graph will determine a permutation of the external lines and vertex operators. From the S-matrix formula we have a standard ordering,

$$\subset f|S|i \supset\sim \langle 0|d_{\overline{n}'}\cdots d_{\overline{1}'}b_{n'}\cdots b_{1'}\Big(\mathrm{N}\big[\overline{\Psi}\Psi\big]\Big)^{n} b_1^\dagger\cdots b_m^\dagger d_{\overline{1}}^\dagger\cdots d_{\overline{m}}^\dagger|0\rangle \tag{3.2.86}$$

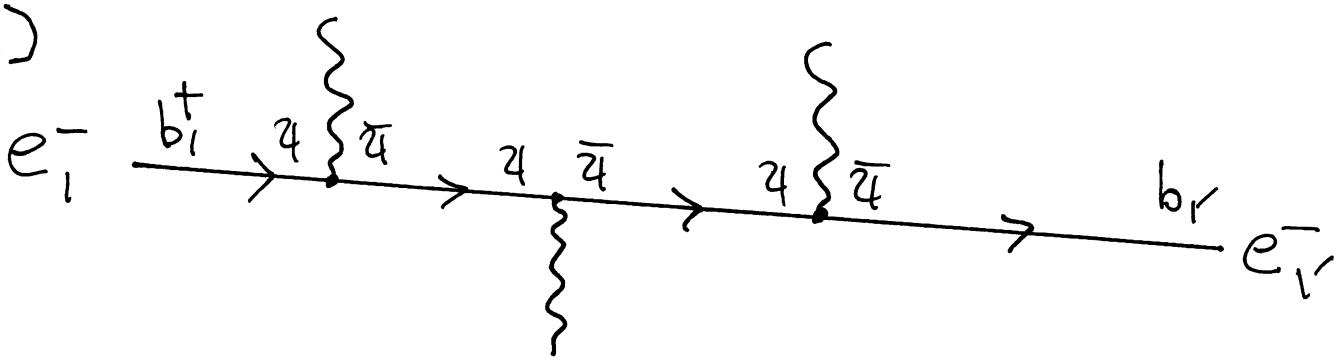
where the interaction Hamiltonian operators are normal ordered:

$$\begin{aligned}\mathrm{N}\big[\overline{\Psi}\Psi\big] = -\mathrm{N}\big[\Psi\overline{\Psi}\big] &= \overline{\Psi}^{-}\Psi^{+} + \overline{\Psi}^{-}\Psi^{-} + \overline{\Psi}^{+}\Psi^{+} - \Psi^{-}\overline{\Psi}^{+} \\ &\sim \big(b^\dagger b + b^\dagger d^\dagger + db - d^\dagger d\big).\end{aligned} \tag{3.2.87}$$

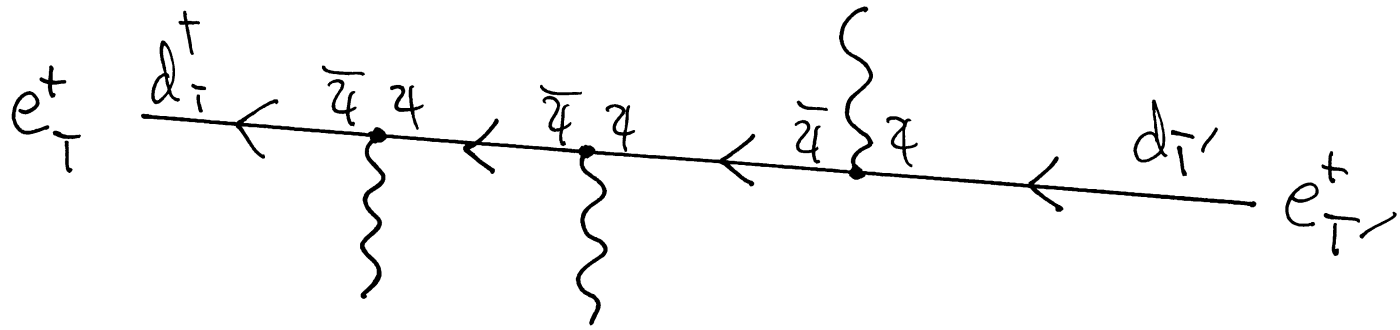
The graph will associate the operators in a string so that each external line can be followed through the graph in the direction of the arrows on the line. This will order the operators so that we go from $b^\dagger$'s or d's through interaction vertices to b's or $d^\dagger$'s. If the number of permutations to return the operators to the above normal order is even, the overall sign on the term is $+1$ ($P = 0$); if odd, then the sign is -1 ($P = 1$).

