§2.3 THE SPIN $\frac{1}{2}$ FERMION FIELD

We next desire to describe non-interacting (free) particles with spin $\frac{1}{2}$. As we know these particles will obey Fermi- Dirac statistics due to the Pauli exclusion principle. Hence, we can expect the commutation rules to be changed for spin $\frac{1}{2}$ fields since the canonical commutation rules implied symmetric wave functions under the interchange of the particles. For antisymmetric wave functions we will need anti-commutation relations for the fermion creation and annihilation operators.

First, let's introduce the field that will be used to describe the spin $\frac{1}{2}$ particles; the four component Dirac (bi-)spinor field, denoted by $\Psi_a(x)$, where $a = 1, 2, 3, 4$. Recall that we can obtain the left-handed and right-handed components of Ψ by multiplying with respect to the γ_5 matrix

$$\Psi_L \equiv \frac{1}{2}(1-\gamma_5)\Psi$$

$$\Psi_R \equiv \frac{1}{2}(1+\gamma_5)\Psi \tag{2.3.1}$$

where $\gamma_5 \equiv +i\gamma^0\gamma^1\gamma^2\gamma^3$ with γ^μ, $\mu = 0, 1, 2, 3$ are the four Dirac γ matrices. The Dirac matrices are four-by-four matrices that obey their defining Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1} \tag{2.3.2}$$

and have the conjugation properties

$${\gamma^0}^\dagger = \gamma^0$$

$${\gamma^i}^\dagger = -\gamma^i \tag{2.3.3}$$

that is

$${\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{2.3.4}$$

(see section 1.2 for a review of the γ-matrices).

The Dirac spinor field $\Psi_a(x)$ has the next simplest non-trivial transformation properties under the restricted Poincare' group (recall that spin zero or scalar fields were invariant under $L_+^\uparrow$ i.e. they transformed according to the identity

$$U^{-1}(a,\Lambda)\Phi(\Lambda x + a)U(a,\Lambda) = 1\Phi(x)\big) .$$

The transformation law is given by the quantum mechanical operator transformation law analogous to equation (1.2.128) and (1.3.28), that is, for $x' = \Lambda x + a$,

$$\begin{aligned}(\psi'_a(x')) &\equiv < A'|\Psi_a(x')|B' > \\ &= L_{ab}(\Lambda) < A|\Psi_b(x)|B > \\ &(= L_{ab}\psi_b(x))\end{aligned} \tag{2.3.5}$$

where $\psi_b(x) \equiv < A|\Psi_b(x)|B >$ and we denote $D_{ab}(S) \equiv L_{ab}$. Since the transformation is implemented by unitary operators $U(a, \Lambda)$ we have

$$U^{-1}(a,\Lambda)\Psi_a(x')U(a,\Lambda) = L_{ab}(\Lambda)\Psi_b(x) \tag{2.3.6}$$

or multiplying on the left by U and on the right by U^{-1}, we obtain

$$U(a,\Lambda)\Psi_a(x)U^{-1}(a,\Lambda) = L^{-1}_{ab}(\Lambda)\Psi_b(\Lambda x + a) \tag{2.3.7}$$

where recall equation (1.2.131)

$$\Lambda^{\mu\nu}\gamma_\mu = L\gamma^\nu L^{-1} \tag{2.3.8}$$

that is

$$\Lambda^{\mu\nu} = \frac{1}{4}\mathrm{Tr}\left[\gamma^\mu L\gamma^\nu L^{-1}\right]. \tag{2.3.9}$$

Since any finite $\mathcal{P}^\uparrow_+$ transformation can be built up from the infintesimal, consider those

$$x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu + a^\nu \tag{2.3.10}$$

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \tag{2.3.11}$$

hence

$$U(a,\Lambda) = e^{ia_\mu\mathcal{P}^\mu}e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}} \approx 1 + ia_\mu\mathcal{P}^\mu - \frac{i}{2}\omega^{\mu\nu}\mathcal{M}^{\mu\nu} \tag{2.3.12}$$

while by equations (1.2.129), (1.2.98), (1.2.107) and (1.2.123)

$$L_{ab} = \delta_{ab} - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}_{ab} \tag{2.3.13}$$

where now

$$(\sigma^{\mu\nu})_{ab} \equiv \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]_{ab} \tag{2.3.14}$$

is the 4×4 spin matrix. Note that

$$L^{-1} = \left(1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \tag{2.3.15}$$

hence

$$\begin{aligned} L\gamma^{\nu}L^{-1} &= \left(1 - \frac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\right)\gamma^{\mu}\left(1 + \frac{i}{4}\omega_{\rho\lambda}\sigma^{\rho\lambda}\right) \\ &= \gamma^{\nu} - \frac{i}{4}\omega_{\alpha\beta}\left[\sigma^{\alpha\beta}, \gamma^{\nu}\right] + O(\omega^2). \end{aligned} \tag{2.3.16}$$

We can evaluate the commutator

$$\begin{aligned} \left[\sigma^{\alpha\beta}, \gamma^{\nu}\right] &= \frac{i}{2}\left[\left[\gamma^{\alpha}, \gamma^{\beta}\right], \gamma^{\nu}\right] \\ &= \frac{i}{2}\left(\left[\gamma^{\alpha}\gamma^{\beta}, \gamma^{\nu}\right] - \left[\gamma^{\beta}\gamma^{\alpha}, \gamma^{\nu}\right]\right) \end{aligned} \tag{2.3.17}$$

by using the identity $[AB, C] = A[B, C]_+ - [A, C]_+ B$, we have

$$\begin{aligned} \left[\sigma^{\alpha\beta}, \gamma^{\nu}\right] &= \frac{i}{2}\left(\gamma^{\alpha}\{\gamma^{\beta}, \gamma^{\nu}\} - \{\gamma^{\alpha}, \gamma^{\nu}\}\gamma^{\beta} - \gamma^{\beta}\{\gamma^{\alpha}, \gamma^{\nu}\} + \{\gamma^{\beta}, \gamma^{\nu}\}\gamma^{\alpha}\right) \\ &= \frac{i}{2}(2)\left(g^{\beta\nu}\gamma^{\alpha} - g^{\alpha\nu}\gamma^{\beta} - g^{\alpha\nu}\gamma^{\beta} + g^{\beta\nu}\gamma^{\alpha}\right) \\ &= 2i\left(g^{\beta\nu}\gamma^{\alpha} - g^{\alpha\nu}\gamma^{\beta}\right). \end{aligned} \tag{2.3.18}$$

So indeed we explicitly check equation (2.3.8)

$$\begin{aligned} L\gamma^{\nu}L^{-1} &= \gamma^{\nu} - \frac{i}{4}(2i)\omega_{\alpha\beta}\left(g^{\beta\nu}\gamma^{\alpha} - g^{\alpha\nu}\gamma^{\beta}\right) \\ &= \gamma^{\nu} + \frac{1}{2}\left(\omega_{\alpha}{}^{\nu}\gamma^{\alpha} - \omega^{\nu}{}_{\beta}\gamma^{\beta}\right) \\ &= \gamma^{\nu} - \omega^{\nu\mu}\gamma_{\mu} \end{aligned} \tag{2.3.19}$$

but $\Lambda^{-1\nu\mu} = g^{\nu\mu} - \omega^{\mu\nu} = \Lambda^{\mu\nu}$ so

$$\begin{aligned} L\gamma^{\nu}L^{-1} &= \Lambda^{-1\nu\mu}\gamma_{\mu} \\ &= \Lambda^{\mu\nu}\gamma_{\mu}. \end{aligned} \tag{2.3.20}$$

Thus, we find for infinitesimal $\mathcal{P}_+^{\uparrow}$ transformations from equation (2.3.7)

$$[\mathcal{P}^{\mu}, \Psi_a(x)] = -i\partial^{\mu}\Psi_a(x)$$

$$[\mathcal{M}^{\mu\nu}, \Psi_a(x)] = -i\left[\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)\Psi_a(x) - \frac{i}{2}\sigma^{\mu\nu}_{ab}\Psi_b(x)\right] \tag{2.3.21}$$

where $D^{\mu\nu}_{ab} \equiv \frac{i}{2}\sigma^{\mu\nu}_{ab}$ according to the notation of equations (1.2.115) and (1.2.116) and obeys the Lorentz algebra, equation (1.2.113). We can contract spinor fields and Dirac matrices in order to make scalars, vectors and tensors. Hence it can be shown that under Lorentz transformations

$$\begin{gathered}
\overline{\Psi}\Psi \text{ is a Lorentz scalar,} \\
\overline{\Psi}\gamma_5\Psi \text{ is a Lorentz pseudoscalar,} \\
\overline{\Psi}\gamma^\mu\Psi \text{ is a Lorentz vector,} \\
\overline{\Psi}\gamma_5\gamma^\mu\Psi \text{ is a Lorentz pseudovector or axial vector,} \\
\overline{\Psi}\sigma^{\mu\nu}\Psi \text{ is a Lorentz tensor,}
\end{gathered} \tag{2.3.22}$$

where $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$.

