

CHAPTER 3. QUANTUM ELECTRODYNAMICS

§3.1 GAUGE INVARIANCE AND THE SCATTERING MATRIX

When describing the interactions of particles we should make certain that the symmetries of Nature are maintained by the description. We will always manifestly maintain Lorentz invariance, hence the Lagrangian density describing the dynamics of our fields will be a Lorentz scalar. If there are other internal symmetries that the particles have these too should be manifest in the Lagrangian. And most important, if the model contains gauge fields that is photon-like vector particles, we must maintain local gauge invariance in the Lagrangian before the gauge choice is made. To make clear these points we will discuss explicitly the interaction of charged massive spin $\frac{1}{2}$ particles with the electromagnetic field, that is photons. To be concrete we will deal with electrons, positrons and photons, this is known as Quantum Electrodynamics or QED. The electrons will have charge $-e$ (i.e. $e = |e|$), the positrons $+e$, their common mass is m , by convention the electrons are called particles and the positrons, anti-particles.

When the particles are widely separated we will imagine them to be non-interacting and as we have seen a collection of non-interacting electrons, positrons and photons can be described by the free Lagrangian $\hat{\mathcal{L}}_0 = N[\mathcal{L}_0]$ with

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(\frac{i}{2}\overleftrightarrow{\not{D}} - m\right)\Psi + \mathcal{L}_g \quad (3.1.1)$$

where $\mathcal{L}_g = -\frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2$ fixes the gauge so that the electromagnetic field can be quantized in a Lorentz invariant manner. As we saw the physical state matrix elements of the observables in the theory $\mathcal{P}^\mu, \mathcal{M}^{\mu\nu}, Q, N$ were determined by $F^{\mu\nu}$ not the gauge potential A^μ . The same physical situation is described by A^μ or $A'^\mu = A^\mu + \partial^\mu \Lambda$, by the gauge equivalent class of photon fields. In order to quantize the theory we choose a particular representative from the gauge equivalent class by means of the Stueckelberg gauge term \mathcal{L}_g and a choice of Λ where $\partial^2 \Lambda = 0$. The physical state matrix elements of the observables were shown to be independent of these choices; we obtained the same answer in the Coulomb or Stueckelberg-Feynman gauge for the energy- momentum and helicity, etc..

Now when interactions between the electrons and photons take place these should do so in a manner that is not determined by each A^μ but only by the gauge equivalence class of A^μ . That is the interactions should be such that the total Lagrangian, before the gauge choice, should be gauge invariant. Indeed since the particles when well separated are non-interacting their free multi-particle states form a complete set of states for the Hilbert space of states (as well as the indefinite metric space \mathcal{V}). Since these are complete we can expand any state, even when the particles are close together and interacting, in terms of the free states. That is, the observables form a complete set of commuting operators; these observables' physical state matrix elements depend only on the gauge equivalence class of a particular A^μ . Hence the physical states of the system at any time depend on the gauge equivalent classes.

Hence the full Lagrangian must also be gauge invariant – that is, depend only on the gauge equivalence class of an A^μ to guarantee the observables also depend upon the equivalence class. Of course our canonical quantization rules are designed for particular fields, not equivalence classes of those fields. Then once again we must choose a member of the equivalence class in order to calculate via the canonical procedure. We will do this in a Lorentz invariant way, by adding to the gauge invariant Lagrangian, a gauge fixing Lagrangian, in our case the Stueckelberg gauge term. The Gupta-Bleuler subsidiary condition will describe the physical subspace of states which the observables leave invariant. The gauge invariance will guarantee that the unphysical degrees of freedom decouple from the physical matrix elements of observables. That is it will demand that the photon couples to a conserved current and hence $\frac{1}{\alpha}\partial^2\partial_\lambda A^\lambda = 0$ as in the free case. Of course at intermediate steps in a calculation we will now produce scalar and longitudinal photons from the interaction, i.e. an electron is surrounded by a cloud of virtual transverse, longitudinal and scalar photons. However, as we have seen in the free case, when we project to the physical subspace all of these effects cancel out so that if we start with physical states we end up after the interaction with physical states only. We never produce scalar or longitudinal photons as physical particles. Alternatively stated, only transverse photons are physical, that QED is sensible (unitary) for such a subspace follows from the Gupta-Bleuler subsidiary condition definition of the physical subspace and current conservation, that is $\partial^2\partial_\lambda A^\lambda = 0$. This is a consequence of the gauge invariance of the theory.

Well then, how do we construct a Lagrangian that is local gauge invariant. That is, what does local gauge invariance mean for the electrons? In the free case we have seen

that the electron Lagrangian is invariant under global (gauge) phase transformations

$$U^{-1}(\Lambda) \Psi_a(x) U(\Lambda) = e^{i\Lambda} \Psi_a(x) \quad (3.1.2)$$

for $\Lambda = \text{constant}$. Then

$$U^{-1}(\Lambda) \mathcal{L}_{f0}(x) U(\Lambda) = \mathcal{L}_{f0}(x) \quad (3.1.3)$$

with

$$\mathcal{L}_{f0} = \left[\bar{\Psi} \left(\frac{i}{2} \overleftrightarrow{\not{\partial}} - m \right) \Psi \right] \quad (3.1.4)$$

and $\hat{\mathcal{L}}_{f0} = \text{N}[\mathcal{L}_{f0}]$.

This global symmetry led to the conservation of charge. By Noether's theorem we found a conserved current $\hat{J}^\mu = \text{N}[J^\mu]$,

$$J^\mu \equiv \bar{\Psi} \gamma^\mu \Psi; \quad (3.1.5)$$

with $\partial_\mu \hat{J}^\mu = 0$ so that $[Q, H] = 0$ with $Q \equiv \int d^3x \hat{J}^0$. Q was the charge operator $Q = (N_- - N_+)$ so that multiplying Q by $-e$ we find $-eQ = e(N_+ - N_-)$ the electric charge operator. Since J^μ is conserved it will be a good candidate for the current with which the photons interact, as we will see.

A local gauge invariance implies that the physics remains unchanged, i.e. the Lagrangian is invariant, if Ψ is multiplied by a different phase factor at every space-time point. That is we demand that the total Lagrangian be invariant under the local gauge transformations

$$\begin{aligned} U^{-1}(\Lambda) A^\mu(x) U(\Lambda) &= A^\mu(x) + \partial^\mu \Lambda(x) \\ U^{-1}(\Lambda) \Psi_a(x) U(\Lambda) &= e^{-ie\Lambda(x)} \Psi_a(x) \\ U^{-1}(\Lambda) \bar{\Psi}_a(x) U(\Lambda) &= \bar{\Psi}_a(x) e^{ie\Lambda(x)} \end{aligned} \quad (3.1.6)$$

where now $\Lambda = \Lambda(x)$ an arbitrary real scalar function of space-time and we have factored the charge of the electron ($-e$) out of the transformation law. First note that we cannot enforce this invariance without the introduction of interactions with the photon field. The mass term in the electron Lagrangian is invariant

$$U^{-1}(\Lambda) \bar{\Psi}(x) \Psi(x) U(\Lambda) = \bar{\Psi}_a(x) e^{+ie\Lambda(x)} e^{-ie\Lambda(x)} \Psi_a(x) = \bar{\Psi}(x) \Psi(x) \quad (3.1.7)$$

but the kinetic terms involve a derivative of the field. Derivatives involve the field at different space-time points, clearly it probes the space-time variation of the field, hence it will involve different phase factors Λ i.e., the derivative of Λ . Thus the gauge variation of the derivative of Ψ is

$$\begin{aligned}
U^{-1}(\Lambda) \partial_\mu \Psi(x) U(\Lambda) &= \partial_\mu U^{-1}(\Lambda) \Psi(x) U(\Lambda) \\
&= \partial_\mu \left(e^{-ie\Lambda(x)} \Psi(x) \right) \\
&= e^{-ie\Lambda(x)} \partial_\mu \Psi(x) - ie e^{-ie\Lambda(x)} \left(\partial_\mu \Lambda(x) \right) \Psi(x).
\end{aligned} \tag{3.1.8}$$

So the kinetic term involves $\bar{\Psi} \gamma^\mu \partial_\mu \Psi$ and the phases now no longer cancel

$$U^{-1}(\Lambda) \bar{\Psi} \gamma^\mu \partial_\mu \Psi U(\Lambda) = \bar{\Psi} \gamma^\mu \partial_\mu \Psi - ie \bar{\Psi} \gamma^\mu \Psi \partial_\mu \Lambda(x). \tag{3.1.9}$$

We see that the variation involves the current times the divergence of Λ . Similarly for $\bar{\Psi}$ the adjoint field

$$U^{-1}(\Lambda) \partial_\mu \bar{\Psi}(x) U(\Lambda) = e^{ie\Lambda} \partial_\mu \bar{\Psi}(x) + \left(ie \partial_\mu \Lambda \right) e^{ie\Lambda} \bar{\Psi}(x) \tag{3.1.10}$$

So that

$$U^{-1}(\Lambda) \partial_\mu \bar{\Psi} \gamma^\mu \Psi U(\Lambda) = \partial_\mu \bar{\Psi} \gamma^\mu \Psi + ie \bar{\Psi} \gamma^\mu \Psi \partial_\mu \Lambda \tag{3.1.11}$$

Hence we find that

$$U^{-1}(\Lambda) \mathcal{L}_{f0} U(\Lambda) = \mathcal{L}_{f0} + e \bar{\Psi} \gamma^\mu \Psi \partial_\mu \Lambda(x) \tag{3.1.12}$$

The electron Lagrangian cannot be made invariant without introducing the photon. Now we immediately see what we must do. Since $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is already invariant we do not have to alter it. We can also couple A_μ to J^μ . Thus we find that

$$\begin{aligned}
U^{-1}(\Lambda) J^\mu A_\mu U(\Lambda) &= U^{-1}(\Lambda) J^\mu U(\Lambda) U^{-1}(\Lambda) A_\mu U(\Lambda) \\
&= J^\mu A_\mu + J^\mu \partial_\mu \Lambda(x)
\end{aligned} \tag{3.1.13}$$

since $U^{-1}(\Lambda) J^\mu U(\Lambda) = J^\mu$, hence

$$\mathcal{L}_{\text{inv}} \equiv \mathcal{L}_0 - e \bar{\Psi} \gamma^\mu \Psi A_\mu \tag{3.1.14}$$

and

$$U^{-1}(\Lambda) \mathcal{L}_{\text{inv}} U(\Lambda) = \mathcal{L}_{\text{inv}}. \tag{3.1.15}$$

The coupling of the conserved electromagnetic current in this way, called minimal coupling or minimal substitution, has a deeper geometric origin. In calculating the derivative of Ψ we are comparing the field at two different space-time points $d\Psi = \Psi(x + dx) - \Psi(x)$. This is not a meaningful comparison since at each point we can independently define the phase of Ψ . There is no reason that the phase conventions for Ψ at x^μ are the same for Ψ at $x^\mu + dx^\mu$. If we could bring the second vector to the same space-time point with the same phase convention as the first and then take their difference we would have a meaningful measure of their change. That is we should parallel transport this vector to the same space-time point. In simply going from point x^μ to $x^\mu + dx^\mu$, the field becomes $\Psi(x + dx) = \Psi(x) + d\Psi = \Psi(x) + \partial_\mu \Psi dx^\mu$. Now we would like to compare this vector (in charge space) at $x + dx$ to Ψ parallel transported from x to $x + dx$. The parallel transported field is denoted $\Psi_{\parallel}(x + dx) = \Psi(x) + \delta\Psi(x)$. The change of Ψ under parallel transport is linearly proportional to the distance of transport dx^μ and to the field Ψ itself since the change of the sum of fields under transport must be the sum of the change of each field. Thus

$$\delta\Psi(x) \equiv -ie A_\mu(x) \Psi(x) dx^\mu \quad (3.1.16)$$

where the constants of proportionality $A_\mu = A_\mu(x)$ depend on the space-time point in question and are called the gauge fields (or gauge connection, those are the internal symmetry space analogs to the Christoffel symbols in gravitational physics). As we will see, these fields $A_\mu(x)$ describe the curvature of the internal symmetry space. If there are no fields ($A_\mu(x) = 0$), then $\delta\Psi = 0$.