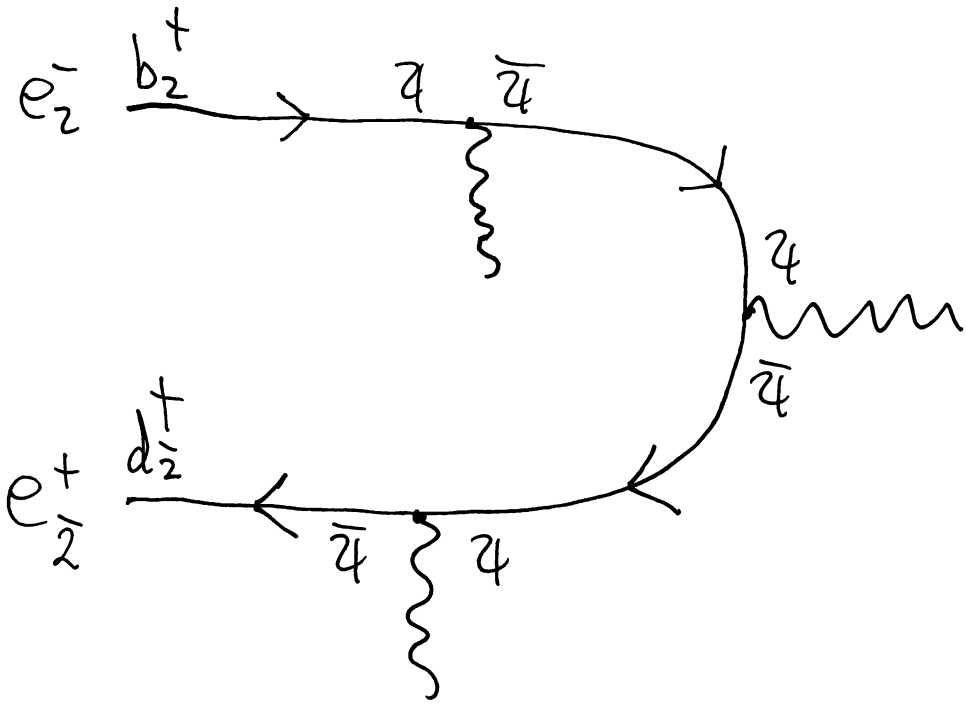
For an example, only showing the initial and final fermions, we have

1i)


e_1^-
$b_1^\dagger$
q
$\bar{q}$
q
$\bar{q}$
q
$\bar{q}$
$b_{1'}$
$e_{1'}^-$

1f)


$e_{\bar{1}}^+$
$d_{\bar{1}}^\dagger$
$\bar{q}$
q
$\bar{q}$
q
$\bar{q}$
q
$d_{\bar{1}'}$
$e_{\bar{1}'}^+$


e_2^-
$b_2^\dagger$
q
$\bar{q}$
q
$\bar{q}$
$e_{\bar{2}}^+$
$d_{\bar{2}}^\dagger$
$\bar{q}$
q

The S-matrix standard ordering is

$$d_{\overline{1}'} b_{1'} \Big(\mathrm{N}\big[\overline{\Psi}\Psi\big] \Big)^9 b_1^\dagger b_2^\dagger d_{\overline{1}}^\dagger d_{\overline{2}}^\dagger. \tag{3.2.88}$$

The graph has permuted the order to read

$$\Big[d_{\overline{1}'} \big(\Psi\overline{\Psi}\big) \big(\Psi\overline{\Psi}\big) \big(\Psi\overline{\Psi}\big) d_{\overline{1}}^\dagger \Big] \Big[b_{1'} \big(\overline{\Psi}\Psi\big) \big(\overline{\Psi}\Psi\big) \big(\overline{\Psi}\Psi\big) b_1^\dagger \Big] \Big[b_2^\dagger \big(\Psi\overline{\Psi}\big) \big(\Psi\overline{\Psi}\big) \big(\Psi\overline{\Psi}\big) d_{\overline{2}}^\dagger \Big]. \tag{3.2.89}$$

To get back to standard order requires six $\Psi \leftrightarrow \overline{\Psi}$ interchanges, a $b_{1'} \leftrightarrow d_{\overline{1}}^\dagger$ interchange, and $d_{\overline{1}}^\dagger$ interchanges with both $b_1^\dagger$ and $b_2^\dagger$. Thus there are a total of nine interchanges; the permutation is odd; and we multiply this graph's contribution to S_{fi} by (-1).