For non-interacting spin $\frac{1}{2}$ particles of mass m, Dirac discovered that the field equation is given by a first order partial differential equation known as the Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\Psi(x) = 0 \tag{2.3.23}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. Writing out the indices we have

$$i\partial^x_\mu \gamma^\mu_{ab}\Psi_b(x) - m\Psi_a(x) = 0, \tag{2.3.24}$$

so we see that the Dirac equation is a matrix differential equation. Since the particles have mass m the relativistic relation $p^2 = m^2$ should be valid, that is, $\Psi(x)$ should also obey the Klein-Gordon equation. To see this consider

$$\begin{aligned}
\left(i\partial\!\!\!/ + m\right)\left(i\partial\!\!\!/ - m\right)\Psi &= \left(-\partial\!\!\!/\partial\!\!\!/ - m^2\right)\Psi \\
&= 0.
\end{aligned} \tag{2.3.25}$$

However,

$$\begin{aligned}
\partial\!\!\!/\partial\!\!\!/ &= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu \\
&= g^{\mu\nu}\mathbf{1}\partial_\mu\partial_\nu = \partial^2
\end{aligned} \tag{2.3.26}$$

and thus,

$$-\left(\partial^2 + m^2\right)\Psi_a(x) = 0, \tag{2.3.27}$$

each component of $\Psi(x)$ obeys the Klein-Gordon equation. Furthermore, if $(i\partial\!\!\!/ - m)\,\Psi = 0$ then taking the hermitian conjugate we obtain

$$\Psi^\dagger \left(-i \overleftarrow{\partial\!\!\!/}^\dagger - m\right) = 0 \tag{2.3.28}$$

that is,

$$\Psi^\dagger \left(-i \overset{leftarrow}{\partial_\mu} \gamma^{\mu^\dagger} - m\right) = 0. \tag{2.3.29}$$

Multiplying by γ^0 and using the relation $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$ yields

$$\Psi^\dagger\gamma^0 \left(-i \overleftarrow{\partial_\mu} \gamma^\mu - m\right) = 0. \tag{2.3.30}$$

Now we define the adjoint Dirac spinor as

$$\overline{\Psi} \equiv \Psi^\dagger\gamma^0 \tag{2.3.31}$$

so that the preceding equation takes the form

$$\overline{\Psi}(x) \left(i \overleftarrow{\partial\!\!\!/} + m\right) = 0 \tag{2.3.32}$$

which is called the adjoint Dirac equation.

Dirac showed that these field equations can be obtained as Euler-Lagrange equations from the Lorentz invariant Dirac Lagrangian

$$\mathcal{L} = \overline{\Psi} \left(\frac{i}{2} \overleftrightarrow{\partial\!\!\!/} - m\right) \Psi. \tag{2.3.33}$$

Notice that $\mathcal{L} = \mathcal{L}^\dagger$, but often we write $\mathcal{L} = \overline{\Psi}\,(i\partial\!\!\!/ - m)\,\Psi$ which is not hermitian but differs from the hermitian Lagrangian by a total divergence $-\frac{i}{2}\partial_\mu\left(\overline{\Psi}\gamma^\mu\Psi\right)$, so the action is the same. Thus, either Lagrangian leads to the same Euler-Lagrange equations of motion,

$$\frac{\partial \mathcal{L}}{\partial\overline{\Psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\overline{\Psi}} = 0 = (i\partial\!\!\!/ - m)\,\Psi, \tag{2.3.34}$$

that is the Dirac equation.

According to our canonical quantization procedure we define the momentum conjugate to Ψ by $\frac{\partial \mathcal{L}}{\partial\partial_0\Psi}$. However, typical of Lagrangians first order in $\frac{\partial}{\partial t}$, the canonically conjugate momenta Π and $\overline{\Pi}$ are proportional to the field variables $\overline{\Psi}$ and Ψ, and hence, the fields and momenta are not independent quantities, i.e. the fields are conjugate to each other. We cannot naively apply our canonical quantization procedure. Dirac found that we must throw all the derivatives to one field when finding its conjugate momenta, choose it as the independent coordinate and then the other field will be the momentum. Either way we throw the derivative, the result is the same momentum-field relation. Thus, throwing the derivatives to $\bar{\Psi}$ by parts, $\mathcal{L} = -i\partial_\mu\bar{\Psi}\gamma^\mu\Psi - m\bar{\Psi}\Psi$ and ignoring the total divergence we define

$$\overline{\Pi}_a(x) \equiv \frac{\partial \mathcal{L}}{\partial\partial_0\overline{\Psi}_a}(x) = -i\gamma^0\Psi(x). \tag{2.3.35}$$

As we will see and as Jordan and Wigner discovered, we must further change the canonical commutation relations used for bosons to anticommutation relations for fermions. The quantization rules become

$$\delta(x^0 - y^0)\{\overline{\Pi}_a(x), \overline{\Psi}_b(y)\} = -i\delta_{ab}\delta^4(x-y) \tag{2.3.36}$$

where $\{A, B\} = [A, B]_+ = AB + BA$, and

$$\delta(x^0 - y^0)\{\bar{\Psi}(x), \bar{\Psi}(y)\} = 0, \tag{2.3.37}$$

and

$$\delta(x^0 - y^0)\{\bar{\Pi}(x), \bar{\Pi}(y)\} = 0 = \delta(x^0 - y^0)\{\Psi(x), \Psi(y)\}. \tag{2.3.38}$$

Now the equal time anti-commutation relations (ETAR), (2.3.36), are

$$\delta(x^0 - y^0)\{-i\gamma^0_{ac}\Psi_c(x), \Psi^\dagger_d(y)\gamma^0_{db}\} = -i\delta_{ab}\delta^4(x-y), \tag{2.3.39}$$

multiplying by $\gamma^0_{ea}\gamma^0_{bf}$, we obtain

$$\delta(x^0 - y^0)\{\Psi_e(x), \Psi^\dagger_f(y)\} = \delta_{ef}\delta^4(x-y) \tag{2.3.40}$$

that is, in summary,

$$\delta(x^0 - y^0)\{\Psi_a(x), \Psi^\dagger_b(y)\} = \delta_{ab}\delta^4(x-y)$$

$$\delta(x^0 - y^0)\{\Psi_a(x), \Psi_b(y)\} = 0. \tag{2.3.41}$$

A few words are in order. The fact that these operators anticommute means that they are not like ordinary "functions" of x, rather they are elements of an infinite dimensional Grassmann algebra (they are anti-commuting numbers). Furthermore, we will see that these are the correct quantization rules for the Dirac fields to obey. They are correct in that they lead to a consistent definition of energy, momentum, and spin of Fermi-Dirac particles. This will become clearer as we proceed.

Naturally we can find the Hamiltonian density given the momenta

$$\begin{aligned}
\mathcal{H} &= \partial_0 \bar{\Psi}\bar{\Pi} - \mathcal{L} \\
&= -i\partial_0 \bar{\Psi}\gamma^0\Psi + \bar{\Psi}\left(i \overleftarrow{\not{\partial}} + m\right)\Psi \\
&= \bar{\Psi}\left(i\gamma^i \frac{\overleftarrow{\partial}}{\partial x^i} + m\right)\Psi.
\end{aligned} \tag{2.3.42}$$

Thus, the Hamiltonian is

$$H = \int d^3x \overline{\Psi}\left(-\frac{i}{2} \overleftrightarrow{\nabla}_i + m\right)\Psi \tag{2.3.43}$$

where we have again made H hermitian by adding total three-divergences. The Heisenberg equations are as before $[H, \Psi(x)] = -i\dot{\Psi}(x)$.

More generally we can apply Noether's Theorem to construct the energy-momentum tensor and angular momentum tensor

$$T^{\mu\nu} \equiv \partial^\nu \Psi \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} + \partial^\nu \overline{\Psi} \frac{\partial \mathcal{L}}{\partial \partial_\mu \overline{\Psi}} - g^{\mu\nu}\mathcal{L} \tag{2.3.44}$$

where now we must be careful taking derivatives with respect to anticommuting variables

$$\begin{aligned}
\frac{\partial}{\partial \Psi_a}\Psi_b &= \delta_{ab} \\
\frac{\partial}{\partial \Psi_a}\overline{\Psi}_b &= 0 \\
\frac{\partial}{\partial \overline{\Psi}_a}\overline{\Psi}_b &= \delta_{ab}
\end{aligned}$$

$$\frac{\partial}{\partial \overline{\Psi}_a} \Psi_b = 0 \tag{2.3.45}$$

and for $\tilde{\Psi}_b$ either Ψ_b or $\bar{\Psi}_b$ and A an arbitrary operator

$$\frac{\partial}{\partial \Psi_a} \tilde{\Psi}_b A = \frac{\partial \tilde{\Psi}_b}{\partial \Psi_a} A - \tilde{\Psi}_b \frac{\partial A}{\partial \Psi_a}. \tag{2.3.46}$$

Similarly for $\frac{\partial}{\partial \overline{\Psi}_a}$. Thus we have the general chain rule for derivatives with respect to anti-commuting quantities

$$\frac{\partial}{\partial \Psi_a} AB = \frac{\partial A}{\partial \Psi_a} B + (-1)^{|A|} A \frac{\partial B}{\partial \Psi_a} \tag{2.3.47}$$

where $|A| = 1$ if A is an odd element of the Grassmann algebra i.e. if A is an anticommuting number or $|A| = 0$ if A is an even element of the Grassmann algebra i.e. if A is a commuting number. Thus the reason for the factors appearing to the left in the expression for $T^{\mu\nu}$ above. We shall also use the hermitian Lagrangian so that $T^{\mu\nu}$ is hermitian

$$\mathcal{L} = \frac{i}{2} \overline{\Psi} \overleftrightarrow{\not{\partial}} \Psi - m \overline{\Psi} \Psi. \tag{2.3.48}$$