Hence we can now compare the two vectors (in charge space) in a meaningful way. Defining their difference as the vector

$$\begin{aligned} D\Psi(x) &\equiv \Psi(x + dx) - \Psi_{\parallel}(x + dx) \\ &= \left(\Psi(x + dx) - \Psi(x) \right) - \left(\Psi_{\parallel}(x + dx) - \Psi(x) \right) \\ &= d\Psi(x) - \delta\Psi(x) \\ &= \left[\partial_\mu \Psi(x) + ie A_\mu(x) \Psi(x) \right] dx^\mu. \end{aligned} \quad (3.1.17)$$

We define the covariant derivative of Ψ as

$$D_\mu \Psi(x) \equiv (\partial_\mu + ie A_\mu(x)) \Psi(x). \quad (3.1.18)$$

That is, a gauge transformation changes the phase of Ψ at x and at $x+dx$ independently. Thus to compute the change on Ψ in an inherent way we would like to move $\Psi(x)$ along

the interval from x^μ to $x^\mu + dx^\mu$ so that it has the same phase change at $x^\mu + dx^\mu$ as $\Psi(x + dx)$. This is precisely how the parallel transport of $\Psi(x)$ from x^μ to $x^\mu + dx^\mu$ is defined; $\Psi_{\parallel}(x + dx)$ is the parallel transport of $\Psi(x)$ from x^μ to $x^\mu + dx^\mu$ so that under a gauge transformation

$$U^{-1}(\Lambda) \Psi_{\parallel}(x + dx) U(\Lambda) = e^{-ie\Lambda(x+dx)} \Psi_{\parallel}(x + dx). \quad (3.1.19)$$

Since $\Psi_{\parallel}(x + dx) \equiv \Psi(x) + \delta\Psi(x)$, this yields a transformation law for $\delta\Psi(x)$, the change of Ψ due to the parallel transport (the change normal to the tangent of the interval contour)

$$U^{-1}(\Lambda) \left(\Psi(x) + \delta\Psi(x) \right) U(\Lambda) = e^{-ie\Lambda(x)} \Psi(x) + U^{-1}(\Lambda) \delta\Psi(x) U(\Lambda) \quad (3.1.20)$$

But

$$\begin{aligned} e^{-ie\Lambda(x+dx)} \Psi_{\parallel}(x + dx) &= e^{-ie\Lambda(x)} \left(1 - ie d\Lambda(x) \right) \left(\Psi(x) + \delta\Psi(x) \right) \\ &= e^{-ie\Lambda(x)} \left(\Psi(x) + \delta\Psi(x) - ie d\Lambda(x) \Psi(x) \right) \end{aligned} \quad (3.1.21)$$

to first order in dx^μ . Hence

$$U^{-1}(\Lambda) \delta\Psi(x) U(\Lambda) = e^{-ie\Lambda(x)} \left(\delta\Psi(x) - ie \partial_\mu \Lambda(x) \Psi(x) dx^\mu \right). \quad (3.1.22)$$

As we have seen this equation plus linear dependence on Ψ of $\delta\Psi$ is satisfied by

$$\delta\Psi(x) = -ie A_\mu(x) \Psi(x) dx^\mu. \quad (3.1.23)$$

Further, the change in Ψ over the interval transforms as Ψ does under a gauge transformation (to first order in dx^μ) by construction

$$\begin{aligned} U^{-1}(\Lambda) D\Psi(x) U(\Lambda) &= e^{-ie\Lambda(x+dx)} D\Psi(x) \\ &= e^{-ie\Lambda(x)} D\Psi(x). \end{aligned} \quad (3.1.24)$$

Thus the covariant derivative is such that

$$U^{-1}(\Lambda) D_\mu \Psi(x) U(\Lambda) = e^{-ie\Lambda(x)} D_\mu \Psi(x) \quad (3.1.25)$$

unlike the ordinary derivative

$$U^{-1}(\Lambda) \partial_\mu \Psi(x) U(\Lambda) = e^{-ie\Lambda(x)} \left(\partial_\mu \Psi(x) - ie \partial_\mu \Lambda(x) \Psi(x) \right). \quad (3.1.26)$$

Now the covariant derivative

$$D_\mu \Psi(x) = (\partial_\mu + ieA_\mu(x))\Psi(x) \quad (3.1.27)$$

describes the rate of change of the field in the presence of gauge fields normal to any curve in space-time. Given any path in space-time $x^\mu = x^\mu(s)$ parameterized by s . The vector $u^\mu \equiv \frac{dx^\mu}{ds}$ is the tangent along the curve. The parallel transport of u^μ along the curve is exactly the change of u^μ along the tangent itself. Hence for u^μ , $D^\mu u^\nu = 0$, i.e. $Du^\mu = 0$ so $u^\mu_{\parallel}(x+dx) = u^\mu(x+dx)$. The parallel displacement of any field $\Psi(x)$ along $x^\mu(s)$ means that the change in the field along the tangent to the curve $\frac{\delta\Psi}{ds}$ should just be equal to the change in Ψ in moving from x to $x+dx$ along the tangent $\frac{d\Psi}{ds}$. That is

$$D_\mu \Psi(x) u^\mu(x) \Big|_{x^\mu=x^\mu(s)} = 0. \quad (3.1.28)$$

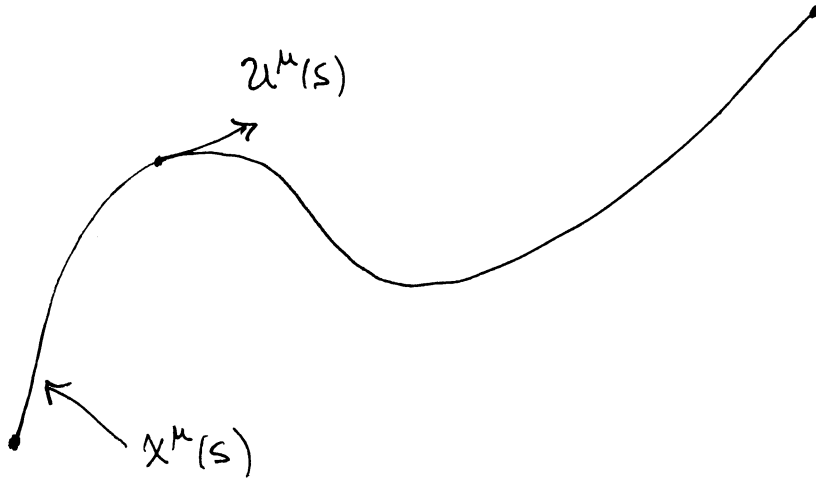


Figure 3.1.1

Since there can be curvature in the charge space when we parallel transport the field $\Psi(x)$ around a closed curve it will not, in general, come back to itself. The change in the field is given by

$$\Delta\Psi = \oint_C \delta\Psi. \quad (3.1.29)$$

That is along any section of the curve, Ψ changes by $ds \frac{d\Psi}{ds} = \frac{\delta\Psi}{\delta s} ds$ along the tangent. So we have

$$\Delta\Psi = \oint_C -ie A_\mu(x) \Psi(x) dx^\mu. \quad (3.1.30)$$

Now, applying Stoke's theorem to the contour integral

$$\oint_C B_\nu dx^\nu = \int_S dS^{\mu\nu} \partial_\mu B_\nu \quad (3.1.31)$$

where S is the directed area bounded by curve C and $dS^{\mu\nu}$ the projection of the area onto the μ - ν plane

$$dS^{\mu\nu} = dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu. \quad (3.1.32)$$

For example suppose the curve is in the x^0 - x^1 plane then

$$\begin{aligned} \oint_C B_\mu dx^\mu &= \oint_C B_0 dx^0 + B_1 dx^1 \\ &= \int_S (dx^0 dx^1 \frac{\partial}{\partial x^1} B_0 + dx^1 dx^0 \frac{\partial}{\partial x^0} B_1) \\ &= \int_S dx^0 dx^1 (\frac{\partial}{\partial x^1} B_0 - \frac{\partial}{\partial x^0} B_1) \\ &= \int_S dx^\mu \wedge dx^\nu \partial_\mu B_\nu \\ &= \frac{1}{2} \int_S dS^{\mu\nu} (\partial_\mu B_\nu - \partial_\nu B_\mu). \end{aligned} \quad (3.1.33)$$

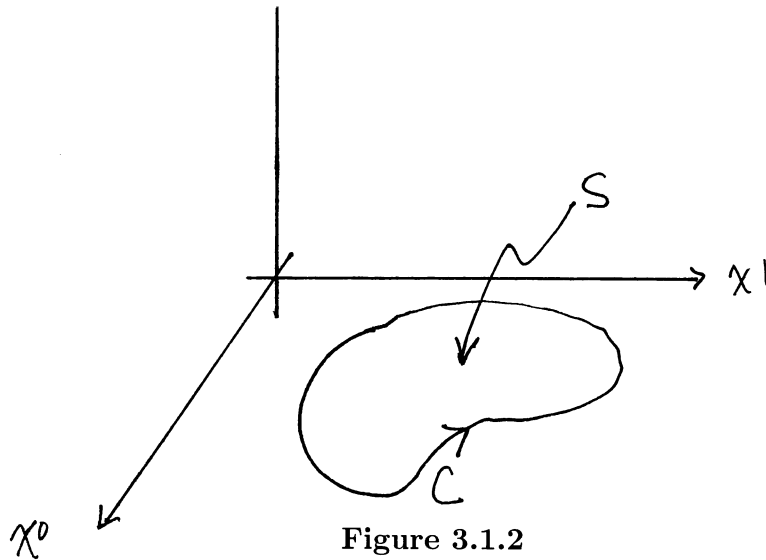


Figure 3.1.2

So for Ψ we find

$$\begin{aligned}\Delta\Psi &= -\frac{ie}{2} \int_S dS^{\mu\nu} [\partial_\mu(A_\nu\Psi) - \partial_\nu(A_\mu\Psi)] \\ &= -\frac{ie}{2} \int_S dS^{\mu\nu} [(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi + A_\nu\partial_\mu\Psi - A_\mu\partial_\nu\Psi].\end{aligned}\tag{3.1.34}$$

But for an infinitesimal contour we can replace the derivative of Ψ inside the contour with the change in Ψ on the contour itself to first order accuracy so since $D_\mu\Psi = 0$ on the contour, $\partial_\mu\Psi = -ieA_\mu\Psi$ on the contour and inside the infinitesimal contour so

$$\Delta\Psi = -\frac{ie}{2} dS^{\mu\nu} \left[(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi - ie(A_\mu A_\nu - A_\nu A_\mu)\Psi \right]\tag{3.1.35}$$

In the abelian charge case we are considering here $A_\mu A_\nu = A_\nu A_\mu$ so that

$$\Delta\Psi = -\frac{ie}{2} dS^{\mu\nu} F_{\mu\nu}\Psi\tag{3.1.36}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature tensor (anti-symmetric field strength tensor) for the charge space.