Consider another example,

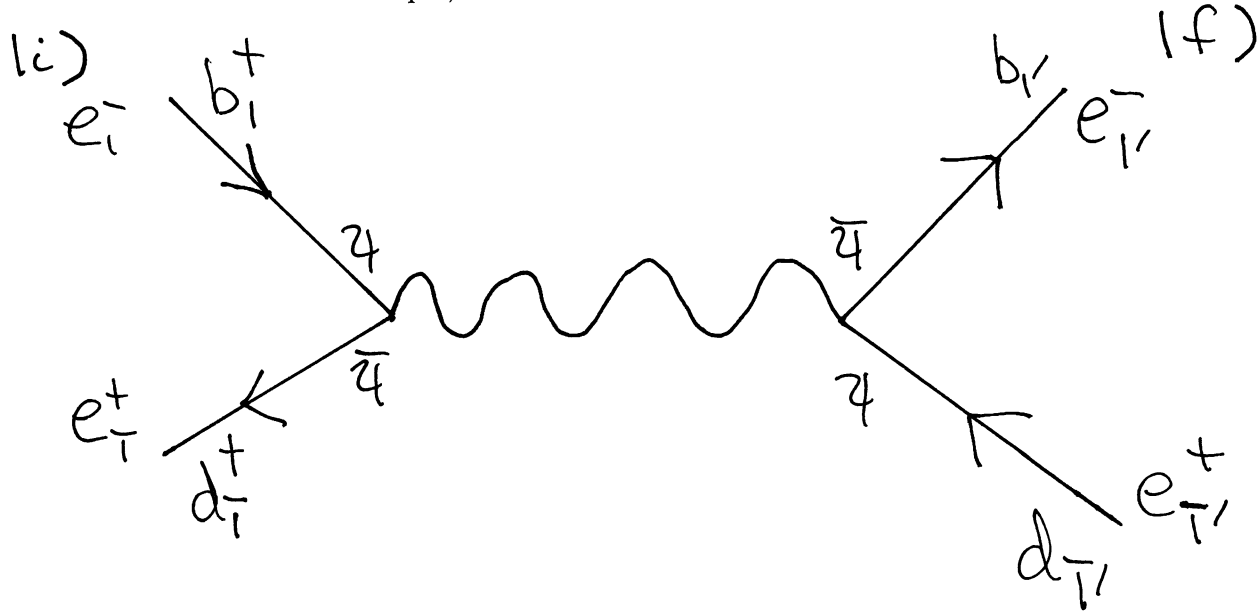


Figure 3.2.11

The standard order is

$$d_{\overline{1}'} b_{1'} \overline{\Psi}\Psi\overline{\Psi}\Psi b_1^\dagger d_{\overline{1}}^\dagger. \tag{3.2.90}$$

The graph's order is

$$\Big[b_1^\dagger \big(\Psi\overline{\Psi}\big) d_{\overline{1}}^\dagger \Big] \Big[b_{1'} \big(\overline{\Psi}\Psi\big) d_{\overline{1}'} \Big]. \tag{3.2.91}$$