So

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \partial_\mu \overline{\Psi}} &= -\frac{i}{2} \gamma^\mu \Psi \\ \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} &= -\frac{i}{2} \overline{\Psi} \gamma^\mu \end{aligned} \tag{2.3.49}$$

where the minus sign appears in the second term since

$$\frac{\partial}{\partial \partial_\mu \Psi} \overline{\Psi} = -\overline{\Psi} \frac{\partial}{\partial \partial_\mu \Psi}. \tag{2.3.50}$$

Hence,

$$T^{\mu\nu} = -\frac{i}{2} \partial^\nu \overline{\Psi} \gamma^\mu \Psi + \frac{i}{2} \overline{\Psi} \gamma^\mu \partial^\nu \Psi - g^{\mu\nu} \mathcal{L}, \tag{2.3.51}$$

the plus sign comes from the relation $\partial^\nu \Psi \overline{\Psi} = -\overline{\Psi} \partial^\nu \Psi$. Thus we secure

$$\begin{aligned} T^{\mu\nu} &= \frac{i}{2} \left[\overline{\Psi} \gamma^\mu \partial^\nu \Psi - \partial^\nu \overline{\Psi} \gamma^\mu \Psi \right] - g^{\mu\nu} \mathcal{L} \\ &= \frac{i}{2} \overline{\Psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \Psi - g^{\mu\nu} \mathcal{L} \end{aligned} . \tag{2.3.53}$$

Now we check directly that this is conserved

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \frac{i}{2}\overline{\Psi}\partial\!\!\!/\partial^\nu\Psi + \frac{i}{2}\overline{\Psi}\overleftarrow{\partial\!\!\!/}\partial^\nu\Psi - \frac{i}{2}\partial^\nu\overline{\Psi}\overleftarrow{\partial\!\!\!/}\Psi \\ &\quad - \frac{i}{2}\partial^\nu\overline{\Psi}\partial\!\!\!/\Psi - \partial^\nu\mathcal{L}.\end{aligned} \tag{2.3.53}$$

Using the field equations $(i\partial\!\!\!/ - m)\Psi = 0$ and $\overline{\Psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right) = 0$, we obtain

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial^\nu\left(\frac{i}{2}\bar{\Psi}\overrightarrow{\partial\!\!\!/}\Psi\right) - i\partial^\nu\bar{\Psi}\overrightarrow{\partial\!\!\!/}\Psi - \frac{i}{2}\partial^\nu\left(\bar{\Psi}\overleftarrow{\partial\!\!\!/}\Psi\right) \\ &\quad + i\bar{\Psi}\overleftarrow{\partial\!\!\!/}\partial^\nu\Psi - \partial^\nu\mathcal{L} \\ &= \partial^\nu\left(\frac{i}{2}\bar{\Psi}\overleftrightarrow{\partial\!\!\!/}\Psi\right) - m\partial^\nu\bar{\Psi}\Psi - m\bar{\Psi}\partial^\nu\Psi - \partial^\nu\mathcal{L} \\ &= \partial^\nu\left(\frac{i}{2}\bar{\Psi}\overleftrightarrow{\partial\!\!\!/}\Psi - m\bar{\Psi}\Psi\right) - \partial^\nu\mathcal{L} = 0.\end{aligned} \tag{2.3.54}$$

Hence, the energy momentum tensor is conserved

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.3.55}$$

Notice however that

$$\begin{aligned}T^{\mu\nu} - T^{\nu\mu} &= \frac{i}{2}\left[\overline{\Psi}\gamma^\mu\partial^\nu\Psi - \overline{\Psi}\gamma^\nu\partial^\mu\Psi - \partial^\nu\overline{\Psi}\gamma^\mu\Psi + \partial^\mu\overline{\Psi}\gamma^\nu\Psi\right] \\ &\neq 0.\end{aligned} \tag{2.3.56}$$

We can put this difference into a more useful form by using the field equations $i\partial\!\!\!/\Psi = m\Psi$ and $i\overline{\Psi}\overleftarrow{\partial\!\!\!/} = -m\overline{\Psi}$. By multiplying the Dirac equation by $\overline{\Psi}\gamma^\mu\gamma^\nu$ and the adjoint equation by $\gamma^\mu\gamma^\nu\Psi$ we obtain

$$\begin{aligned}i\overline{\Psi}\gamma^\mu\gamma^\nu\gamma^\rho\partial_\rho\Psi &= m\overline{\Psi}\gamma^\mu\gamma^\nu\Psi \\ i\partial_\rho\overline{\Psi}\gamma^\rho\gamma^\mu\gamma^\nu\Psi &= -m\overline{\Psi}\gamma^\mu\gamma^\nu\Psi.\end{aligned} \tag{2.3.57}$$

Hence, adding we find

$$\overline{\Psi}\gamma^\mu\gamma^\nu\gamma^\rho\partial_\rho\Psi + \partial_\rho\overline{\Psi}\gamma^\rho\gamma^\mu\gamma^\nu\Psi = 0. \tag{2.3.58}$$

This can be further simplified by exploiting the γ-matrix identities. Recall

$$\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu} - i\sigma^{\mu\nu}$$

$$\gamma_5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$$

$$\{\gamma^{\rho}, \sigma^{\mu\nu}\} = 2\epsilon^{\rho\mu\nu\lambda}\gamma_5\gamma_\lambda \tag{2.3.59}$$

and the identity we need

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} = g^{\mu\nu}\gamma^{\rho} - g^{\mu\rho}\gamma^{\mu} + g^{\nu\rho}\gamma^{\mu} - i\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\lambda. \tag{2.3.60}$$

Substituting this into the Dirac equation (2.3.58) it becomes

$$\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial^{\nu}} \Psi - \overline{\Psi}\gamma^{\nu} \overleftrightarrow{\partial^{\mu}} \Psi = i\partial_\rho\epsilon^{\rho\mu\nu\lambda}\overline{\Psi}\gamma_5\gamma_\lambda\Psi - g^{\mu\nu}\partial_\rho\left(\overline{\Psi}\gamma^{\rho}\Psi\right). \tag{2.3.61}$$

Now then equation (2.3.56) yields

$$T^{\mu\nu} - T^{\nu\mu} = -\frac{1}{2}\partial_\rho\epsilon^{\rho\mu\nu\lambda}\overline{\Psi}\gamma_5\gamma_\lambda\Psi - \frac{i}{2}g^{\mu\nu}\partial_\rho\left(\overline{\Psi}\gamma^{\rho}\Psi\right) \tag{2.3.62}$$

but again adding the Dirac and adjoint Dirac equations we have

$$i\partial_\rho\left(\overline{\Psi}\gamma^{\rho}\Psi\right) = m\overline{\Psi}\Psi - m\overline{\Psi}\Psi = 0. \tag{2.3.63}$$

So

$$T^{\mu\nu} - T^{\nu\mu} = -\frac{1}{2}\partial_\rho\epsilon^{\rho\mu\nu\lambda}\bar{\Psi}\gamma_5\gamma_\lambda\Psi. \tag{2.3.64}$$

According to Belinfante's improvement procedure

$$T^{\mu\nu} - T^{\nu\mu} = \partial_\rho H^{\rho\mu\nu} \tag{2.3.65}$$

and we identify

$$H^{\rho\mu\nu} = -\frac{1}{2}\epsilon^{\rho\mu\nu\lambda}\overline{\Psi}\gamma_5\gamma_\lambda\Psi. \tag{2.3.66}$$

The symmetric Belinfante energy momentum tensor is

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} - \partial_\rho G^{\rho\mu\nu} \tag{2.3.67}$$

where

$$
\begin{aligned}
G^{\rho\mu\nu} &= \frac{1}{2}\left[H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho}\right] \\
&= \frac{1}{2}(-\frac{1}{2})\left[\epsilon^{\rho\mu\nu\lambda} + \epsilon^{\mu\nu\rho\lambda} + \epsilon^{\nu\mu\rho\lambda}\right]\overline{\Psi}\gamma_5\gamma_\lambda\Psi \\
&= \frac{1}{2}H^{\rho\mu\nu}
\end{aligned} \tag{2.3.68}
$$

and

$$
\begin{aligned}
\partial_\rho G^{\rho\mu\nu} &= \frac{1}{2}\partial_\rho H^{\rho\mu\nu} \\
&= -\frac{1}{4}\partial_\rho \epsilon^{\rho\mu\nu\lambda}\overline{\Psi}\gamma_5\gamma_\lambda\Psi
\end{aligned}
$$

$$
\partial_\rho G^{\rho\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\Psi - \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial^\mu}\Psi\right]. \tag{2.3.69}
$$