Since $D_\mu\Psi$ has the same charge space “vector” character as Ψ , by construction we have that they transform the same way under gauge transformations, if

$$U^{-1}(\Lambda)\Psi(x)U(\Lambda) = e^{-ie\Lambda(x)}\Psi(x)\tag{3.1.37}$$

then

$$U^{-1}(\Lambda)D_\mu\Psi(x)U(\Lambda) = e^{-ie\Lambda(x)}D_\mu\Psi(x).\tag{3.1.38}$$

This as we know implies that

$$U^{-1}(\Lambda)A_\mu(x)U(\Lambda) = A_\mu(x) + \partial_\mu\Lambda(x).\tag{3.1.39}$$

That is

$$\begin{aligned}U^{-1}(\Lambda)D_\mu\Psi(x)U(\Lambda) &= \partial_\mu U^{-1}(\Lambda)\Psi(x)U(\Lambda) + ieU^{-1}(\Lambda)A_\mu(x)\Psi(x)U(\Lambda) \\ &= e^{-ie\Lambda(x)}\partial_\mu\Psi(x) - ie e^{-ie\Lambda(x)}\partial_\mu\Lambda(x)\Psi(x) \\ &\quad + ie e^{-ie\Lambda(x)}U^{-1}(\Lambda)A_\mu U(\Lambda)\Psi(x) \\ &\equiv e^{-ie\Lambda(x)}D_\mu\Psi(x) \\ &= e^{-ie\Lambda}\partial_\mu\Psi + ie e^{-ie\Lambda}A_\mu\Psi\end{aligned}\tag{3.1.40}$$

which finally yields

$$ie U^{-1}(\Lambda) A_\mu(x) U(\Lambda) = ie \left(A_\mu(x) + \partial_\mu \Lambda(x) \right). \quad (3.1.41)$$

Further since $\Delta\Psi$ has the same character as Ψ we find that

$$U^{-1}(\Lambda) F_{\mu\nu}(x) U(\Lambda) = e^{-ie\Lambda(x)} F_{\mu\nu}(x) e^{+ie\Lambda(x)}, \quad (3.1.42)$$

and, since it is an abelian phase factor,

$$= F_{\mu\nu}(x) \quad (3.1.43)$$

as can be checked directly from the definition of $F_{\mu\nu}$ and the A_μ -transformation law. Note that the commutator of covariant derivatives yields the curvature

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu] \\ &= ie F_{\mu\nu} \end{aligned} \quad (3.1.44)$$

while the Jacobi identity

$$[[D_\mu, D_\nu], D_\rho] + [[D_\nu, D_\rho], D_\mu] + [[D_\rho, D_\mu], D_\nu] = 0 \quad (3.1.45)$$

implies

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0, \quad (3.1.46)$$

which in the abelian case yields

$$\epsilon^{\lambda\rho\mu\nu} \partial_\rho F_{\mu\nu} = 0, \quad (3.1.47)$$

the charge space Bianchi identity. Lately there has been a re-emergence of interest in the speculation that this charge space curvature can be unified with space-time curvature in a higher dimensional “space-time” unified field theory; this is known as Kaluza- Klein theory.

Hence when we consider the evolution of the field Ψ in space-time we should consider as physically meaningful the covariant changes in Ψ ; $D\Psi$. Hence the Lagrangian describing the evolution of the matter field is given by

$$\mathcal{L}_{f\text{inv}} = \frac{i}{2} \bar{\Psi} \gamma^\mu D_\mu \Psi - \frac{i}{2} \overline{D}_\mu \bar{\Psi} \gamma^\mu \Psi - m \bar{\Psi} \Psi \quad (3.1.48)$$

where if $D_\mu \Psi = \partial_\mu \Psi + ieA_\mu \Psi$ then $(D_\mu \Psi)^\dagger = \partial_\mu \Psi^\dagger - ieA_\mu \Psi^\dagger$. So

$$\begin{aligned}\bar{D}_\mu \bar{\Psi} &\equiv (D_\mu \Psi)^\dagger \gamma^0 \\ &= (\partial_\mu - ieA_\mu) \bar{\Psi}.\end{aligned}\tag{3.1.49}$$

Note that since

$$U^{-1}(\Lambda) \Psi(x) U(\Lambda) = e^{-ie\Lambda(x)} \Psi(x)\tag{3.1.50}$$

$$U^{-1}(\Lambda) D_\mu \Psi(x) U(\Lambda) = e^{-ie\Lambda(x)} D_\mu \Psi(x)\tag{3.1.51}$$

and

$$U^{-1}(\Lambda) \bar{\Psi}(x) U(\Lambda) = e^{+ie\Lambda(x)} \bar{\Psi}(x).\tag{3.1.52}$$

So

$$U^{-1}(\Lambda) \bar{D}_\mu \bar{\Psi}(x) U(\Lambda) = e^{+ie\Lambda(x)} \bar{D}_\mu \bar{\Psi}(x)\tag{3.1.53}$$

and we find that $\mathcal{L}_{f_{\text{inv}}}$ is gauge invariant

$$U^{-1}(\Lambda) \mathcal{L}_{f_{\text{inv}}}(x) U(\Lambda) = \mathcal{L}_{f_{\text{inv}}}.\tag{3.1.54}$$

Also writing out the covariant derivatives

$$\begin{aligned}\mathcal{L}_{f_{\text{inv}}} &= \bar{\Psi} \left(\frac{i}{2} \overleftrightarrow{\not{\partial}} - m \right) \Psi - e \bar{\Psi} \gamma^\mu \Psi A_\mu \\ &= \mathcal{L}_{f_0} - e \bar{\Psi} \gamma^\mu \Psi A_\mu\end{aligned}\tag{3.1.55}$$

the same result as we found with the minimal coupling prescription earlier.

In general we add to this the kinetic energy terms for the gauge fields

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\Psi} \gamma^\mu D_\mu \Psi - \frac{i}{2} \bar{D}_\mu \bar{\Psi} \gamma^\mu \Psi - m \bar{\Psi} \Psi\tag{3.1.56}$$

for the total gauge invariant Lagrangian. Since it is gauge invariant the dynamics depend only on the gauge equivalence classes of A_μ as required.

We could add to \mathcal{L}_{inv} other gauge invariant terms such as $\bar{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu}$ or $(F_{\mu\nu} F^{\mu\nu})^2$, etc. Those are excluded for two reasons: first, for the experiments we will discuss these terms are not present; secondly, such terms will lead to uncontrollable numbers of infinite terms appearing in the S-matrix, removable only at the price of destroying predictability of the theory – i.e., an infinite number of parameters will appear only to be determined by an infinite number of experiments! This situation is described

as the theory being non-renormalizable. Only renormalizable theories can be given a consistent interpretation at all energy scales. It can be shown but is beyond the material covered in this course that only dimension 4 and less field monomials lead to renormalizability. Since A_μ has dimension 1 (in inverse mass units) Ψ , $\frac{3}{2}$ and ∂_μ , 1 we find that $\bar{\Psi}\Psi$ is dimension 3, $\bar{\Psi}\gamma^\mu D_\mu\Psi$ is dimension 4, and so are all renormalizable. $\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$ is dimension 5 and makes the theory non-renormalizable. Similarly for other terms.

Thus we find that the gauge invariant Lagrangian describing the interaction of the electrons, positrons and photons is given by

$$\mathcal{L}_{\text{inv}} = \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu\Psi - \frac{i}{2}\overline{D_\mu\Psi}\gamma^\mu\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (3.1.57)$$

In order to quantize the fields according to our canonical quantization procedure we must pick a representative from the equivalence class of fields A_μ . This we do in a Lorentz invariant manner by again working in the Stueckelberg gauge. The total Lagrangian then becomes

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_g \quad (3.1.58)$$

where $\mathcal{L}_g = -\frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2$ (also of dimension 4 so renormalizable).

The Euler-Lagrange equations describing the space-time development of the fields are now:

$$1.) \frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = 0 \implies$$

$$\partial_\mu F^{\mu\nu}(x) + \frac{1}{\alpha}\partial^\nu\partial_\lambda A^\lambda(x) = e\bar{\Psi}(x)\gamma_\nu\Psi(x) \quad (3.1.59)$$

$$2.) \frac{\partial\mathcal{L}}{\partial\Psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi} = 0 \implies$$

$$i\gamma^\mu D_\mu\Psi(x) - m\Psi(x) = 0, \quad (3.1.60)$$

or explicitly writing out the covariant derivative

$$(i\rlap{/}\partial - m)\Psi(x) = e\gamma^\mu A_\mu(x)\Psi(x) \quad (3.1.61)$$

3.) And the adjoint equation $\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\Psi}} = 0 \implies$

$$i\overleftarrow{D}_\mu \bar{\Psi}(x)\gamma^\mu + m\bar{\Psi}(x) = 0 \quad (3.1.62)$$

or expanding the covariant derivative

$$\bar{\Psi}(x)\left(i\overleftarrow{\not{D}} + m\right) = -e\bar{\Psi}(x)A_\mu(x)\gamma^\mu. \quad (3.1.63)$$

These are coupled, non-linear, partial differential equations. Needless to say we cannot solve these equations analytically. We will utilize the interaction picture perturbative solution shortly.

First let's recall the canonical commutation relations. Since the interaction terms are time independent we see that the canonical momenta and therefore the ETCR and ETAR are the same as for fermions and photons in the free case,

$$\begin{aligned} \Pi^\mu &\equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\alpha}g^{\mu 0}\partial_\lambda A^\lambda \\ \bar{\Pi} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\Psi}}} = -i\gamma^0\Psi \\ \Pi &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = -i\bar{\Psi}\gamma^0. \end{aligned} \quad (3.1.64)$$

The ETCR and ETAR are

$$\delta(x^0 - y^0)\left\{\bar{\Pi}_a(x), \bar{\Psi}_b(y)\right\} = -i\delta_{ab}\delta^4(x - y)$$

or

$$\delta(x^0 - y^0)\left\{\Pi_a(x), \Psi_b(y)\right\} = -i\delta_{ab}\delta^4(x - y)$$

and

$$\delta(x^0 - y^0)\left[\Pi^\mu(x), A^\nu(y)\right] = -ig^{\mu\nu}\delta^4(x - y) \quad (3.1.65)$$

All other ET(anti-)commutators vanish. Substituting the definition of the momenta we find, as usual,

$$\begin{aligned} \delta(x^0 - y^0)\left\{\Psi_a(x), \Psi_b^\dagger(y)\right\} &= \delta_{ab}\delta^4(x - y) \\ \delta(x^0 - y^0)\left[\dot{A}^\mu(x), A^\nu(y)\right] &= ig^{\mu\nu}\left[1 + (\alpha - 1)g^{\mu 0}\right]\delta^4(x - y) \quad (\not{\Sigma}_\mu) \\ \delta(x^0 - y^0)\left[\dot{A}^\mu(x), \dot{A}^i(y)\right] &= i(1 - \alpha)\partial_x^i\delta^4(x - y). \end{aligned} \quad (3.1.66)$$

All other ETCR and ETAR vanish.