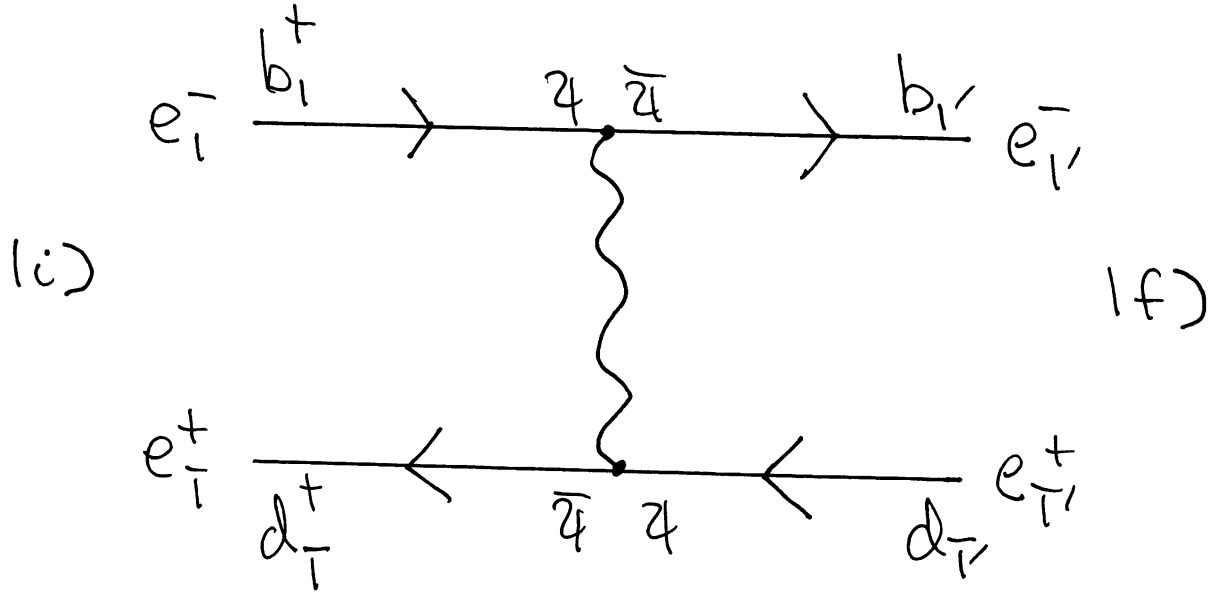


Figure 3.2.12

To return to standard order requires two permutations. Therefore the graph has an overall sign of $+1$.

Similarly, for the same process, we have Figure 3.2.12.

The graph's order is

$$\left[b_{1'}(\overline{\Psi}\Psi)b_1^\dagger\right]\left[d_{\bar{1}'}(\Psi\overline{\Psi})d_{\bar{1}}^\dagger\right]. \tag{3.2.92}$$

An odd number of interchanges returns us to standard order. Hence this contribution receives a sign of (-1).

Following these rules the S-matrix for any process can be swiftly written down.

Before working out a few examples, let's just summarize the results of our long analysis.

Given a Lagrangian for QED in the Feynman-Steuckelberg gauge $\alpha = 1$,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\lambda A^\lambda\right)^2 + \frac{i}{2}\overline{\Psi}\gamma^\mu D_\mu\Psi - \frac{i}{2}\overline{D}_\mu\overline{\Psi}\gamma^\mu\Psi - m\overline{\Psi}\Psi \\ &= +\frac{1}{2}A_\mu\partial^2 A^\mu + \overline{\Psi}\left(i\vec{\not\partial} - m\right)\Psi - e\overline{\Psi}\gamma^\mu\Psi A_\mu \\ &\quad - \frac{1}{2}\partial_\mu\left[A_\nu\partial^\mu A^\nu + A^\mu\overleftrightarrow{\partial}_\nu A^\nu + i\overline{\Psi}\gamma^\mu\Psi\right].\end{aligned} \tag{3.2.93}$$

Since <u>total</u> divergences do not matter we can ignore the last term. The Lagrangian is then written as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ where

$$\begin{aligned}\mathcal{L}_0 &= +\frac{1}{2}A_\mu\partial^2 A^\mu + \overline{\Psi}\left(i\vec{\not\partial} - m\right)\Psi \\ \mathcal{L}_I &= -e\overline{\Psi}\gamma^\mu\Psi A_\mu.\end{aligned} \tag{3.2.94}$$

We then transform to the interaction picture to calculate

$$S_{fi} = \frac{\subset f|\mathrm{T}e^{+i\int d^4x\,\mathrm{N}\left[\mathcal{L}_I^{IP}(x)\right]}|i\supset}{\langle 0|\mathrm{T}e^{+i\int d^4x\,\mathrm{N}\left[\mathcal{L}_I^{IP}(x)\right]}|0\rangle}. \tag{3.2.95}$$