Thus, according to (2.3.52), (2.3.67) and (2.3.69), the symmetric Belinfante energy momentum tensor becomes

$$
\Theta^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\Psi - g^{\mu\nu}\mathcal{L} - \frac{i}{4}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\Psi + \frac{i}{4}\overline{\Psi}\gamma^\nu \overleftrightarrow{\partial^\mu}\Psi
$$

$$
\Theta^{\mu\nu} = \frac{i}{4}\overline{\Psi}\left[\gamma^\mu \overleftrightarrow{\partial^\nu} + \gamma^\nu \overleftrightarrow{\partial^\mu}\right]\Psi - g^{\mu\nu}\mathcal{L}. \tag{2.3.70}
$$

It is conserved

$$
\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0 \tag{2.3.71}
$$

and symmetric

$$
\Theta^{\mu\nu} = \Theta^{\nu\mu}. \tag{2.3.72}
$$

Note all this follows from Belinfante's general prescription, equations (2.1.157), (2.1.158) and (2.1.159)

$$
H^{\rho\mu\nu} \equiv \Pi_r^\rho D_{rs}^{\mu\nu}\Phi_s \tag{2.3.73}
$$

which in our case yields

$$
\begin{aligned}
H^{\rho\mu\nu} &= D^{\mu\nu}\Psi\frac{\partial\mathcal{L}}{\partial\partial^\rho\Psi} + \overline{\Psi D}^{\mu\nu}\frac{\partial\mathcal{L}}{\partial\partial^\rho\overline{\Psi}} \\
&= \frac{i}{2}\overline{\Psi}\gamma^\rho D^{\mu\nu}\Psi - \frac{i}{2}\overline{\Psi D}^{\mu\nu}\gamma^\rho\Psi.
\end{aligned} \tag{2.3.74}
$$

However, $D^{\mu\nu} = \frac{i}{2}\sigma^{\mu\nu}$ and $\overline{D}^{\mu\nu} = \gamma^0 D^{\mu\nu\dagger}\gamma^0 = -\frac{i}{2}\sigma^{\mu\nu}$, so that

$$\begin{aligned} H^{\rho\mu\nu} &= -\frac{1}{4}\overline{\Psi}\{\gamma^\rho, \sigma^{\mu\nu}\}\Psi \\ &= -\frac{1}{2}\epsilon^{\rho\mu\nu\lambda}\overline{\Psi}\gamma_5\gamma_\lambda\Psi. \end{aligned} \tag{2.3.75}$$

The generator of translations is given by

$$\begin{aligned} \mathcal{P}^\mu &\equiv \int d^3x \Theta^{0\mu} = \int d^3x T^{0\mu} \\ &= \int d^3x \left(\frac{i}{2}\overline{\Psi}\gamma^0 \overleftrightarrow{\partial^\mu} \Psi - g^{0\mu}\mathcal{L}\right). \end{aligned} \tag{2.3.76}$$

Consequently, the Hamiltonian is as given by the Legendre transform of $\mathcal{L}$, equation (2.3.43)

$$\begin{aligned} H = \mathcal{P}^0 = \int d^3x &\left(\frac{i}{2}\overline{\Psi}\gamma^0\dot{\Psi} - \frac{i}{2}\partial_0\overline{\Psi}\gamma^0\Psi \right. \\ &\left. -\frac{i}{2}\overline{\Psi}\gamma^0\dot{\Psi} + \frac{i}{2}\partial_0\overline{\Psi}\gamma^0\Psi - \frac{i}{2}\overline{\Psi}\gamma^i \overleftrightarrow{\partial_i} \Psi + m\overline{\Psi}\Psi\right) \\ &= \int d^3x\overline{\Psi}\left(-\frac{i}{2}\gamma^i \overleftrightarrow{\partial_i} + m\right)\Psi \\ &= \int d^3x\frac{i}{2}\left(\Psi^\dagger\dot{\Psi} - \dot{\Psi}^\dagger\Psi\right) \end{aligned} \tag{2.3.77}$$

since $\mathcal{L} = 0$ upon use of the Dirac equations, and

$$\left[\mathcal{P}^0, \Psi(x)\right] = -i\dot{\Psi}(x). \tag{2.3.78}$$

In addition, we have

$$\begin{aligned} \vec{\mathcal{P}} &= \int d^3x\frac{i}{2}\overline{\Psi}\gamma^0 \overleftrightarrow{\nabla} \Psi \\ &= \int d^3x \left(i\overline{\Psi}\gamma^0\vec{\nabla}\Psi\right) \\ &= \int d^3x \left(i\Psi^\dagger\vec{\nabla}\Psi\right) \end{aligned} \tag{2.3.79}$$

so that, since $\vec{\mathcal{P}}$ is time independent,

$$\begin{aligned} \left[\vec{\mathcal{P}}, \Psi(x)\right] &= i\int_{y^0=x^0} d^3y \left[\Psi^\dagger(y)\vec{\nabla}_y\Psi(y), \Psi(x)\right] \\ &= -i\int_{y^0=x^0} d^3y\{\Psi^\dagger(y), \Psi(x)\}\vec{\nabla}_y\Psi(y). \end{aligned} \tag{2.3.80}$$

Using the ETAR equation (2.3.41) this yields

$$\left[\vec{\mathcal{P}}, \Psi(x)\right] = -i\vec{\nabla}\Psi(x). \tag{2.3.81}$$

Thus, we have as we should for space-time translations

$$[\mathcal{P}^\mu, \Psi(x)] = -i\partial^\mu \Psi(x). \tag{2.3.82}$$

According to Belinfante's procedure the angular momentum tensor is given by

$$M^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}$$

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{2.3.83}$$

and

$$\mathcal{M}^{\mu\nu} \equiv \int d^3x M^{0\mu\nu} \tag{2.3.84}$$

which yields

$$[\mathcal{M}^{\mu\nu}, \Psi(x)] = -i\left[\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\Psi(x) - \frac{i}{2}\sigma^{\mu\nu}\Psi(x)\right]. \tag{2.3.85}$$

Finally, since Ψ is a complex field we also have a phase invariance because $\mathcal{L}$ is made from $\Psi^\dagger\Psi$. As in the complex scalar case we introduce unitary operators $U(\alpha) = e^{i\alpha Q}$ transforming the phase of Ψ and $\overline{\Psi}$

$$U^\dagger(\alpha)\Psi(x)U(\alpha) \equiv e^{i\alpha}\Psi(x)$$

$$U^\dagger(\alpha)\overline{\Psi}(x)U(\alpha) \equiv e^{-i\alpha}\overline{\Psi}(x). \tag{2.3.86}$$

Since $U(\alpha) = e^{i\alpha Q}$, (2.3.86) implies

$$[Q, \Psi(x)] = -\Psi(x)$$

$$\left[Q, \overline{\Psi}(x)\right] = +\overline{\Psi}(x). \tag{2.3.87}$$

Since $\mathcal{L}$ is invariant under this transformation

$$\delta\mathcal{L} = \delta\overline{\mathcal{L}} = U^\dagger(\alpha)\mathcal{L}U(\alpha) - \mathcal{L} = 0 \tag{2.3.88}$$

we have by Noether's Theorem a conserved current in terms of the dynamical variables Ψ and $\overline{\Psi}$ whose zero component gives the charge Q

$$J^\mu \equiv i\left(\overline{\Psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\Psi}} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\Psi\right)$$

$$J^\mu = i\left(-\frac{i}{2}\overline{\Psi}\gamma^\mu\Psi - \frac{i}{2}\overline{\Psi}\gamma^\mu\Psi\right)$$

$$J^\mu = \overline{\Psi}\gamma^\mu\Psi. \tag{2.3.89}$$

Checking explicitly the conservation of this current we have using the Dirac equations

$$\begin{aligned}\partial_\mu J^\mu &= -i\overline{\Psi}i\partial\!\!\!/\Psi - i\overline{\Psi}i\overleftarrow{\partial\!\!\!/}\Psi + im\overline{\Psi}\Psi - im\overline{\Psi}\Psi \\ &= -i\overline{\Psi}\left(i\overrightarrow{\partial\!\!\!/} - m\right)\Psi - i\overline{\Psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right)\Psi = 0.\end{aligned} \tag{2.3.90}$$

Hence the U(1) phase symmetry current is conserved

$$\partial_\mu J^\mu = 0. \tag{2.3.91}$$

By Noether's theorem the U(1) charge is

$$Q \equiv \int d^3x J^0(x) = \int d^3x\overline{\Psi}\gamma^0\Psi$$

$$Q = \int d^3x\Psi^\dagger\Psi. \tag{2.3.92}$$

Further $\dot{Q} = 0$ since $\delta\mathcal{L} = 0$, or more generally, the internal and space-time symmetry generators commute

$$[Q, \mathcal{P}^\mu] = 0$$

$$[Q, \mathcal{M}^{\mu\nu}] = 0. \tag{2.3.93}$$

Finally we check explicitly that Q given by equation (2.3.92) indeed induces phase transformations on the Dirac fields

$$\begin{aligned}[Q, \Psi(x)] &= \int_{y^0=x^0} d^3y\left[\overline{\Psi}(y)\gamma^0\Psi(y), \Psi(x)\right] \\ &= -\int_{y^0=x^0} d^3y\{\Psi_a^\dagger(y), \Psi(x)\}\Psi_a(y)\end{aligned}$$

$$[Q, \Psi(x)] = -\Psi(x) \tag{2.3.93}$$

and similarly

$$\left[Q, \bar{\Psi}(x)\right] = +\bar{\Psi}(x) \tag{2.3.94}$$

as required.