Note that taking the divergence of (3.1.59) implies that

$$\frac{1}{\alpha} \partial^2 \partial_\lambda A^\lambda(x) = e \partial_\mu (\bar{\Psi} \gamma^\mu \Psi)(x) \quad (3.1.67)$$

Adding equations (3.1.61) and (3.1.63) after multiplying (3.1.61) by $\bar{\Psi}$ and (3.1.63) by Ψ implies

$$i \partial_\mu (\bar{\Psi} \gamma^\mu \Psi) = 0. \quad (3.1.68)$$

Thus $\partial_\lambda A^\lambda$ obeys the free wave equation $\frac{1}{\alpha} \partial^2 \partial_\lambda A^\lambda(x) = 0$. The physical states of the theory will again be defined in a Lorentz covariant manner by the Gupta-Bleuler subsidiary condition:

$$|\Phi \rangle \text{ is a physical state if } \partial_\lambda A^{\lambda+}(x) |\Phi \rangle = 0. \quad (3.1.69)$$

As previously, the global symmetries of the Lagrangian imply conservative laws via Noether's theorem. Poincare invariance implies that the energy-momentum tensor and angular momentum tensor are conserved

$$\begin{aligned} T^{\mu\nu} &= \partial^\nu \Psi \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} + \partial^\nu \bar{\Psi} \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\Psi}} + \partial^\nu A_\lambda \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\lambda} - g^{\mu\nu} \mathcal{L} \\ &= \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \Psi - F^{\mu\lambda} \partial^\nu A_\lambda - \frac{1}{\alpha} (\partial_\lambda A^\lambda) \partial^\nu A^\mu - g^{\mu\nu} \mathcal{L}. \end{aligned} \quad (3.1.70)$$

Explicitly we can check that $\partial_\mu T^{\mu\nu} = 0$ and that the translation generator is $\mathcal{P}^\mu \equiv \int d^3x T^{0\mu}$ such that

$$\begin{aligned} [\mathcal{P}^\mu, \Psi(x)] &= -i \partial^\mu \Psi(x) \\ [\mathcal{P}^\mu, \bar{\Psi}(x)] &= -i \partial^\mu \bar{\Psi}(x) \\ [\mathcal{P}^\mu, A^\nu(x)] &= -i \partial^\mu A^\nu(x). \end{aligned} \quad (3.1.71)$$

As previously $T^{\mu\nu} \neq T^{\nu\mu}$ and we can apply the Belinfante procedure to define a symmetric conserved tensor

$$\begin{aligned} \Theta^{\mu\nu} &\equiv T^{\mu\nu} - \partial_\rho G^{\rho\mu\nu} \\ G^{\rho\mu\nu} &= \frac{1}{2} [H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho}] \end{aligned} \quad (3.1.72)$$

where

$$\begin{aligned} H^{\rho\mu\nu} &\equiv \Pi_r^\rho D_{rs}^{\mu\nu} \phi_s \\ &= -\frac{1}{4} \bar{\Psi} \left\{ \gamma^\rho, \sigma^{\mu\nu} \right\} \Psi + F^{\rho\mu} A^\nu - F^{\rho\nu} A^\mu + \frac{1}{\alpha} (\partial_\lambda A^\lambda) (g^{\rho\mu} A^\nu - g^{\rho\nu} A^\mu). \end{aligned} \quad (3.1.73)$$

Thus

$$H^{\rho\mu\nu} = -\frac{1}{2} \epsilon^{\rho\mu\nu\lambda} \bar{\Psi} \gamma_5 \gamma_\lambda \Psi + F^{\rho\mu} A^\nu - F^{\rho\nu} A^\mu + \frac{1}{\alpha} (\partial_\lambda A^\lambda) (g^{\rho\mu} A^\nu - g^{\rho\nu} A^\mu). \quad (3.1.74)$$

Hence

$$\partial_\rho G^{\rho\mu\nu} = -\frac{1}{4} \partial_\rho \epsilon^{\rho\mu\nu\lambda} \bar{\Psi} \gamma_5 \gamma_\lambda \Psi + \partial_\rho \left[F^{\rho\mu} A^\nu + \frac{1}{\alpha} (\partial_\lambda A^\lambda) (g^{\mu\nu} A^\rho - g^{\rho\nu} A^\mu) \right]. \quad (3.1.75)$$

It is left as an exercise to find the final form of $\Theta^{\mu\nu}$ and to show that $\partial_\mu \Theta^{\mu\nu} = 0$, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and that $\mathcal{P}^\mu = \int d^3x \theta^{0\mu} = \int d^3x T^{0\mu}$.

With $\Theta^{\mu\nu}$ the angular momentum tensor becomes

$$M^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}, \quad (3.1.76)$$

and it is conserved – i.e., $\partial_\mu M^{\mu\nu\rho} = 0$. The Lorentz transformation operator then is given by

$$\mathcal{M}^{\mu\nu} \equiv \int d^3x M^{0\mu\nu}, \quad (3.1.77)$$

and it can be checked that

$$\begin{aligned} [\mathcal{M}^{\mu\nu}, \Psi(x)] &= -i \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) \Psi(x) - \frac{i}{2} \sigma^{\mu\nu} \Psi(x) \right] \\ [\mathcal{M}^{\mu\nu}, \bar{\Psi}(x)] &= -i \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) \bar{\Psi}(x) + \frac{i}{2} \bar{\Psi}(x) \sigma^{\mu\nu} \right] \\ [\mathcal{M}^{\mu\nu}, A^\lambda(x)] &= -i \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) A^\lambda(x) + (g^{\mu\lambda} g^{\nu\rho} - g^{\nu\lambda} g^{\mu\rho}) A_\rho(x) \right] \end{aligned} \quad (3.1.78)$$

all as before.

As we have already indicated the Lagrangian still conserves charge, is globally gauge invariant so that $J^\mu = \bar{\Psi} \gamma^\mu \Psi$ is conserved – i.e., $\partial_\mu J^\mu = 0$. Hence $Q \equiv -e \int d^3x \hat{J}^0(x)$ generates global charge transformations

$$Q = -e \int d^3x N[\Psi^\dagger(x) \Psi(x)]. \quad (3.1.79)$$

Using the ETCR and ETAR we have

$$\begin{aligned} [Q, \Psi(x)] &= +e\Psi(x) \\ [Q, \bar{\Psi}(x)] &= -e\bar{\Psi}(x) \\ [Q, A^\mu(x)] &= 0. \end{aligned} \tag{3.1.80}$$

We can no longer simply Fourier transform the interacting fields in terms of $\frac{d^3k}{2\omega_k}$ since the fields will now create single particles as well as any number of pairs and virtual photons thus k^2 is not restricted to just $k^2 = m^2$ or $k^2 = 0$ as in the non-interacting case. To further study the field theory we will make use of the interaction picture.

The field operators we have been considering above are in the Heisenberg picture, that is they are time dependent

$$\begin{aligned} [H, \phi(x)] &= -i\frac{\partial}{\partial t}\phi(x) && \text{with } \phi(x) \in \{\Psi, \bar{\Psi}, A^\mu\} \\ [H, \Pi(x)] &= -i\frac{\partial}{\partial t}\Pi(x) && \Pi(x) \in \{\Pi, \bar{\Pi}, \Pi^\mu\}. \end{aligned} \tag{3.1.81}$$

That is, since H is time independent, $\dot{H} = [H, H] = 0$, and the time translation operator is $U(a, \Lambda)$ with $\Lambda^{\mu\nu} = g^{\mu\nu}$ and $a^\mu = (t, \vec{0})$

$$U(a^\mu = (t, \vec{0}), 1) \equiv U^\dagger(t) = e^{+iHt} \quad (\text{i.e., } U(t) = e^{-iHt}) \tag{3.1.82}$$

so that time translations give

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iHt} \phi(0, \vec{x}) e^{-iHt} \\ \Pi(t, \vec{x}) &= e^{iHt} \Pi(0, \vec{x}) e^{-iHt}. \end{aligned} \tag{3.1.83}$$

On the other hand state vectors $|A\rangle$ in the Heisenberg picture are time independent

$$\frac{\partial}{\partial t}|A\rangle = 0. \tag{3.1.84}$$

The other extreme is the Schrödinger picture in which the field operators are time independent while the states carry all the time dependence. Identifying the two pictures at $t = 0$ we have the Schrödinger fields $\phi^S(\vec{x})$, $\Pi^S(\vec{x})$ and states $|A(t)\rangle_S$ given by

$$\phi^S(\vec{x}) = \phi(0, \vec{x}) = e^{-iHt} \phi(t, \vec{x}) e^{+iHt}$$

$$\Pi^S(\vec{x}) = \Pi(0, \vec{x}) = e^{-iHt} \Pi(t, \vec{x}) e^{+iHt} \quad (3.1.85)$$

and

$$|A(t)\rangle_S = e^{-iHt} |A(0)\rangle_S = e^{-iHt} |A\rangle \quad (3.1.86)$$

where the Heisenberg quantum fields $\phi(t, \vec{x})$ and states $|A\rangle$ are written without postscripts. Since the two pictures are unitarily related the matrix elements of operators are preserved

$$\begin{aligned} {}_S\langle A(t)|\phi^S(\vec{x})|B(t)\rangle_S &= \langle A|U^\dagger(t)\phi^S(\vec{x})U(t)|B\rangle \\ &= \langle A|\phi(t, \vec{x})|B\rangle. \end{aligned} \quad (3.1.87)$$

Further since $H^S = e^{-iHt} H e^{+iHt} = H$ we have $\frac{\partial}{\partial t} \phi^S(\vec{x}) = 0 = \frac{\partial}{\partial t} \Pi^S(\vec{x})$ and states obey the Schrödinger equation

$$i \frac{\partial}{\partial t} |A(t)\rangle_S = H |A(t)\rangle_S. \quad (3.1.88)$$

Intermediate to the Schrödinger and Heisenberg pictures is the interaction picture introduced by Dirac. The Hamiltonian is split into a free, non-interacting part, H_0 , plus an interacting part, H_I , in the Heisenberg picture

$$H = H_0 + H_I. \quad (3.1.89)$$

In the Schrödinger picture

$$H_0^S = H_0(t=0). \quad (3.1.90)$$

The interaction picture is defined so that the state vectors evolve only according to the interaction Hamiltonian (in the interaction picture); that is the free Hamiltonian time evolution is removed from them. Hence we define

$$|A(t)\rangle_{IP} \equiv e^{+iH_0^S t} |A(t)\rangle_S = e^{+iH_0^S t} e^{-iHt} |A\rangle. \quad (3.1.91)$$

So that matrix elements are preserved

$${}_{IP}\langle A(t)|\phi^{IP}(t, \vec{x})|B(t)\rangle_{IP} = {}_S\langle A(t)|\phi^S(\vec{x})|B(t)\rangle_S \quad (3.1.92)$$

we have that operators are related by

$$\phi^{IP}(t, \vec{x}) = e^{iH_0^S t} \phi^S(\vec{x}) e^{-iH_0^S t}$$

$$\Pi^{IP}(t, \vec{x}) = e^{+iH_0^S t} \Pi^S(\vec{x}) e^{-iH_0^S t}. \quad (3.1.93)$$

Applying this to H_0^S we find

$$H_0^{IP} = e^{+iH_0^S t} H_0^S e^{-iH_0^S t} = H_0^S. \quad (3.1.94)$$

We can similarly relate the interaction picture to the Heisenberg picture via the Schrödinger picture

$$\begin{aligned} \phi^{IP}(t, \vec{x}) &= e^{iH_0^S t} \phi^S(\vec{x}) e^{-iH_0^S t} \\ &= e^{iH_0^S t} e^{-iHt} \phi(t, \vec{x}) e^{+iHt} e^{-iH_0^S t}, \end{aligned} \quad (3.1.95)$$

also

$$\Pi^{IP}(t, \vec{x}) = e^{iH_0^S t} e^{-iHt} \Pi(t, \vec{x}) e^{+iHt} e^{-iH_0^S t} \quad (3.1.96)$$

and

$$\begin{aligned} |A(t)\rangle_{IP} &= e^{iH_0^S t} |A(t)\rangle_S \\ &= e^{iH_0^S t} e^{-iHt} |A\rangle. \end{aligned} \quad (3.1.97)$$

Note we have taken all pictures to agree at $t = 0$

$$\begin{aligned} \phi^{IP}(0, \vec{x}) &= \phi^S(\vec{x}) = \Pi(0, \vec{x}) \\ \Pi^{IP}(0, \vec{x}) &= \Pi^S(\vec{x}) = \Pi(0, \vec{x}) \\ |A(0)\rangle_{IP} &= |A(0)\rangle_S = |A\rangle. \end{aligned} \quad (3.1.98)$$

Also if there is no interaction, $H_I = 0$, the interaction and Heisenberg pictures are the same as expected from the construction of the interaction picture.