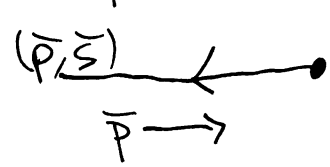
The initial and final states are made from the fields in $\mathcal{L}_0$, i.e. from the free fields. Thus

$$\begin{aligned}A^{\mu\,IP}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{\lambda=0}^{3}\epsilon^\mu(k,\lambda)\left[a_{(\lambda)}(\vec{k})\,e^{-ikx} + a^\dagger_{(\lambda)}(\vec{k})\,e^{+ikx}\right] \\ \Psi^{IP}(x) &= \int \frac{d^3p}{(2\pi)^3 2\omega_p}\sum_{s=1}^{2}\left[b_s(\vec{p})\,u^{(s)}(\vec{p})\,e^{-ipx} + d^\dagger_s(\vec{p})\,v^{(s)}(\vec{p})\,e^{+ipx}\right] \\ \overline{\Psi}^{IP}(x) &= \int \frac{d^3p}{(2\pi)^3 2\omega_p}\sum_{s=1}^{2}\left[d_s(\vec{p})\,\overline{v}^{(s)}(\vec{p})\,e^{-ipx} + b^\dagger_s(\vec{p})\,\overline{u}^{(s)}(\vec{p})\,e^{+ipx}\right].\end{aligned} \tag{3.2.96}$$

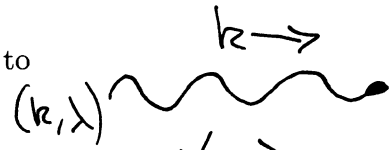
So we have the following mathematical and graphical representations for the particles:

Incoming electrons with factor $u^{(s)}(\vec{p})$ corresponding to

Incoming positrons with factor $\overline{v^{(s)}}(\vec{p})$ corresponding to

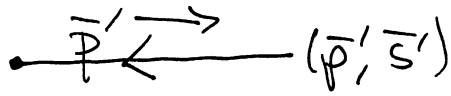


Incoming photons with factor $\epsilon_\mu(k,\lambda)$ corresponding to

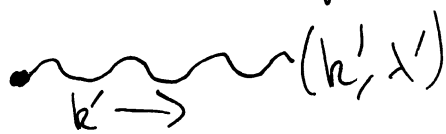


Outgoing electrons with factor $\overline{u}^{(s')}(\vec{p}\,')$ corresponding to

Outgoing positrons with factor $v^{(\overline{s}')}(\vec{\overline{p}}')$ corresponding to

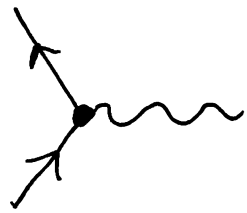


Outgoing photons with factor $\epsilon_\mu(k',\lambda')$ corresponding to



just the plane wave expansion momentum space wavefunctions.

These interact at vertices via the interaction Lagrangian $-e\overline{\Psi}\gamma^\mu\Psi A_\mu$ in the exponent of S. Each vertex corresponds to a factor $-ie\gamma^\mu$ arising from the expansion of that exponential, represented graphically as



Vertices are joined by propagating fermions and photons. These are just $(+i)$ times the inverse of the free Euler-Lagrange differential operators of the field equations in momentum space. So, referring to $\mathcal{L}_0$ we see that for photons $(+i)$ times the inverse of ∂^2 is

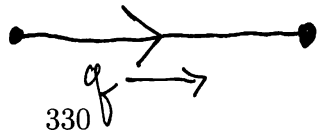
$$\frac{-i}{q^2+i\epsilon}g^{\mu\nu}$$

for fermions $(+i)$ times the inverse of $(i\partial\!\!\!/ - m)$ is

$$\frac{i}{\not{q}-m+i\epsilon}.$$

Hence we have the corresponding internal line factors and their graphical elements

$$\frac{i}{\not{q}-m+i\epsilon} \iff$$



$\frac{-i}{q^2+i\epsilon} g^{\mu\nu} \Longleftrightarrow$ 

Each internal line has its own momentum integral, but at each vertex there is an energy-momentum conserving delta function. Since there are L lines, and V vertices we have $m = L - V + 1$ ($+1$ for the one overall energy-momentum delta function that factors out) internal momentum integrals left – this is just the number of loops in the diagram.

Hence we have written S_{fi} as a sum over Feynman diagrams with the above rules that, with a little experience, can be rapidly obtained just by looking at the Lagrangian.

Finally let's apply these QED rules to calculate the S-matrix for some other process.

1.) First recall the transition amplitude for Compton scattering. Let's re-derive it using our rules.