Thus we have obtained the space-time properties of the Dirac field theory according to the canonical quantization procedure. To interpret this quantum field theory in terms of particle states and associated creation and annihilation operators, we Fourier transform $\Psi(x)$ to momentum space, that is, expand it in terms of plane wave solutions of the Dirac equation

$$\Psi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\Psi}(k). \tag{2.3.95}$$

As we have seen Ψ is a solution to the Klein-Gordon equation,

$$\left(\partial^2 + m^2\right) \Psi = 0, \tag{2.3.96}$$

implying that

$$\tilde{\Psi}_a(k) = (2\pi)\delta\left(k^2 - m^2\right) w_a(\vec{k}, k^0). \tag{2.3.97}$$

Using $\delta(k^2 - m^2) = \frac{1}{2}\left[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)\right]$, this yields

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ikx} w(\vec{k}, +k^0) + e^{+ikx} w(-\vec{k}, -k^0)\right], \tag{2.3.98}$$

where $k^0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$, as usual, is understood. In addition to the Klein-Gordon equation, Ψ also obeys the Dirac equation, $(i\partial\!\!\!/ - m)\Psi(x) = 0$,

$$(i\partial\!\!\!/ - m)\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ikx}(k\!\!\!/ - m) w(\vec{k}, +k^0)\right.$$
$$\left. - e^{+ikx}(k\!\!\!/ + m) w(-\vec{k}, -k^0)\right] = 0. \tag{2.3.99}$$

Hence w_a obeys the 4×4 matrix equation

$$\begin{aligned} (k\!\!\!/ - m) w(\vec{k}, +k^0) &= 0 \\ (k\!\!\!/ + m) w(-\vec{k}, -k^0) &= 0. \end{aligned} \tag{2.3.100}$$

It is convenient to separate the c-number spinor solutions to the Dirac equation from the q-number operator coefficients. We define

$$w_a(\vec{k}, +k^0) \equiv b(\vec{k}) u_a(\vec{k})$$

$$w_a(-\vec{k}, -k^0) \equiv d^\dagger(\vec{k}) v_a(\vec{k}) \tag{2.3.101}$$

where b and $d^\dagger$ are q-number operators, i.e. creation and annihilation operators, while $u_a(\vec{k})$ and $v_a(\vec{k})$ are c-number spinor solutions to the momentum space Dirac equations

$$(\not{k} - m) u(\vec{k}) = 0$$

$$(\not{k} + m) v(\vec{k}) = 0. \tag{2.3.102}$$

Since Ψ is complex, $w_a^\dagger(\vec{k}, +k^0)$ and $w_a(-\vec{k}, -k^0)$ are independent, thus the coefficients b and d and the column vectors u and v are independent also.

In order to solve (2.3.102) for u and v we go to the rest frame of the particle, that is consider these equations at $\vec{k} = 0$ and $k^0 = m$. So equations (2.3.102) become

$$m\left(\gamma^0 - 1\right) u(0) = 0$$

$$m\left(\gamma^0 + 1\right) v(0) = 0. \tag{2.3.103}$$

That is we obtain the γ^0 Dirac matrix eigenvalue equations

$$\gamma^0 u(0) = +u(0)$$

$$\gamma^0 v(0) = -v(0). \tag{2.3.104}$$

Hence, u and v are the $+1$ and -1 eigenvectors of γ^0 respectively. Recall that $\gamma^{0^2} = 1$, so γ^0 has two pairs of eigenvalues ± 1 as seen directly in the Dirac representation where γ^0 is diagonal. $u(0)$ denotes the pair of $+1$ eigenvalue eigenvectors and $v(0)$ the -1 pair. Since the γ^0 eigenvalues are degenerate we need another label for $u(\vec{k})$ and $v(\vec{k})$ to denote the independent eigenvectors. It is conventional as to how to choose the orthogonal pair of degenerate γ^0 eigenvectors. We choose them to be eigenvectors of σ^{12} also, these are simultaneously diagonalizable since $\left[\gamma^0, \sigma^{ij}\right] = 0$. As we will see σ^{ij} will give the three components of the spin of the particle at rest, σ^{12} being the projection of the spin onto the x^3 axis. So the four independent vectors in the $\vec{k}$-frame are denoted by $u^{(s)}(\vec{k})$ and $v^{(s)}(\vec{k})$ where $s = 1, 2$ labels the two u and the two v vectors. We take the rest frame vectors to obey the eigenvalue equations (2.3.104) as stated

$$\gamma^0 u^{(s)}(0) = +u^{(s)}(0)$$

$$\gamma^0 v^{(s)}(0) = -v^{(s)}(0) \tag{2.3.105}$$

as well as the σ^{12} eigenvalue equations, note σ^{12} has eigenvalues ± 1 also,

$$\sigma^{12} u^{(s)}(0) = (3-2s) u^{(s)}(0) = (-1)^{s+1} u^{(s)}(0)$$

$$\sigma^{12} v^{(s)}(0) = (3-2s) v^{(s)}(0) = (-1)^{s+1} v^{(s)}(0), \tag{2.3.106}$$

that is for instance

$$\sigma^{12} u^{(1)}(0) = +u^{(1)}(0)$$

$$\sigma^{12} u^{(2)}(0) = -u^{(2)}(0). \tag{2.3.107}$$

Furthermore, we have that $u^{(r)}(0)$ and $v^{(r)}(0)$ are orthogonal and we normalize them, by convention, to $2m$. So we take

$$u^{(r)}(0)^\dagger u^{(s)}(0) = 2m\delta_{rs}$$

$$v^{(r)}(0)^\dagger v^{(s)}(0) = 2m\delta_{rs}$$

$$u^{(r)}(0)^\dagger v^{(s)}(0) = v^{(r)}(0)^\dagger u^{(s)}(0) = 0. \tag{2.3.108}$$

Recall that in the Dirac representation

$$\gamma^0 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.3.109}$$

while for $i \neq j$

$$\begin{aligned} \sigma^{ij} &= i\gamma^i\gamma^j \\ &= -i \begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \end{aligned} \tag{2.3.110}$$

Hence,

$$\sigma^{12} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.3.111}$$

Thus the simultaneous (γ^0, σ^{12}) eigenvectors are

$$u^{(1)}(0) = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u^{(2)}(0) = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v^{(1)}(0) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v^{(2)}(0) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.3.112}$$

We also have creation and annihilation operators for each of these eigenmodes. Hence, the expressions (2.3.101) for $w_a(\vec{k}, +k^0)$ and $w_a(-\vec{k}, -k^0)$ become a sum over the $\vec{k}$-frame eigenvector basis $u^{(s)}(\vec{k}), v^{(s)}(\vec{k})$

$$w_a(\vec{k}, +k^0) \equiv \sum_{s=1}^{2} b_s(\vec{k}) u_a^{(s)}(\vec{k})$$

$$w_a(-\vec{k}, -k^0) \equiv \sum_{s=1}^{2} d_s^\dagger(\vec{k}) v_a^{(s)}(\vec{k}). \tag{2.3.113}$$

The Dirac field Fourier transform, equation (2.3.98), becomes

$$\Psi_a(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left[b_s(\vec{k}) u_a^{(s)}(\vec{k}) e^{-ikx} + d_s^\dagger(\vec{k}) v_a^{(s)}(\vec{k}) e^{+ikx} \right]. \tag{2.3.114}$$

(Compare this with the complex scalar field expansion, equation (2.2.266)

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{+kx}\right].) \qquad (2.3.115)$$

Now $u^{(s)}(\vec{k})$ and $v^{(s)}(\vec{k})$ still must be solutions of the Dirac equation

$$(\not{k} - m)u^{(s)}(\vec{k}) = 0$$
$$(\not{k} + m)v^{(s)}(\vec{k}) = 0. \qquad (2.3.116)$$

Since $(\not{k}+m)(\not{k}-m) = (\not{k}-m)(\not{k}+m) = k^2 - m^2 = 0$ we can solve the above equations (alternatively we could just boost from the rest frame $u^{(s)}(0)$ and $v^{(s)}(0)$ to momentum $\vec{k}$ by means of (SL(2,C)) Lorentz transformation $S(\vec{k})$ to obtain the same answer)

$$u^{(s)}(\vec{k}) = A(\vec{k})(\not{k}+m)u^{(s)}(0)$$
$$v^{(s)}(\vec{k}) = B(\vec{k})(\not{k}-m)v^{(s)}(0) \qquad (2.3.117)$$

with A and B arbitrary normalization constants chosen by convention. We choose u and v just to be the (SL(2,C)) Lorentz transformation matrices to the moving frame $\vec{k}$ since the $u(0)$ and $v(0)$ are already normalized at rest as we desire. Hence,

$$A(\vec{k}) \equiv \frac{1}{\sqrt{2m(\omega_k+m)}} \equiv -B(\vec{k}). \qquad (2.3.118)$$