Now differentiating the field and momentum definitions for the interaction picture we find

$$\begin{aligned} -i \frac{\partial}{\partial t} \phi^{IP}(t, \vec{x}) &= [H_0^S, \phi^{IP}(t, \vec{x})] \\ -i \frac{\partial}{\partial t} \Pi^{IP}(t, \vec{x}) &= [H_0^S, \Pi^{IP}(t, \vec{x})] \end{aligned} \quad (3.1.99)$$

The fields obey the Heisenberg equations of motion with time evolution determined by the free Hamiltonian in the Schrödinger picture, H_0^S . But the time independent H_0^S is $H_0^S = H_0^{IP}$, and we find

$$[H_0^{IP}, \phi^{IP}(t, \vec{x})] = -i \frac{\partial}{\partial t} \phi^{IP}(t, \vec{x})$$

$$\left[H_0^{IP}, \Pi^{IP}(t, \vec{x}) \right] = -i \frac{\partial}{\partial t} \Pi^{IP}(t, \vec{x}). \quad (3.1.100)$$

Further, the time derivative of the state vectors in the interaction picture is given by

$$i \frac{\partial}{\partial t} |A(t)\rangle_{IP} = -H_0^S |A(t)\rangle_{IP} + e^{iH_0^S t} i \frac{\partial}{\partial t} |A(t)\rangle_S \quad (3.1.101)$$

from the definition of $|A(t)\rangle_{IP}$. In the Schrodinger picture

$$\begin{aligned} i \frac{\partial}{\partial t} |A(t)\rangle_S &= H^S |A(t)\rangle_S \\ &= H^S e^{-iH_0^S t} e^{+iH_0^S t} |A(t)\rangle_S \\ &= H^S e^{-iH_0^S t} |A(t)\rangle_{IP}. \end{aligned} \quad (3.1.102)$$

So

$$i \frac{\partial}{\partial t} |A(t)\rangle_{IP} = [e^{iH_0^S t} H^S e^{-iH_0^S t} - H_0^S] |A(t)\rangle_{IP} \quad (3.1.103)$$

But $H^S = H_0^S + H_I^S$ and $e^{iH_0^S t} H_0^S e^{-iH_0^S t} = H_0^S$ and $H_I^{IP} = e^{iH_0^S t} H_I^S e^{-iH_0^S t}$. Thus

$$i \frac{\partial}{\partial t} |A(t)\rangle_{IP} = H_I^{IP} |A(t)\rangle_{IP}. \quad (3.1.104)$$

Thus $|A(t)\rangle_{IP}$ obeys the Schrodinger equation with the time evolution given by the interaction picture interaction Hamiltonian. Note however that although $H_0^S = H_0^{IP} = H_0(t=0)$ is time independent H_I^{IP} is not! Since

$$\begin{aligned} H_I^{IP} &= e^{iH_0 t} H_I^S e^{-iH_0 t} \\ &= e^{iH_0 t} H_I(t=0) e^{-iH_0 t} = H_I^{IP}(t), \end{aligned} \quad (3.1.105)$$

we have $-i\dot{H}_I^{IP} = [H_0^{IP}, H_I^{IP}] \neq 0$.

Note also that the equal time CR and AR for Π and ϕ are

$$\left[\Pi(t, \vec{x}), \phi(t, \vec{y}) \right]_{\pm} = -i\delta^3(\vec{x} - \vec{y}) \quad (3.1.106)$$

(+ if AR, - if CR). Since the interaction picture fields are related to the Heisenberg fields by a unitary transformation they obey the same relations

$$\phi^{IP}(t, \vec{x}) = e^{iH_0^S t} e^{-iHt} \phi(t, \vec{x}) e^{+iHt} e^{-iH_0^S t}$$

$$\Pi^{IP}(t, \vec{x}) = e^{iH_0^S t} e^{-iHt} \Pi(t, \vec{x}) e^{+iHt} e^{-iH_0^S t} \quad (3.1.107)$$

and $U = e^{iH_0^S t} e^{-iHt}$, $U^\dagger = e^{+iHt} e^{-iH_0^S t}$ with $UU^\dagger = 1$. Hence multiplying the commutation relations by U from the left and U^\dagger from the right implies

$$\left[\Pi^{IP}(t, \vec{x}), \phi^{IP}(t, \vec{y}) \right]_{\pm} = -i\delta^3(\vec{x} - \vec{y}). \quad (3.1.108)$$

Further Π^{IP} and ϕ^{IP} obey the same equations of motion as the free fields since

$$\begin{aligned} -i \frac{\partial}{\partial t} \phi^{IP}(x) &= [H_0^{IP}, \phi^{IP}(x)] \\ -i \frac{\partial}{\partial t} \Pi^{IP}(x) &= [H_0^{IP}, \Pi^{IP}(x)]. \end{aligned} \quad (3.1.109)$$

Thus $\phi^{IP}(x)$ and $\Pi^{IP}(x)$ can be expanded as free-field creation and annihilation operators in momentum space as we have studied in detail.

Continuing with the time evolution of the states, the interaction picture state at time t is related to the state at time t_0 by

$$\begin{aligned} |A(t)\rangle_{IP} &= e^{iH_0^S t} e^{-iHt} |A\rangle \\ &= e^{iH_0^S t} e^{-iHt} e^{+iHt_0} e^{-iH_0^S t_0} |A(t_0)\rangle_{IP} \\ |A(t)\rangle_{IP} &= e^{iH_0^S t} e^{-iH(t-t_0)} e^{-iH_0^S t_0} |A(t_0)\rangle_{IP}. \end{aligned} \quad (3.1.110)$$

This combination of exponentials defines the time evolution operator

$$U(t, t_0) \equiv e^{iH_0^S t} e^{-iH(t-t_0)} e^{-iH_0^S t_0}. \quad (3.1.111)$$

This operator relates interaction picture states at time t_0 to those at t

$$|A(t)\rangle_{IP} = U(t, t_0) |A(t_0)\rangle_{IP}. \quad (3.1.112)$$

Note also that the relationship of operators in the Heisenberg and interaction pictures are given by

$$\phi^{IP}(t, \vec{x}) = U(t, 0) \phi(t, \vec{x}) U^{-1}(t, 0). \quad (3.1.113)$$

Either from the above definitions of $U(t, t_0)$ or from the IP Schrodinger equation we find

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I^{IP}(t) U(t, t_0) \quad (3.1.114)$$

along with the initial condition $U(t_0, t_0) = 1$.

Recall that we can solve this equation for the time evolution operator by iteration as seen by re-expressing the equation and initial conditions as an integral equation

$$U(t, t_0) = 1 - i \int_{t_0}^t dt_1 H_I^{IP}(t_1) U(t, t_0). \quad (3.1.115)$$

The solution is (see chapter 1.3)

$$\begin{aligned} U(t, t_0) &= \text{T}e^{-i \int_{t_0}^t dt' H_I^{IP}(t')} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n \text{T}H_I^{IP}(t_1) \dots H_I^{IP}(t_n). \end{aligned} \quad (3.1.116)$$

In our field theories we have that $H = \int d^3x \mathcal{H}(x)$ and $H_o = \int d^3x \mathcal{H}_o(x)$, $H_I = \int d^3x \mathcal{H}_I(x)$ with $\mathcal{H} = \mathcal{H}_o + \mathcal{H}_I$. Hence in the interaction picture these become

$$\begin{aligned} \mathcal{H}_I^{IP}(t, \vec{x}) &= U(t, 0) \mathcal{H}_I(t, \vec{x}) U^{-1}(t, 0). \\ \mathcal{H}_o^{IP}(t, \vec{x}) &= U(t, 0) \mathcal{H}_o(t, \vec{x}) U^{-1}(t, 0). \end{aligned} \quad (3.1.117)$$

Thus we can write the time evolution operator as

$$\begin{aligned} U(t, t_0) &= \text{T}e^{-i \int_{t_0}^t d^4x H_I^{IP}(x)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t d^4x_1 \dots \int_{t_0}^{t_{n-1}} d^4x_n \text{T}H_I^{IP}(x_1) \dots H_I^{IP}(x_n). \end{aligned} \quad (3.1.118)$$

The utility of the interaction picture now becomes apparent. In high energy physics we are typically interested in predicting the results of scattering experiments. That is given some initial state $|A(t_0)\rangle$ it evolves into the state $|A(t)\rangle$ at a later time where

$$|A(t)\rangle = U(t, t_0) |A(t_0)\rangle. \quad (3.1.119)$$

We then ask what is the probability amplitude that the state evolved into a specific state $|B(t)\rangle$ at time t , that transition probability amplitude is

$$\langle B(t) | A(t) \rangle = \langle B(t) | U(t, t_0) | A(t_0) \rangle. \quad (3.1.120)$$

The question then arises as to how we should specify the initial states $|A(t_0)\rangle$ and the final states $|B(t)\rangle$. When experiments are performed the initial states are prepared in the remote past ($t \rightarrow -\infty$) so that the states contain definite numbers of particles that are spatially separated and non-interacting. That is we will assume that at early times $H_I^{IP}(t)$ is switched off so that the interaction picture and Heisenberg picture are the same. The free interaction picture fields will then create and annihilate the particles of the theory with the states just being the Fock space states of the free Hamiltonian H_0 built up from the free fields. As the time proceeds forward the interaction turns on and these free or bare states become dressed with their clouds of virtual particles evolving according to the Schrödinger equation with $H_I^{IP} \neq 0$ now. As these dressed particle states approach the interaction region complicated scattering, annihilation and creation processes go on. As time continues the old and new particles separate from the collision region and their interactions drop off and only their virtual cloud remains. As time proceeds we will assume that we can turn off all interactions again so that only the free bare particles remain. Again these final states in the remote future ($t \rightarrow +\infty$) will be states of definite numbers of non-interacting particles. That is in the remote future we have $H_I^{IP}(t \rightarrow +\infty) \rightarrow 0$ and the interaction picture is again the same as the Heisenberg picture with dynamics given by H_0 . The interaction picture free field operators will again create and annihilate the final particles and the totality of final states is built up as the Fock space of these operators. Since every initial state is the final state of the experiment before it, it makes sense that the final state space is the same as the initial state space. It is the Fock space of free particles. This process of turning off the interaction and turning it on again is implemented by there being a time dependence for the coupling constants of the theory. For instance in QED we can imagine the electric charge $e = e(t)$ to have a time dependence such as shown in Figure 3.1.3.