We have the process

$$e^-(p,s) + \gamma(k,\lambda) \longrightarrow e^-(p',s') + \gamma(k',\lambda'). \tag{3.2.97}$$

Next we write out the possible graphs with their momentum routing in Figure 3.2.13.

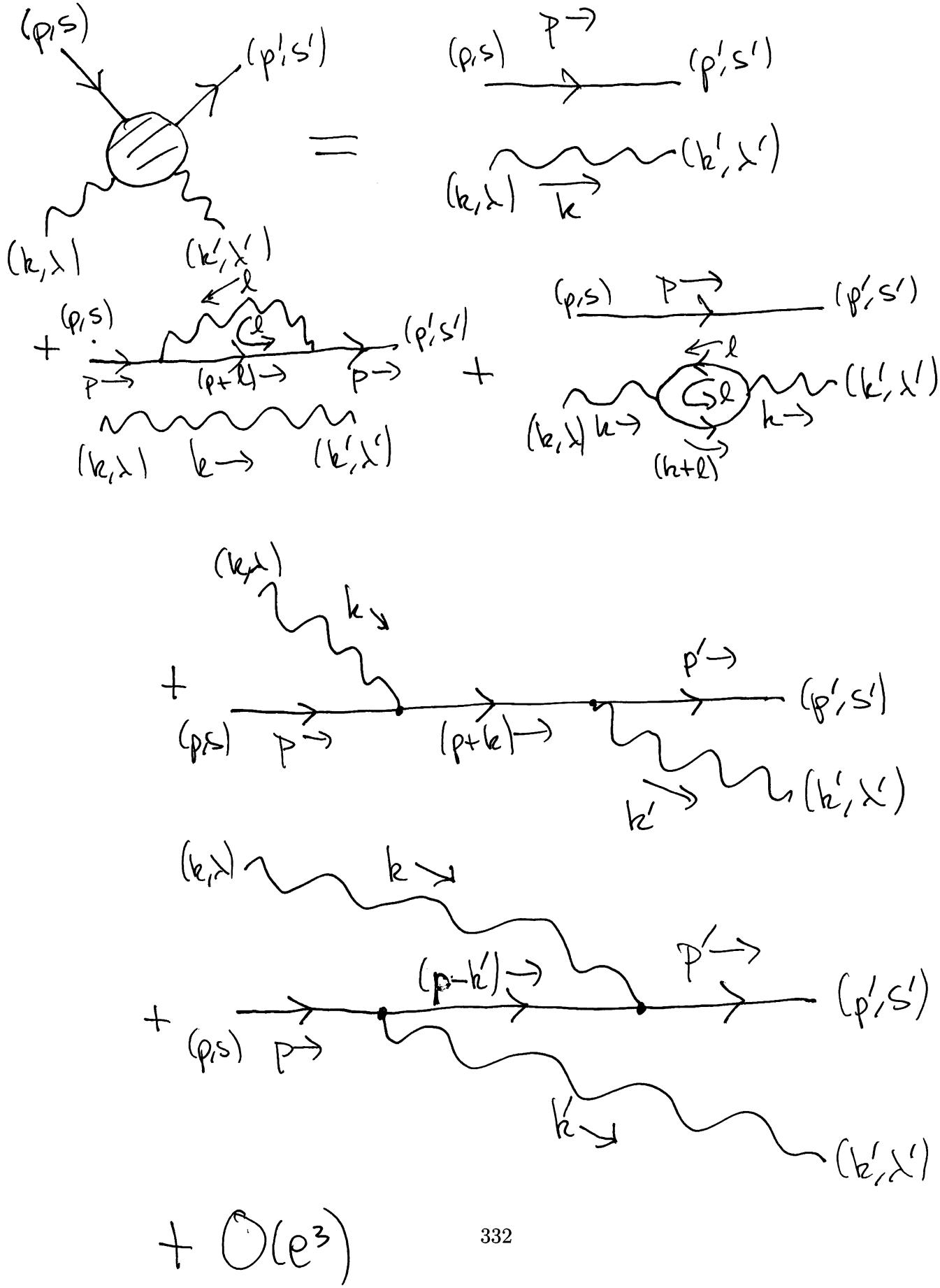


Figure 3.2.13

The S_{fi} can be written down immediately from our rules:

$$
\begin{aligned}
&S(e^-\gamma \to e^-\gamma) \\
&\quad = (2\pi)^3 2\omega_k \delta_{\lambda\lambda'} \delta^3(\vec{k}-\vec{k}')(2\pi)^3 2\omega_p \delta_{ss'} \delta^3(\vec{p}-\vec{p}\,') \\
&\qquad + (2\pi)^3 2\omega_k \delta_{\lambda\lambda'} \delta(\vec{k}-\vec{k}')(2\pi)^4 \delta(p-p')(-ie)^2 \int \frac{d^4l}{(2\pi)^4} \\
&\qquad\quad \times \left[\bar{u}^{s'}(\vec{p}\,')\gamma^\mu \frac{i}{\not{p}+\not{l}-m}\gamma^\nu u^{(s)}(\vec{p})\right]\left(\frac{-ig_{\mu\nu}}{l^2}\right) \\
&\qquad - (2\pi)^3 2\omega_p \delta_{ss'} \delta^3(\vec{p}-\vec{p}\,')(2\pi)^4 \delta(k-k')(-ie)^2 \int \frac{d^4l}{(2\pi)^4} \\
&\qquad\quad \times \mathrm{Tr}\left[\not{\epsilon}(k',\lambda')\frac{i}{\not{k}+\not{l}-m}\not{\epsilon}(k,\lambda)\frac{i}{\not{l}-m}\right] \\
&\qquad + (2\pi)^4 \delta^4(p+k-p'-k')\,(-ie)^2 \\
&\qquad\quad \times \left[\overline{u}^{(s')}(\vec{p}\,')\not{\epsilon}(k',\lambda')\frac{i}{\not{p}+\not{k}-m}\not{\epsilon}(k,\lambda)\,u^{(s)}(\vec{p})\right. \\
&\qquad\qquad \left. + \overline{u}^{(s')}(\vec{p}\,')\not{\epsilon}(k,\lambda)\frac{i}{\not{p}-\not{k}'-m}\not{\epsilon}(k',\lambda')\,u^{(s)}(\vec{p})\right]
\end{aligned}
\tag{3.2.98}
$$

Since we are interested in scattering with $(k,\lambda) \neq (k',\lambda')$ and $(p,s) \neq (p',s')$ only the last two terms are non-zero. This, of course agrees with our previous result, equation (3.2.50), which we derived from first principles.

2.) Secondly, let's consider the elastic scattering of electrons and positrons

$$
e^+(\overline{p},\overline{s}) + e^-(p,s) \longrightarrow e^+(\overline{p}',\overline{s}') + e^-(p',s'). \tag{3.2.99}
$$

This is called Bhabha scattering. The Feynman diagrams contributing to the process are shown in Figure 3.2.14.