Consequently,

$$u^{(s)}(\vec{k}) = \frac{(\not{k}+m)}{\sqrt{2m(\omega_k+m)}}u^{(s)}(0)$$
$$v^{(s)}(\vec{k}) = \frac{(-\not{k}+m)}{\sqrt{2m(\omega_k+m)}}v^{(s)}(0) \qquad (2.3.119)$$

where $u^{(s)}(0)$ and $v^{(s)}(0)$ are given above in the Dirac representation. To see this consider the Lorentz boost to go from the rest frame of the particle to the frame in which the particle has momentum $\vec{k}$. That is

$$k^\mu = \Lambda^{\mu\nu}(\vec{k})k_\nu^{rest}, \qquad (2.3.120)$$

with $k^\mu_{rest} = (m,0,0,0)$. Hence $\Lambda^{\mu\nu}(\vec{k})$ is given by

$$\Lambda^0{}_0(\vec{k}) = \cosh\beta$$

$$\Lambda^i{}_0(\vec{k}) = \Lambda^0{}_i(\vec{k}) = \frac{k^i}{|\vec{k}|}\sinh\beta$$

$$\Lambda^i{}_j(\vec{k}) = \delta^i_j + \frac{k^i}{|\vec{k}|}\frac{k^j}{|\vec{k}|}[\cosh\beta - 1], \tag{2.3.121}$$

where $\tanh\beta = \frac{|\vec{k}|}{\omega_k}$. Correspondingly the SL(2,C) transformation, denoted $S(\vec{k}) = W(\vec{k})$, that takes the spinor $u^{(s)}(0)$ from the rest frame to the $\vec{k}$-frame, $u^{(s)}(\vec{k})$, is

$$W(\vec{k}) = \frac{(\not{k} + m)}{\sqrt{2m(\omega_k + m)}}, \tag{2.3.122}$$

such that

$$u^{(s)}(\vec{k}) = W(\vec{k})u^{(s)}(0). \tag{2.3.123}$$

Likewise, we have

$$v^{(s)}(\vec{k}) = \bar{W}(\vec{k})v^{(s)}(0), \tag{2.3.124}$$

with

$$\bar{W}(\vec{k}) = \frac{(-\not{k} + m)}{\sqrt{2m(\omega_k + m)}} = \gamma_5 W(\vec{k})\gamma_5. \tag{2.3.125}$$

Recalling the relation between SL(2,C) transformations and Lorentz transformation matrices, equation (1.2.131), $\Lambda^{\mu\nu}(\vec{k})\gamma_\mu = W(\vec{k})\gamma^\nu W^{-1}(\vec{k})$, or in a more direct form

$$\Lambda^{\mu\nu}(\vec{k}) = \frac{1}{4}\mathrm{Tr}[\gamma^\mu W \gamma^\nu W^{-1}]. \tag{2.3.126}$$

Using equation (2.3.125) we see that $\bar{W}(\vec{k})$ yields the same Lorentz transformation matrix. Thus using the explicit form for $W(\vec{k})$, equation (2.3.122), we verify that the Lorentz transformation obtained in equation (2.3.126) is indeed the same as in equation (2.3.120) and (2.3.121). Thus, the Fourier transform breaks up into positve and negative frequency components

$$\Psi(x) = \Psi^+(x) + \Psi^-(x)$$

$$\Psi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{s=1}^{2} b_s(\vec{k})u_a^{(s)}(\vec{k})e^{-ikx}$$

$$\Psi^-(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{s=1}^{2} d_s^\dagger(\vec{k})v_a^{(s)}(\vec{k})e^{+ikx} \tag{2.3.127}$$

where the positive frequency plane wave solutions to the Dirac equation are

$$U^{(s)}_{\vec{k}}(x) \equiv u^{(s)}(\vec{k})e^{-ikx} \tag{2.3.128}$$

and the negative frequency solutions are

$$V^{(s)}_{\vec{k}}(x) \equiv v^{(s)}(\vec{k})e^{+ikx} \tag{2.3.129}$$

with

$$(i\partial\!\!\!/ - m)U^{(s)}_{\vec{k}}(x) = 0 = (i\partial\!\!\!/ - m)V^{(s)}_{\vec{k}}(x). \tag{2.3.130}$$

Now we desire to invert these Fourier transforms and to do so we need the orthogonality properties of the plane wave solutions u and v. First consider the hermitian conjugate (in Dirac matrix space)

$$u^{(r)^\dagger}(\vec{k}) = \frac{1}{\sqrt{2m(\omega_k + m)}}u^{(r)^\dagger}(0)(k\!\!\!/^\dagger + m), \tag{2.3.131}$$

but $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$ so that

$$u^{(r)^\dagger}(\vec{k}) = \frac{1}{\sqrt{2m(\omega_k + m)}}u^{(r)^\dagger}(0)(\gamma^0 k\!\!\!/\gamma^0 + m). \tag{2.3.132}$$

However,

$$\gamma^0 u^{(r)}(0) = u^{(r)}(0), \tag{2.3.133}$$

hence

$$u^{(r)^\dagger}(0)\gamma^0 = u^{(r)^\dagger}(0) \tag{2.3.134}$$

and so

$$u^{(r)^\dagger}(\vec{k}) = u^{(r)^\dagger}(0)\frac{(k\!\!\!/ + m)}{\sqrt{2m(\omega_k + m)}}\gamma^0. \tag{2.3.135}$$

Thus, the orthogonality properties are given by

$$\begin{aligned} u^{(r)^\dagger}(\vec{k})u^{(s)}(\vec{k}) &= u^{(r)^\dagger}(0)\frac{(k\!\!\!/ + m)\gamma^0(k\!\!\!/ + m)}{2m(\omega_k + m)}u^{(s)}(0) \\ &= u^{(r)^\dagger}(0)\frac{(-k\!\!\!/ + m + 2\gamma^0 k^0)(k\!\!\!/ + m)}{2m(\omega_k + m)}u^{(s)}(0) \\ &= u^{(r)^\dagger}(0)\frac{2w_k\gamma^0(k\!\!\!/ + m)}{2m(\omega_k + m)}u^{(s)}(0) \\ &= u^{(r)^\dagger}(0)\frac{\omega_k(\omega_k + m)}{m(\omega_k + m)}u^{(s)}(0) + u^{(r)^\dagger}(0)\frac{\omega_k\gamma^0\gamma^i k_i}{m(\omega_k + m)}u^{(s)}(0). \end{aligned} \tag{2.3.136}$$

But $\gamma^0\gamma^i\gamma^0 = -\gamma^i$, so that

$$\begin{aligned} u^{(r)^\dagger}(0)\gamma^0\gamma^i\gamma^0 u^{(s)}(0) &= u^{(r)^\dagger}(0)\gamma^i u^{(s)}(0) \\ &= -u^{(r)^\dagger}(0)\gamma^i u^{(s)}(0), \end{aligned} \tag{2.3.137}$$

hence,

$$u^{(r)^\dagger}(0)\gamma^i u^{(s)}(0) = 0. \tag{2.3.138}$$

Since we normalized at rest to $2m$,

$$u^{(r)^\dagger}(0)u^{(s)}(0) = 2m\delta_{rs}, \tag{2.3.139}$$

we find

$$u^{(r)^\dagger}(\vec{k})u^{(s)}(\vec{k}) = 2\omega_k\delta_{rs}. \tag{2.3.140}$$

Similarly for the negative energy spinors $v^{(s)}(\vec{k})$ and

$$v^{(r)\dagger}(\vec{k}) = -v^{(r)\dagger}(0)\frac{(-\not{k}+m)}{\sqrt{2m(\omega_k+m)}}\gamma^0, \tag{2.3.141}$$

which leads to

$$v^{(r)^\dagger}(\vec{k})v^{(s)}(\vec{k}) = 2\omega_k\delta_{rs}. \tag{2.3.142}$$

Furthermore, we have

$$\begin{aligned} u^{(r)^\dagger}(\vec{k})v^{(s)}(-\vec{k}) &= u^{(r)^\dagger}(0)\frac{(\not{k}+m)\gamma^0(-\gamma^0\not{k}\gamma^0+m)}{2m(\omega_k+m)}v^{(s)}(0) \\ &= u^{(r)^\dagger}(0)\frac{(\not{k}+m)(-\not{k}+m)\gamma^0}{2m(\omega_k+m)}v^{(s)}(0) = 0. \end{aligned} \tag{2.3.143}$$

Thus we secure

$$u^{(r)^\dagger}(\vec{k})v^{(s)}(-\vec{k}) = 0 = v^{(r)^\dagger}(\vec{k})u^{(s)}(-\vec{k}). \tag{2.3.144}$$

Equivalently we could introduce adjoint spinors

$$\begin{aligned} \overline{u}^{(r)}(\vec{k}) &\equiv u^{(r)^\dagger}(\vec{k})\gamma^0 \\ \overline{v}^{(r)}(\vec{k}) &\equiv v^{(r)^\dagger}(\vec{k})\gamma^0. \end{aligned} \tag{2.3.145}$$

Then, as usual, if

$$(\not{k}-m)u^{(r)}(\vec{k}) = 0 \tag{2.3.146}$$

taking the hermitian conjugate yields

$$\begin{aligned} u^{(r)^\dagger}(\vec{k})(\not{k}^\dagger - m) &= 0 \\ &= u^{(r)^\dagger}(\vec{k})(\gamma^0 \not{k} \gamma^0 - m) \end{aligned}. \tag{2.3.147}$$