Initially we have our bare free particle states given by the free field creation operators on the vacuum, these are eigenstates of $H_0, \vec{P}_0, \vec{J}_0$ or $\frac{\vec{J}_0 \cdot \vec{k}}{|\vec{k}|}$ the complete commuting set of Poincaré generators that are made from the free interaction picture fields which for $t > |T|$ coincides with the Heisenberg picture. Then as the time proceeds the particles become fully charged and are surrounded by a cloud of virtual particles, pairs and photons and evolve further in time according to the full interaction Hamiltonian in the interaction picture $H_I^{IP}(t)$.

As the particles spatially approach each other they also interact with each other by the exchange of the virtual quanta surrounding themselves and given by $H_I^{IP}(t)$. As

they spatially separate as time proceeds their momenta and spins have changed and possibly some have disappeared and new particles have been created and they stop interacting with each other. As the charge begins to vanish again the particles don't interact even with their virtual cloud until eventually we have bare free particle final states given by the free field creation operators on the vacuum again, exactly in the same manner as the initial states. This turning on and off of the interaction is called the adiabatic hypothesis. In quantum mechanics with finite range potentials we were able to show the equivalence between using localized wavepacket initial and final states and plane wave states with the adiabatic hypothesis. That is since the plane waves are spread over all space they overlap the support of the potential and the particles are always interacting. Hence we must turn the potential off initially and finally in order to simulate the non-interaction of separated wavepacket states. In order to avoid emitting or absorbing energy through the Fourier transform of the potential's time dependence during this turning off and on period, it must be done slowly, that is adiabatically. In field theory the particles are actually always interacting with their own cloud of virtual particles and it will be necessary to further account for these self-interaction effects. This is done through self-energy and charge renormalization, a topic covered in the next course on field theory in which the adiabatic hypothesis will be more carefully formulated in terms of asymptotic conditions.

So we now have a description of our initial and final states. We are interested in describing a scattering process from some initial state to some final state. Let's denote the (bare) initial and final states by rounded brackets just to keep in mind our adiabatic hypothesis procedure. Thus initially we have some state $|i \rhd = |i(t \rightarrow -\infty) \rangle_{IP}$ made from the initial vacuum state $|0 \rhd = |0(t \rightarrow -\infty) \rangle_{IP}$ by the action of creation operators of the free IP fields. This state evolves in time according to the interaction Hamiltonian in the IP with the adiabatic hypothesis time dependence of the coupling constants, thus at time t the state is

$$|i(t) \rangle_{IP} = U(t, -\infty)|i \rhd . \quad (3.1.121)$$

We desire the transition probability amplitude for this initial state to evolve into a specified final state at $t \rightarrow +\infty$. The final state, defined for late times, is denoted by $|f \rhd = |f(t \rightarrow +\infty) \rangle_{IP}$ again $|f \rhd$ is made by the action of the free IP field creation operators on the final vacuum $|0 \rhd = |0(t \rightarrow +\infty) \rangle_{IP}$ which is the same ground state as the initial vacuum, the no bare particle state as written.

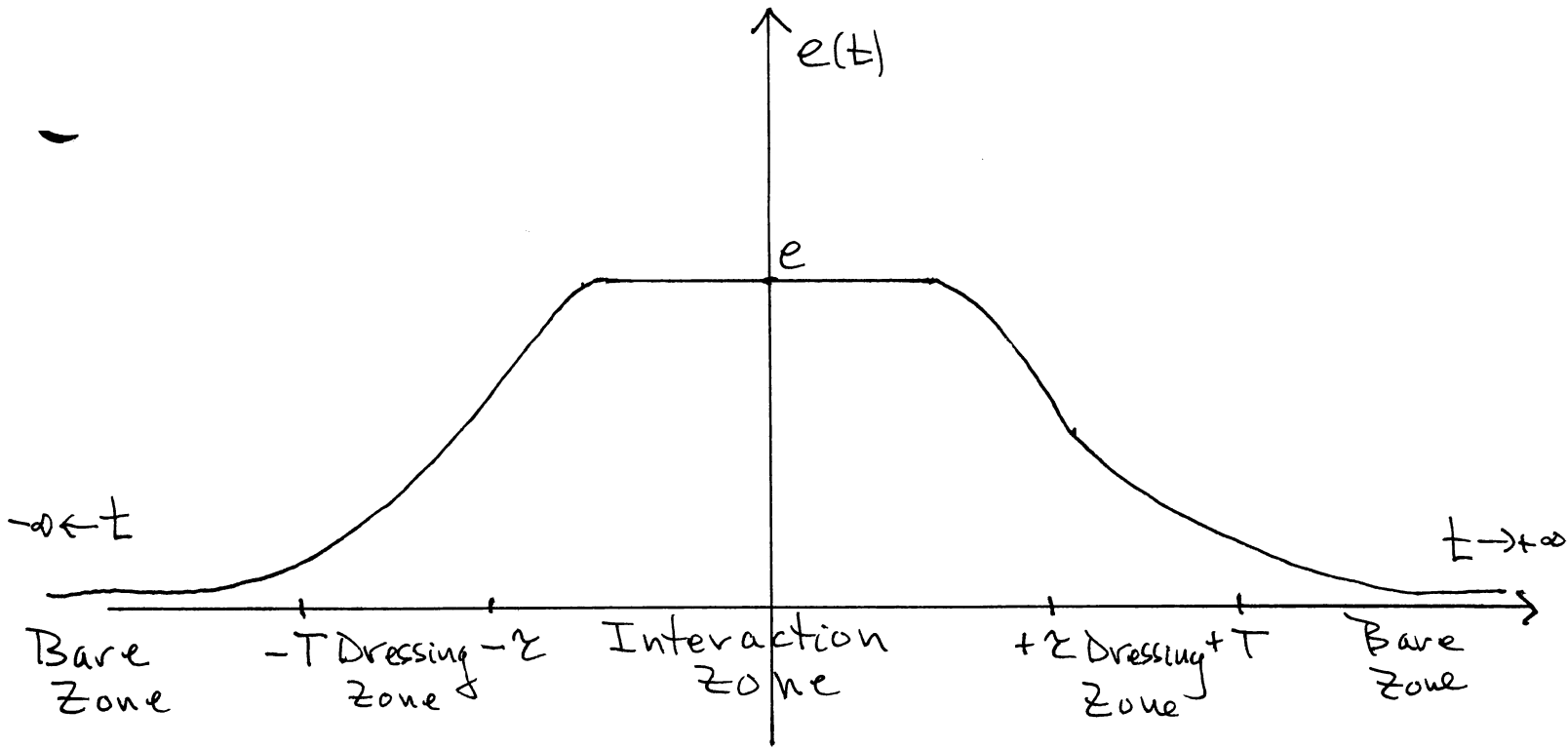


Figure 3.1.3

The transition probability amplitude is given by

$${}_{IP}\langle f(t) | i(t) \rangle_{IP}. \quad (3.1.122)$$

Since $|f(t)\rangle_{IP}$ and $|i(t)\rangle_{IP}$ both evolve according to $H_I^{IP}(t)$, this is independent of time, the unitary time evolution operator leaves probabilities unchanged so

$$\begin{aligned} {}_{IP}\langle f(t) | i(t) \rangle_{IP} &= \langle f | i(t \rightarrow +\infty) \rangle_{IP} \\ &= \langle f | U(+\infty, -\infty) | i \rangle. \end{aligned} \quad (3.1.123)$$

This transition probability amplitude is called the S -matrix element

$$S_{fi} \equiv \langle f | U(+\infty, -\infty) | i \rangle. \quad (3.1.124)$$

The time evolution operator that takes us from initial times to final times is called the S -operator

$$S = U(+\infty, -\infty) = T e^{-i \int_{-\infty}^{+\infty} d^4x \mathcal{H}_I^{IP}(x)}. \quad (3.1.125)$$

Then the S -matrix element is precisely that the initial and final state matrix element of the S -operator. As we will see, the initial and final vacuum states may differ by a

phase. This additional phase difference will then be present between each initial and final state. Hence we might as well factor it out of the definition of the S -operator. So if

$$\begin{aligned} |0(t \rightarrow +\infty)\rangle_{IP} &= e^{i\varphi} |0(t \rightarrow -\infty)\rangle_{IP} \\ &\equiv e^{i\varphi} |0\rangle, \end{aligned}$$

then the initial to final vacuum transition amplitude is given by

$${}_{IP}\langle 0(t \rightarrow -\infty)|0(t \rightarrow +\infty)\rangle_{IP} = e^{+i\varphi} = {}_{\subset}0|U(+\infty, -\infty)|0\rangle_{\supset}.$$

Thus factoring this phase out of the S -operator we define it as

$$\begin{aligned} S &\equiv \frac{U(+\infty, -\infty)}{{}_{\subset}0|U(+\infty, -\infty)|0\rangle_{\supset}} \\ &= \frac{T e^{-i \int_{-\infty}^{+\infty} d^4x \mathcal{H}_I^{IP}(x)}}{{}_{\subset}0|T e^{-i \int_{-\infty}^{+\infty} d^4x \mathcal{H}_I^{IP}(x)}|0\rangle_{\supset}}. \end{aligned}$$

Correspondingly the S matrix elements are given by

$$S_{fi} = \frac{{}_{\subset}f|T e^{-i \int_{-\infty}^{+\infty} d^4x \mathcal{H}_I^{IP}(x)}|i\rangle_{\supset}}{{}_{\subset}0|T e^{-i \int_{-\infty}^{+\infty} d^4x \mathcal{H}_I^{IP}(x)}|0\rangle_{\supset}}.$$

We will evaluate $e^{i\varphi}$ in section 3.2 but for now let's choose the final vacuum to be the same as the initial vacuum, that is the phase to be zero.

We are now in a position to apply these techniques to calculating scattering operator matrix elements in QED. Let's recall the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{g}} \quad (3.1.126)$$

where

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Psi}\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - \frac{i}{2}(\partial_\mu - ieA_\mu)\bar{\Psi}\gamma^\mu\Psi - m\bar{\Psi}\Psi \quad (3.1.127)$$

and

$$\mathcal{L}_{\text{g}} = -\frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2. \quad (3.1.128)$$

These are Heisenberg picture fields. In order to find our interaction picture fields we must separate the Hamiltonian into free H_0 and interaction parts H_I . Since the

interaction in QED is non-derivative this takes a simple form. Recall the Hamiltonian density is

$$\mathcal{H} = \dot{\phi}\Pi - \mathcal{L} = T^{00} \quad (3.1.129)$$

writing $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 + \bar{\Psi}\left(\frac{i}{2}\overleftrightarrow{\not{D}} - m\right)\Psi, \quad (3.1.130)$$

that is all bilinear terms in the fields in \mathcal{L} , and

$$\mathcal{L}_I = -eA_\mu\bar{\Psi}\gamma^\mu\Psi \quad (3.1.131)$$

we see that $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ with the non-interacting Hamiltonian made from the free Lagrangian \mathcal{L}_0 , $\mathcal{H}_0 = \dot{\phi}\Pi - \mathcal{L}_0$. Recall for non-derivative coupling

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}_0}{\partial\dot{\phi}}, \quad (3.1.132)$$

so the interaction Hamiltonian is made purely from the interaction Lagrangian \mathcal{L}_I , i.e. $\mathcal{H}_I = -\mathcal{L}_I$.