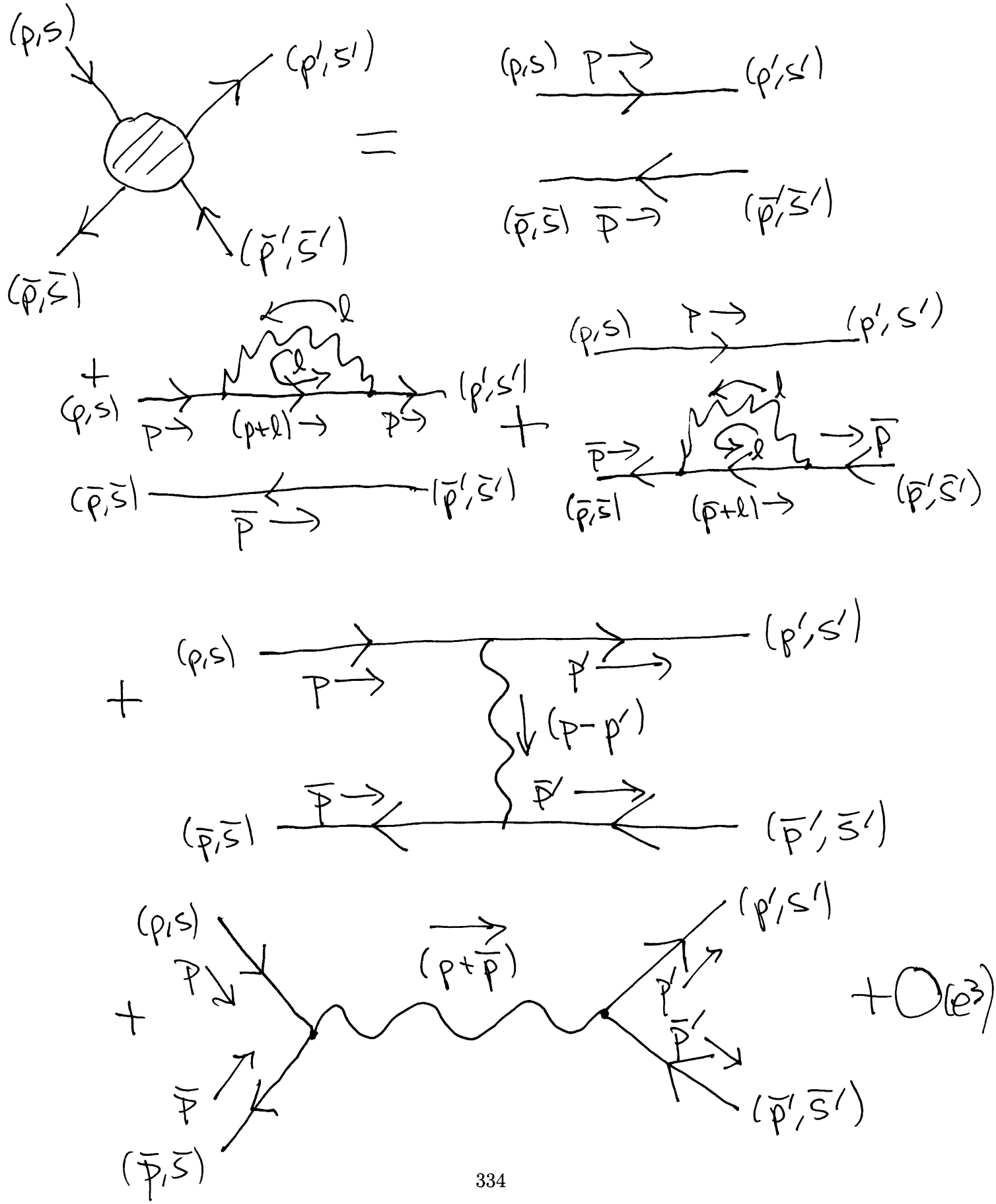

(p,s)
(p',s')
(p̄,s̄)
(p̄',s̄')
=
(p,s) p→ (p',s')
(p̄,s̄) p̄→ (p̄',s̄')
+
l
(p+l)→
p→
p→
(p,s)
(p',s')
(p̄,s̄)
p̄→
(p̄',s̄')
(p,s) p→ (p',s')
p̄→
p̄→
(p̄,s̄)
(p+l)→
(p̄',s̄')
(p,s)
p→
(p',s')
p'→
(p-p')
p̄→
p̄'→
(p̄,s̄)
(p̄',s̄')
(p,s)
p→
p̄→
(p̄,s̄)
(p+p̄)
p'→
p̄'→
(p',s')
(p̄',s̄')
+O(e³)

Figure 3.2.14

Using the Feynman rules we find the S-matrix element for this process again taking $(p,s) \neq (p',s')$ and $(\bar{p},\bar{s}) \neq (\bar{p}',\bar{s}')$ so that only the last two graphs yield a non-zero contribution to the S-matrix element,

$$
\begin{aligned}
& S(e^+e^- \to e^+e^-) \\
&= (2\pi)^4 \delta^4(p+\overline{p}-(p'+\overline{p}')\,(-ie)^2\Big[-\overline{u}^{(s')}(\vec{p}\,')\,\gamma^\mu\,u^{(s)}(\vec{p}) \\
&\qquad \times\left(\frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\right)\overline{v}^{(\bar{s})}(\vec{\overline{p}})\,\gamma^\nu\,v^{(\bar{s}')}(\vec{\overline{p}}\,') \\
&+\overline{u}^{(s')}(\vec{p}\,')\,\gamma^\mu\,\overline{v}^{(\bar{s}')}(\vec{\overline{p}}\,')\left(\frac{-ig_{\mu\nu}}{(p+\overline{p})^2+i\epsilon}\right)\overline{v}^{(\bar{s})}(\vec{\overline{p}})\,\gamma^\nu\,u^{(s)}(\vec{p})\Big]. \qquad (3.2.100)
\end{aligned}
$$