Multiplying by γ^0 implies

$$\overline{u}^{(r)}(\vec{k})(\not{k} - m) = 0, \tag{2.3.148}$$

and similarly $(\not{k} + m)v^{(r)}(\vec{k}) = 0$ leads to

$$\overline{v}^{(r)}(\vec{k})(\not{k} + m) = 0. \tag{2.3.149}$$

Hence, we can proceed as before, so just summarizing the results here for the spinor and adjoint spinor orthogonality relations in the $\vec{k}$- frame

$$\overline{u}^{(r)}(\vec{k})u^{(s)}(\vec{k}) = \delta_{rs}2m$$

$$\overline{v}^{(r)}(\vec{k})v^{(s)}(\vec{k}) = -\delta_{rs}2m$$

$$\overline{u}^{(r)}(\vec{k})v^{(s)}(\vec{k}) = 0 = \overline{v}^{(r)}(\vec{k})u^{(s)}(\vec{k}). \tag{2.3.150}$$

Besides orthogonality the spinors are complete; that is we have for positive energy spinors

$$\sum_{s=1}^{2} u^{(s)}(\vec{k})\overline{u}^{(s)}(\vec{k}) = (\not{k} + m) \tag{2.3.151}$$

where we used the fact that $u^{(s)}(0)$ are the $+1$ eigenvectors of γ^0 (see (2.3.112)) and thus

$$\sum_{s=1}^{2} u^{(s)}(0)u^{(s)^\dagger}(0) = \frac{1}{2}(1 + \gamma^0)2m. \tag{2.3.152}$$

It then follows that

$$\sum_{s=1}^{2} u^{(s)}(\vec{k})\overline{u}^{(s)}(\vec{k}) = \frac{2m}{(2m)(\omega_k+m)}(\not{k}+m)\frac{(1+\gamma^0)}{2}(\not{k}^\dagger+m)\gamma^0$$

$$= \frac{2m}{(2m)(\omega_k+m)}(\not{k}+m)\frac{(1+\gamma^0)}{2}(\not{k}+m)$$

$$= \frac{(2m)\frac{1}{2}}{(2m)(\omega_k+m)}\left[(\not{k}+m)(\not{k}+m)+2\not{k}k^0\right.$$

$$\left.-k^2\gamma^0+m^2\gamma^0+m\{\not{k},\gamma^0\}\right] \qquad (2.3.153)$$

$$= \frac{2m}{(4m)(\omega_k+m)}[(\not{k}+m)(\not{k}+m)+2\omega_k(\not{k}+m)]$$

$$= \frac{2m}{(4m)(\omega_k+m)}[k^2+m^2+2m\not{k}+2\omega_k\not{k}+2\omega_k m]$$

$$= \frac{2m}{(4m)(\omega_k+m)}[2m(m+\not{k})+2\omega_k(\not{k}+m)]$$

$$= \not{k}+m.$$

Similarly the negative energy spinors obey

$$\sum_{s=1}^{2} v^{(s)}(\vec{k})\overline{v}^{(s)}(\vec{k}) = (\not{k}-m) \qquad (2.3.154)$$

since $v^{(s)}(0)$ are the -1 eigenvalue eigenvectors of γ^0 (see equation (2.3.112)) and thus

$$\sum_{s=1}^{2} v^{(s)}(0)v^{(s)^\dagger}(0) = \frac{1}{2}(1-\gamma^0)2m. \qquad (2.3.155)$$

Thus, we have completeness of the spinors

$$\sum_{s=1}^{2}\frac{1}{2m}[u^{(s)}(\vec{k})\overline{u}^{(s)}(\vec{k}) - v^{(s)}(\vec{k})\overline{v}^{(s)}(\vec{k})] = \mathbf{1}, \qquad (2.3.156)$$

where $\mathbf{1}$ is the 4×4 identity matrix. (Note that we have chosen our spinors to be eigenstates of γ^0 and σ^{12} at rest. Alternatively, we could have chosen another combination of 4×4 γ matrices at rest or in another frame. Mandl and Shaw have chosen u and v to be eigenvectors of the helicity operator

$$h \equiv \frac{\vec{k}\cdot\vec{\sigma}}{|\vec{k}|} \qquad (2.3.157)$$

where $\vec{\sigma} \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12})$ and of course the Dirac equation $\not{k}u = mu$ and $\not{k}v = -mv$ in the $\vec{k}$-frame, not at rest as we have chosen. Also note that $[\not{k}, h] = 0$. Each choice are linear combinations, boosts, and rotations of the other.)

We can now invert the Fourier transforms of the Dirac field

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left[b_s(\vec{k}) u^{(s)}(\vec{k}) e^{-ikx} + d_s^\dagger(\vec{k}) v^{(s)}(\vec{k}) e^{+ikx} \right] \tag{2.3.158}$$

and the adjoint Dirac field

$$\overline{\Psi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left[d_s(\vec{k}) \overline{v}^{(s)}(\vec{k}) e^{-ikx} + b_s^\dagger(\vec{k}) \overline{u}^{(s)}(\vec{k}) e^{+ikx} \right] \tag{2.3.159}$$

to obtain

$$\begin{aligned}
b_s(\vec{k}) &= \int d^3x e^{+ikx} \overline{u}^{(s)}(\vec{k}) \gamma^0 \Psi(x) \\
d_s^\dagger(\vec{k}) &= \int d^3x e^{-ikx} \overline{v}^{(s)}(\vec{k}) \gamma^0 \Psi(x) \\
b_s^\dagger(\vec{k}) &= \int d^3x \overline{\Psi}(x) \gamma^0 u^{(s)}(\vec{k}) e^{-ikx} \\
d_s(\vec{k}) &= \int d^3x \overline{\Psi}(x) \gamma^0 v^{(s)}(\vec{k}) e^{+ikx}.
\end{aligned} \tag{2.3.160}$$

The canonical anti-commutation relations (CAR) can now be obtained for b and d and their hermitian conjugates by using the ETAR and the fact that they are time independent, $\dot{b} = 0 = \dot{d}$, for example

$$\begin{aligned}
\left\{ b_r(\vec{p}), b_s^\dagger(\vec{k}) \right\} &= \int_{x^0=y^0} d^3x d^3y e^{ipx} e^{-iky} \left(\overline{u}^{(r)}(\vec{p}) \gamma^0 \right)_a \left(u^{(s)}(\vec{k}) \right)_b \left\{ \Psi_a(x), \Psi_b^\dagger(y) \right\} \\
&= \int_{x^0=y^0} d^3x d^3y e^{ipx} e^{-iky} \left(\overline{u}^{(r)}(\vec{p}) \gamma^0 \right)_a \left(u^{(s)}(\vec{k}) \right)_b \delta_{ab} \delta^3(\vec{x} - \vec{y}) \\
&= \int d^3x e^{i(p-k)x} \overline{u}^{(r)}(\vec{p}) \gamma^0 u^{(s)}(\vec{k}) \\
&= (2\pi)^3 \delta^3(\vec{p} - \vec{k}) e^{i(\omega_p - \omega_k)x^0} u^{(r)^\dagger}(\vec{p}) u^{(s)}(\vec{p})
\end{aligned} \tag{2.3.161}$$

but $\delta^3(\vec{p} - \vec{k}) e^{i(\omega_p - \omega_k)x^0} = \delta^3(\vec{p} - \vec{k})$ and $u^{(r)^\dagger}(\vec{p}) u^{(s)}(\vec{p}) = 2\omega_p \delta_{rs}$, so that

$$\{ b_r(\vec{p}), b_s^\dagger(\vec{k}) \} = (2\pi)^3 (2\omega_k) \delta_{rs} \delta^3(\vec{p} - \vec{k}). \tag{2.3.162}$$

Similarly we find

$$\{d_r(\vec{p}), d_s^\dagger(\vec{k})\} = (2\pi)^3(2\omega_k)\delta_{rs}\delta^3(\vec{p}-\vec{k}). \tag{2.3.163}$$

Since Ψ and Ψ anticommute as do $\overline{\Psi}$ and $\overline{\Psi}$ we have

$$\{b_r(\vec{p}), b_s(\vec{k})\} = 0 = \{d_r(\vec{p}), d_s(\vec{k})\} \tag{2.3.164}$$

and

$$\{b_r(\vec{p}), d_s^\dagger(\vec{k})\} = 0 \tag{2.3.165}$$

and since u and v are orthogonal we also find

$$\{b_r(\vec{p}), d_s(\vec{k})\} = 0, \tag{2.3.166}$$

the adjoints of the above expressions vanish as well.