Thus in the interaction picture the operators are all Heisenberg operators multiplied by the time evolution operators $U(t,0)O(t,\vec{x})U^{-1}(t,0) = O^{IP}(t,\vec{x})$. Thus

$$\begin{aligned} \mathcal{H}_0^{IP} &= \mathcal{H}_0(\phi^{IP}, \Pi^{IP}) = \dot{\phi}^{IP}\Pi^{IP} - \mathcal{L}_0(\phi^{IP}, \Pi^{IP}) \\ \mathcal{H}_I^{IP} &= \mathcal{H}_I(\phi^{IP}) = -\mathcal{L}_I(\phi^{IP}) = eA_\mu^{IP}\bar{\Psi}^{IP}\gamma^\mu\Psi^{IP}. \end{aligned} \quad (3.1.133)$$

That is, for the interaction picture, we just replace in the expressions for the operators in terms of the fields and momenta in the Heisenberg picture the fields and momenta in the interaction picture. The S -operator is then given by

$$S = \text{T}e^{-i\int d^4x \mathcal{H}_I^{IP}} = \text{T}e^{+i\int d^4x \mathcal{L}_I^{IP}} \quad (3.1.134)$$

or

$$S = \text{T}e^{-ie\int d^4x A_\mu^{IP}\bar{\Psi}^{IP}\gamma^\mu\Psi^{IP}}. \quad (3.1.135)$$

According to the interaction picture the field operators obey the free equations of motion

$$\left[H_0^{IP}, \phi^{IP}(x)\right] = -i\frac{\partial}{\partial t}\phi^{IP}(x)$$

$$\left[H_0^{IP}, \Pi^{IP}(x) \right] = -i \frac{\partial}{\partial t} \Pi^{IP}(x). \quad (3.1.136)$$

Equivalently rather than the Heisenberg equations of motion we can use the Euler-Lagrange equations of motion from \mathcal{L}_0^{IP}

$$\begin{aligned} 1) \quad & \frac{\partial \mathcal{L}_0^{IP}}{\partial A_\nu^{IP}} - \partial_\mu \frac{\partial \mathcal{L}_0^{IP}}{\partial \partial_\mu A_\nu^{IP}} = 0 = \partial_\mu F^{IP\mu\nu} + \frac{1}{\alpha} \partial^\nu \partial_\lambda A^{IP\lambda} \\ 2) \quad & \frac{\partial \mathcal{L}_0^{IP}}{\partial \bar{\Psi}^{IP}} - \partial_\mu \frac{\partial \mathcal{L}_0^{IP}}{\partial \partial_\mu \bar{\Psi}^{IP}} = 0 = (i\rlap{/}\partial - m) \Psi^{IP} \\ 3) \quad & \frac{\partial \mathcal{L}_0^{IP}}{\partial \Psi^{IP}} - \partial_\mu \frac{\partial \mathcal{L}_0^{IP}}{\partial \partial_\mu \Psi^{IP}} = 0 = \bar{\Psi}^{IP} (i\overleftarrow{\rlap{/}\partial} + m). \end{aligned} \quad (3.1.137)$$

As before in order to minimize the algebra to follow we will work in the Feynman gauge $\alpha = 1$. The fields then obey the free field equations that we have analyzed previously

$$\begin{aligned} \partial^2 A^{IP\mu}(x) &= 0 \\ (i\rlap{/}\partial - m) \Psi^{IP} &= 0 \\ \bar{\Psi}^{IP} (i\overleftarrow{\rlap{/}\partial} + m) &= 0. \end{aligned} \quad (3.1.138)$$

We can Fourier transform these fields in terms of the plane wave solutions to the wave equation and Dirac equation

$$\begin{aligned} A^{IP\mu}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^3 \epsilon^\mu(k, \lambda) [a_{(\lambda)}(\vec{k}) e^{-ikx} + a_{(\lambda)}^\dagger(\vec{k}) e^{+ikx}] \\ \Psi^{IP}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^2 [b_s(\vec{k}) u^{(s)}(\vec{k}) e^{-ikx} + d_s^\dagger(\vec{k}) v^{(s)}(\vec{k}) e^{+ikx}] \\ \bar{\Psi}^{IP}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^2 [d_s(\vec{k}) \bar{v}^{(s)}(\vec{k}) e^{-ikx} + b_s^\dagger(\vec{k}) \bar{u}^{(s)}(\vec{k}) e^{+ikx}]. \end{aligned} \quad (3.1.139)$$

The interaction picture fields are related to the Heisenberg picture fields by a unitary transformation, hence they obey the ETCR,

$$\begin{aligned} \delta(x^0 - y^0) \left\{ \Psi_a^{IP}(x), \bar{\Psi}_b^{IP}(y) \right\} &= \gamma_{ab}^0 \delta^4(x - y) \\ \delta(x^0 - y^0) \left[\dot{A}^{IP\mu}(x), A^{IP\nu} \right] &= +ig^{\mu\nu} \delta^4(x - y), \end{aligned} \quad (3.1.140)$$

all other ETCR vanishing. As usual these imply the creation and annihilation operator algebra for a, b, d :

$$\begin{aligned}
\{b_r(\vec{p}), b_s^\dagger(\vec{k})\} &= (2\pi)^3 2\omega_k \delta_{rs} \delta^3(\vec{p} - \vec{k}) \\
\{d_r(\vec{p}), d_s^\dagger(\vec{k})\} &= (2\pi)^3 2\omega_k \delta_{rs} \delta^3(\vec{p} - \vec{k}) \\
[a_{(\lambda)}(\vec{p}), a_{(\rho)}^\dagger(\vec{k})] &= -g_{\lambda\rho} (2\pi)^3 2\omega_k \delta^3(\vec{p} - \vec{k})
\end{aligned} \tag{3.1.141}$$

with all other commutators vanishing;

$$\begin{aligned}
\{b_r(\vec{p}), b_s(\vec{k})\} &= 0 = \{d_r(\vec{p}), d_s(\vec{k})\} \\
\{b_r(\vec{p}), d_s(\vec{k})\} &= 0 = \{b_r(\vec{p}), d_s^\dagger(\vec{k})\} \\
[a_{(\lambda)}(\vec{p}), a_{(\rho)}(\vec{k})] &= 0 = [a_{(\lambda)}(\vec{p}), b_s(\vec{k})] \\
[a_{(\lambda)}(\vec{p}), d_s(\vec{k})] &= 0 = [a_{(\lambda)}(\vec{p}), b_s^\dagger(\vec{k})] \\
[a_{(\lambda)}(\vec{p}), d_s^\dagger(\vec{k})] &= 0.
\end{aligned} \tag{3.1.142}$$

The state vectors in the interaction picture evolve in time according to the interaction Hamiltonian or equivalently by the time evolution operator $U(t, t_0)$. According to the adiabatic hypothesis for initial and final times ($t \rightarrow \pm\infty$) the interaction Hamiltonian vanishes $H_I^{IP}(t) \xrightarrow{t \rightarrow \pm\infty} 0$. Hence the interaction picture states are described by eigenstates of $\mathcal{P}_o^{IP\mu}$ and $\vec{\mathcal{J}}_o^{IP}$ and Q_o^{IP} as well as the number operators N_o^{IP} . Recall that in general the single particle states in the theory are the eigenstates of the energy-momentum four vector \mathcal{P}^μ and the helicity $\frac{\vec{\mathcal{J}} \cdot \vec{k}}{|\vec{k}|}$ or spin projection \mathcal{J}_3 in the rest frame of massive particles and whatever other charges might be present like the electric charge Q . In the interaction picture

$$\begin{aligned}
\mathcal{P}^{IP\mu} &= \mathcal{P}_0^{IP\mu} + \mathcal{P}_I^{IP\mu} \\
\mathcal{M}^{IP\mu\nu} &= \mathcal{M}_0^{IP\mu\nu} + \mathcal{M}_I^{IP\mu\nu} \\
Q^{IP} &= Q_0^{IP} + Q_I^{IP}
\end{aligned} \tag{3.1.143}$$

where \mathcal{O}_0^{IP} operators are made from \mathcal{L}_0^{IP} as if H_I^{IP} were zero and \mathcal{O}_I^{IP} are the additional terms due to $H_I^{IP} \neq 0$. Note that in QED we have that

$$\mathcal{P}^{IP0} = H^{IP} = H_0^{IP} + H_I^{IP} \tag{3.1.144}$$

in general, and that since there is no derivative coupling

$$\vec{\mathcal{P}}^{IP} = \vec{\mathcal{P}}_0^{IP} \quad \text{or } \vec{\mathcal{P}}_I^{IP} = 0; \quad (3.1.145)$$

the momentum has the same form in the free as the interacting theory. Similarly (with the $-e$ factor not taken to zero, it is just the unit of charge)

$$Q^{IP} = Q_0^{IP} \quad (3.1.146)$$

and

$$\mathcal{M}^{ijIP} = \mathcal{M}_0^{ijIP}. \quad (3.1.147)$$

The point is that although we can construct H^{IP} in terms of the fields of the theory we do not know how to construct the eigenstates of H^{IP} directly from the (free IP) fields at arbitrary time. However as we see above in the interaction picture according to the adiabatic hypothesis $H_I^{IP}(t) \xrightarrow{t \rightarrow \pm\infty} 0$ then $\mathcal{P}^{IP\mu} = \mathcal{P}_0^{IP\mu}$, $\mathcal{M}^{ijIP} = \mathcal{M}_0^{ijIP}$, and $Q^{IP} = Q_0^{IP}$ are expressed in terms of the free IP fields and have the free non-interacting form. The eigenstates of these operators we know how to form from the creation operators acting on the vacuum. Of course once we have the eigenstates they remain eigenstates. Since they are a basis of the state space we can expand any state in terms of them. They evolve in time according to $U(t, \pm\infty)$, formerly then we can find the eigenstate at any time.

However we are only explicitly interested in constructing the initial and final states. The S -matrix elements ($U(+\infty, -\infty)$) will then be evaluated perturbatively; that is the time evolution of these eigenstates will be evaluated perturbatively. So the totality of initial and final states is determined for $H_I^{IP}(t \rightarrow \pm\infty) = 0$. Thus the free Lagrangian \mathcal{L}_0^{IP} describes the fields used for the Hilbert space construction. \mathcal{L}_0^{IP} is just the free field theory we have been studying in detail the last few chapters:

$$\mathcal{L}_0^{IP} = -\frac{1}{4}F_{\mu\nu}^{IP}F^{IP\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^{IP\lambda})^2 + \bar{\Psi}^{IP}\left(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m\right)\Psi^{IP}. \quad (3.1.148)$$

The general space \mathcal{V} of states is found from the action of a^\dagger , b^\dagger , and d^\dagger on the vacuum. Thus the lowest energy state is found to be $|0\rangle$ so that it is defined to be the state such that

$$\begin{aligned} a_{(\lambda)}(\vec{k})|0\rangle &= 0 & \text{all } \lambda, \vec{k} \\ b_s(\vec{k})|0\rangle &= 0 & \text{all } s, \vec{k} \\ d_s(\vec{k})|0\rangle &= 0 & \text{all } s, \vec{k} \end{aligned} \quad (3.1.149)$$

(Of course we must first normal order the Lagrangian, $T^{\mu\nu}$, $M^{\mu\nu\rho}$, J^μ to define the vacuum thusly. Since in the interaction picture the fields are just as free fields we can normal order \mathcal{L} , \mathcal{L}_0 , \mathcal{L}_I , $T^{\mu\nu}$, $T_0^{\mu\nu}$, $T_I^{\mu\nu}$, etc.. So everywhere we replace products of fields with normal products – this will change things as far as the interaction goes and field equations, etc.. More on this later.)

The one particle states are

$$\begin{aligned}
|\vec{k}, (\lambda) \rangle &= a_{(\lambda)}^\dagger(\vec{k})|0 \rangle && \text{for "photon" states} \\
|\vec{k}, \frac{(-1)^{s+1}}{2}, - \rangle &= b_s^\dagger(\vec{k})|0 \rangle && \text{for electron states} \\
|\vec{k}, \frac{(-1)^s}{2}, + \rangle &= d_s^\dagger(\vec{k})|0 \rangle && \text{for positron states}
\end{aligned} \tag{3.1.150}$$

(Note the charge of the electron states is $-e$, positron states $+e$, we have multiplied the previous Q by $-e$.) As before these are eigenstates of energy and momentum

$$\mathcal{P}^{I P \mu} |\vec{k}, \dots \rangle = k^\mu |\vec{k}, \dots \rangle \tag{3.1.151}$$

of charge

$$\begin{aligned}
Q|\vec{k}, (\lambda) \rangle &= 0 && \text{photon has no electric charge} \\
Q|\vec{k}, \pm \frac{1}{2}, - \rangle &= -e|\vec{k}, \pm \frac{1}{2}, - \rangle && \text{electron has } -e \text{ charge} \\
Q|\vec{k}, \pm \frac{1}{2}, + \rangle &= +e|\vec{k}, \pm \frac{1}{2}, + \rangle && \text{positron has } +e \text{ charge.}
\end{aligned} \tag{3.1.152}$$

The photon has helicity $\pm 1, 0, 0$ hence

$$\frac{\vec{\mathcal{J}} \cdot \vec{k}}{|\vec{k}|} |\vec{k}, (\lambda) \rangle = \begin{cases} 0 & \text{for } \lambda = 0, 3 \\ \pm |\vec{k}, (\lambda) \rangle & \text{for } \lambda = \pm. \end{cases} \tag{3.1.153}$$

Also, the electron and positron have spin $\pm \frac{1}{2}$ on the third axis at rest

$$\begin{aligned}
\mathcal{J}_3 |\vec{0}, \pm \frac{1}{2}, - \rangle &= \pm \frac{1}{2} |\vec{0}, \pm \frac{1}{2}, - \rangle \\
\mathcal{J}_3 |\vec{0}, \pm \frac{1}{2}, + \rangle &= \pm \frac{1}{2} |\vec{0}, \pm \frac{1}{2}, + \rangle.
\end{aligned} \tag{3.1.154}$$

The photon $N_{(\lambda)}$, electron N_{e^-} and positron N_{e^+} number operators defined as

$$\begin{aligned}
N_{(\lambda)} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (-g_{\lambda\lambda}) a_{(\lambda)}^\dagger(\vec{k}) a_{(\lambda)}(\vec{k}) \quad (\Sigma_\lambda) \\
N_{e^-} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^2 b_s^\dagger(\vec{k}) b_s(\vec{k}) \\
N_{e^+} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^2 d_s^\dagger(\vec{k}) d_s(\vec{k})
\end{aligned} \tag{3.1.155}$$

yield the number of respective particles in each state :

$$\begin{aligned}
N_{(\lambda)}|0\rangle &= N_{e^-}|0\rangle = N_{e^+}|0\rangle = 0 \\
N_{(\lambda)}|\vec{k}, (\rho)\rangle &= \delta_{\lambda\rho}|\vec{k}, (\rho)\rangle \\
N_{e^+}|\vec{k}, \pm\frac{1}{2}, +\rangle &= |\vec{k}, \pm\frac{1}{2}, +\rangle \\
N_{e^-}|\vec{k}, \pm\frac{1}{2}, -\rangle &= |\vec{k}, \pm\frac{1}{2}, -\rangle \\
N_{(\lambda)}|\vec{k}, \pm\frac{1}{2}, \pm\rangle &= 0 = N_{e^\pm}|\vec{k}, (\lambda)\rangle \\
N_{e^+}|\vec{k}, \pm\frac{1}{2}, -\rangle &= 0 = N_{e^-}|\vec{k}, \pm\frac{1}{2}, +\rangle.
\end{aligned} \tag{3.1.156}$$

The multiparticle states are made from repeated application of the creation operators, the l -photon, m -electron, n -positron state is

$$\begin{aligned}
|(\vec{k}_1, \lambda_1), \dots, (\vec{k}_l, \lambda_l); (\vec{l}_1, \frac{(-1)^{s_1+1}}{2}, -), \dots; (\vec{p}_1, \frac{(-1)^{r_1}}{2}, +), \dots\rangle \\
= a_{(\lambda_1)}^\dagger(\vec{k}_1) \cdots b_{s_1}^\dagger(\vec{l}_1) \cdots d_{r_1}^\dagger(\vec{p}_1) \cdots |0\rangle.
\end{aligned} \tag{3.1.157}$$

The inner product of states in \mathcal{V} is defined as usual through the commutator and with $\langle 0|0\rangle = 1$,

$$\begin{aligned}
\langle \vec{k}, \frac{(-1)^{r+1}}{2}, - | \vec{k}', \frac{(-1)^{r'+1}}{2}, - \rangle &= (2\pi)^3 2\omega_k \delta_{rr'} \delta^3(\vec{k} - \vec{k}') \\
\langle \vec{k}, \frac{(-1)^r}{2}, + | \vec{k}', \frac{(-1)^{r'}}{2}, + \rangle &= (2\pi)^3 2\omega_k \delta_{rr'} \delta^3(\vec{k} - \vec{k}')
\end{aligned} \tag{3.1.158}$$

but

$$\langle \vec{k}, \lambda | \vec{k}', \lambda' \rangle = -g_{\lambda\lambda'} (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'), \quad (3.1.159)$$

all others vanishing.

As usual the full space \mathcal{V} is an indefinite metric space – it contains unphysical photons, the scalar ($\lambda = 0$) and longitudinal ($\lambda = 3$) modes. The physical subspace of states is defined by means of the Gupta- Bleuler subsidiary condition. The condition is

$$\partial_\lambda A^{\lambda+}(x) |\Phi\rangle = 0 \quad (3.1.160)$$

if $|\Phi\rangle$ is a physical state. Transforming to the interaction picture yields

$$\begin{aligned} 0 &= e^{iH_0^S t} e^{-iHt} \partial_\lambda A^{\lambda+}(t, \vec{x}) e^{iHt} e^{-iH_0^S t} e^{iH_0^S t} e^{-iHt} |\Phi\rangle \\ &= \partial_\lambda A^{IP\lambda+}(t, \vec{x}) |\Phi(t)\rangle_{IP}. \end{aligned} \quad (3.1.161)$$

Since $\partial_\lambda A^{IP\lambda}$ always obeys the free wave equation $\partial^2 \partial_\lambda A^{IP\lambda} = 0$, this condition reduces to

$$\left[a_{(0)}(\vec{k}) - a_{(3)}(\vec{k}) \right] |\Phi(t)\rangle_{IP} = 0 \quad (3.1.162)$$

and for initial and final states $t \rightarrow \pm\infty$, $|\Phi(t)\rangle_{IP} \rightarrow |\Phi\rangle$. Thus the physical subspace of states involves only states which obey

$$a_{(0)}(\vec{k}) |\Phi\rangle = a_{(3)}(\vec{k}) |\Phi\rangle. \quad (3.1.163)$$

The most general such vector is of the form

$$|\Phi\rangle = |\Phi_{tr}\rangle + |\hat{\Phi}\rangle \quad (3.1.164)$$

where $|\Phi_{tr}\rangle$ consists of e^\pm and transverse $\lambda = 1, 2$ photons only. While

$$|\hat{\Phi}\rangle = \left[1 + \sum_{n=1}^{\infty} \int \frac{d^3 k_1 \cdots d^3 k_n}{(2\pi)^3 2\omega_{k_1} \cdots (2\pi)^3 2\omega_{k_n}} C(\vec{k}_1 \dots \vec{k}_n) \mathcal{G}^\dagger(\vec{k}_1) \cdots \mathcal{G}^\dagger(\vec{k}_n) \right] |0\rangle \quad (3.1.165)$$

and

$$\mathcal{G}^\dagger(\vec{k}) \equiv a_{(0)}^\dagger(\vec{k}) - a_{(3)}^\dagger(\vec{k}). \quad (3.1.166)$$

As we saw we could always choose the gauge Λ with $\partial^2 \Lambda = 0$ such that $|\hat{\Phi}\rangle = |0\rangle$, that is

$$|\Phi\rangle = |\Phi_{tr}\rangle. \quad (3.1.167)$$

Hence we can now construct the complete set of initial and final physical states forming a Hilbert space

$$\mathcal{H}_{\text{phys}} = |0\rangle \oplus \{|\vec{k}, \lambda = 1 \text{ or } 2\rangle, |\vec{k}, \pm \frac{1}{2}, \pm\rangle\} \oplus \dots \quad (3.1.168)$$

For example, suppose we have an initial state of one electron with momentum \vec{p} , spin $+\frac{1}{2}$ and one photon of momentum \vec{k} , linearly polarized in the $\lambda = 2$ direction

$$\begin{aligned} |i\rangle &= |(\vec{p}, +\frac{1}{2}, -); (\vec{k}, 2)\rangle = |\vec{p}, +\frac{1}{2}, -\rangle |\vec{k}, 2\rangle \\ &= b_1^\dagger(\vec{p}) a_{(2)}^\dagger(\vec{k}) |0\rangle. \end{aligned} \quad (3.1.169)$$

Or an initial state of a positron of momentum \vec{p} , spin $+\frac{1}{2}$ and an electron of momentum \vec{q} , spin $-\frac{1}{2}$

$$\begin{aligned} |i\rangle &= |(\vec{q}, -\frac{1}{2}, -); (\vec{p}, +\frac{1}{2}, +)\rangle = |\vec{q}, -\frac{1}{2}, -\rangle |\vec{p}, +\frac{1}{2}, +\rangle \\ &= b_2^\dagger(\vec{q}) d_2^\dagger(\vec{p}) |0\rangle. \end{aligned} \quad (3.1.170)$$

Similarly suppose we have a final state with a electron of momentum \vec{p} , spin $+\frac{1}{2}$ and one photon of momentum \vec{k} , linearly polarized in the $\lambda = 2$ direction

$$\begin{aligned} |f\rangle &= |(\vec{p}, +\frac{1}{2}, -); (\vec{k}, 2)\rangle \\ &= b_1^\dagger(\vec{p}) a_{(2)}^\dagger(\vec{k}) |0\rangle \end{aligned} \quad (3.1.171)$$

the same state as the first initial state example. Again the final state Hilbert space is the same as the initial state Hilbert space which makes sense physically since the initial states of the experiment are the final states of the experiment before it.

We are now ready to calculate transition probability amplitudes, that is S -matrix elements. The S -operator is given by

$$S = \text{T}e^{-i \int d^4x \mathcal{H}_I^P}. \quad (3.1.172)$$

First recall that S is restricted Lorentz invariant

$$U^{-1}(a, \Lambda) S U(a, \Lambda) = S \quad (3.1.173)$$

i.e.

$$\left[S, U(a, \Lambda) \right] = 0. \quad (3.1.174)$$

This follows from the fact that $\int d^4x \mathcal{H}_I^{IP}$ is invariant and then the only question is whether the time ordering can be changed by a Lorentz transformation. This can happen only for space-like separated points in $T\mathcal{H}_I^{IP}(x_1) \cdots \mathcal{H}_I^{IP}(x_n)$, however by the microcausality property of our fields

$$\left[\mathcal{H}_I^{IP}(x_1), \mathcal{H}_I^{IP}(x_2) \right] = 0 \quad \text{for } (x_1 - x_2)^2 < 0. \quad (3.1.175)$$

Thus the order does not matter and S is invariant.