The expressions for the Hamiltonian, momentum, and charge operators can now be Fourier transformed. As usual we obtain these quantities by starting with

$$\mathcal{P}^\mu = \int d^3x \frac{i}{2}\overline{\Psi}\gamma^0 \overleftrightarrow{\partial^\mu}\, \Psi$$

$$Q = \int d^3x \overline{\Psi}\gamma^0\Psi$$

$$\begin{aligned}\mathcal{M}^{\mu\nu} &= \int d^3x \left[x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu}\right] \\ &= \int d^3x \frac{i}{4}\overline{\Psi}\Big[\gamma^0 x^\mu \overleftrightarrow{\partial^\nu} - \gamma^0 x^\nu \overleftrightarrow{\partial^\mu} + (x^\mu\gamma^\nu - x^\nu\gamma^\mu)\gamma^0 \overleftrightarrow{\partial^0} \\ &+ \left(g^{0\mu}\gamma^\nu - g^{0\mu}\gamma^\nu\right)\Big]\Psi.\end{aligned} \tag{2.3.167}$$

First we analyze the energy momentum operator

$$\begin{aligned}
\mathcal{P}^\mu &= \int d^3x \frac{i}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} \sum_{r=1}^{2} \sum_{s=1}^{2} \Big[d_r(\vec{k}) \overline{v}^{(r)}(\vec{k}) e^{-ikx} \\
&+ b_r^\dagger(\vec{k}) \overline{u}^{(r)}(\vec{k}) e^{+ikx} \Big] \gamma^0 \overset{\leftrightarrow}{\partial}{}_x^\mu \Big[b_s(\vec{l}) u^{(s)}(\vec{l}) e^{-ilx} + d_s^\dagger(\vec{l}) v^{(s)}(\vec{l}) e^{+ilx} \Big] \\
&= \frac{i}{2} \int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} \sum_{r,s=1}^{2} \Big[d_r(\vec{k}) \overline{v}^{(r)}(\vec{k}) e^{-ikx} \\
&+ b_r^\dagger(\vec{k}) \overline{u}^{(r)}(\vec{k}) e^{+ikx} \Big] \gamma^0 \Big[-il^\mu b_s(\vec{l}) u^{(s)}(\vec{l}) e^{-ilx} + il^\mu d_s^\dagger(\vec{l}) v^{(s)}(\vec{l}) e^{+ilx} \Big] \\
&+ \Big[ik^\mu d_r(\vec{k}) \overline{v}^{(r)}(\vec{k}) e^{-ikx} - ik^\mu b_r^\dagger(\vec{k}) \overline{u}^{(r)}(\vec{k}) e^{+ikx} \Big] \gamma^0 \\
&\times \Big[b_s(\vec{l}) u^{(s)}(\vec{l}) e^{-ilx} + d_s^\dagger(\vec{l}) v^{(s)}(\vec{l}) e^{+ilx} \Big] \\
&= \frac{i}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \Big[d_r(\vec{k}) d_r^\dagger(\vec{k}) ik^\mu - ik^\mu b_r^\dagger(\vec{k}) b_r(\vec{k}) \Big] \times 2.
\end{aligned} \tag{2.3.168}$$

So we find for the energy-momentum operator

$$\mathcal{P}^\mu = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} k^\mu \Big[b_s^\dagger(\vec{k}) b_s(\vec{k}) - d_s(\vec{k}) d_s^\dagger(\vec{k}) \Big]. \tag{2.3.169}$$

The charge operator is found similarly to be

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \Big[b_s^\dagger(\vec{k}) b_s(\vec{k}) + d_s(\vec{k}) d_s^\dagger(\vec{k}) \Big]. \tag{2.3.170}$$

With an analogous expression for $\mathcal{M}^{\mu\nu}$ which we will not bother to calculate here.

We see that the Hamiltonian is given by

$$H = \mathcal{P}^0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \omega_k \Big[b_s^\dagger(\vec{k}) b_s(\vec{k}) - d_s(\vec{k}) d_s^\dagger(\vec{k}) \Big]. \tag{2.3.171}$$

First suppose that d and $d^\dagger$ obeyed the CCR rather than the CAR, then we can write H as

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \omega_k \Big[b_s^\dagger(\vec{k}) b_s(\vec{k}) - d_s^\dagger(\vec{k}) d_s(\vec{k}) \Big]$$

$$-\int \frac{d^3k\omega_k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} [d_s(\vec{k}), d_s^\dagger(\vec{k})] \tag{2.3.172}$$

where now we have that $[d_r(\vec{k}), d_r^\dagger] = (2\pi)^3 2\omega_k \delta_{rr} \delta(\vec{k} - \vec{k})$, the CCR. Thus H would be given by

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \omega_k \left[b_s^\dagger(\vec{k}) b_s(\vec{k}) - d_s^\dagger(\vec{k}) d_s(\vec{k}) \right] - E_0 \tag{2.3.173}$$

with

$$E_0 \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \omega_k [(2\pi)^3 2\omega_k \delta_{ss} \delta(\vec{k} - \vec{k})], \tag{2.3.174}$$

the infinite constant harmonic oscillator zero point energy for the four oscillators corresponding to the $b^{(s)}(\vec{k})$ and $d^{(s)}(\vec{k})$ degrees of freedom. The point is that if b and d obey CCR the Hamiltonian is unbounded below; it can become as negative as you like by adding more and more "d" particles to a state. Hence, there is no state with lowest energy, if $|E>$ is the lowest energy state $d_r^\dagger(\vec{k})|E>$ is ω_k lower in energy! Hence, we have no stable ground state. The only consistent interpretation of the field theory is to demand that b and d obey the CAR. Thus,

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \omega_k \left[b_s^\dagger(\vec{k}) b_s(\vec{k}) + d_s^\dagger(\vec{k}) d_s(\vec{k}) \right] - E_0. \tag{2.3.175}$$

As in the scalar case the infinite constant E_0 is eliminated by a more careful definition of the Lagrangian and the energy-momentum tensor by means of the Wick normal product. As before, the Wick product of Fermi fields is defined so that the annihilation operators b and d are to the right of the creation operators $b^\dagger$ and $d^\dagger$. Since these are Grassmann algebra valued we include a $(+1)$ if the fields are ordered in an even permutation of the original field order and (-1) if the fields are ordered in an odd permutation of the original field order. Thus, denoting the Wick product as $N[\quad]$ or $:\quad:$, we have for instance

$$N[\Psi(x)\overline{\Psi}(y)] \equiv \Psi^+(x)\overline{\Psi}^+(y) + \Psi^-(x)\overline{\Psi}^-(y) + \Psi^-(x)\overline{\Psi}^+(y) - \overline{\Psi}^-(y)\Psi^+(x). \tag{2.3.176}$$

Recall the ordinary product is simply

$$\begin{aligned} \Psi(x)\overline{\Psi}(y) &= \Psi^+(x)\overline{\Psi}^+(y) + \Psi^-(x)\overline{\Psi}^-(y) + \Psi^-(x)\overline{\Psi}^+(y) + \Psi^+(x)\overline{\Psi}^-(y) \\ &= N[\Psi(x)\overline{\Psi}(x)] + \left\{ \Psi^+(x), \overline{\Psi}^-(y) \right\}. \end{aligned} \tag{2.3.177}$$

In general the Wick product of Fermi fields is defined by its interchange property

$$N[\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n)] = (-1)^P N[\tilde{\Psi}(x_{i_1})\cdots\tilde{\Psi}(x_{i_n})] \tag{2.3.178a}$$

where $\tilde{\Psi}$ is Ψ or $\overline{\Psi}$ and P is the parity of the permutation of $(1, 2, \cdots, n)$ into $(i_1, \cdots, i_n)$, its linearity

$$\begin{aligned} N[(\tilde{\Psi}(x) + a\tilde{\Psi}'(x))\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n)] &= N[\tilde{\Psi}(x)\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n)] \\ &+ aN[\tilde{\Psi}'(x)\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n)], \end{aligned} \tag{2.3.178b}$$

and its definition in the standard order

$$\begin{aligned} &N[\tilde{\Psi}^-(x_1)\cdots\tilde{\Psi}^-(x_m)\tilde{\Psi}^+(y_1)\cdots\tilde{\Psi}^+(y_n)] \\ &= +\tilde{\Psi}^-(x_1)\cdots\tilde{\Psi}^-(x_m)\tilde{\Psi}^+(y_1)\cdots\tilde{\Psi}^+(y_n). \end{aligned} \tag{2.3.179}$$

Thus, the correctly defined coincident point composite operators are just defined by normal ordering the ordinary products of fields, for example

$$\begin{aligned} \hat{\mathcal{L}} &= N[\mathcal{L}] \\ \hat{T}^{\mu\nu} &= N[T^{\mu\nu}] \\ \hat{J}^{\mu} &\equiv N[J^{\mu}] \\ \hat{M}^{\mu\nu\rho} &\equiv N[M^{\mu\nu\rho}]. \end{aligned} \tag{2.3.180}$$

Since these differ from the naive expressions by infinite constants, none of our field equations, CAR, CCR of charges, and conservation equations change. So now

$$\begin{aligned} \mathcal{P}^{\mu} &= \int d^3x N[\Theta^{0\mu}] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} k^{\mu} \left[b_s^{\dagger}(\vec{k})b_s(\vec{k}) + d_s^{\dagger}(\vec{k})d_s(\vec{k})\right]. \end{aligned} \tag{2.3.181}$$

Since b and d obey CAR, Q can become negative instead of being positive definite, as is reasonable for a charge

$$\begin{aligned} Q &= \int d^3x N[J^0] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left[b_s^{\dagger}(\vec{k})b_s(\vec{k}) - d_s^{\dagger}(\vec{k})d_s(\vec{k})\right]. \end{aligned} \tag{2.3.182}$$

The operators $b_s^\dagger(\vec{k})$ create particles with energy-momentum k^μ and charge $+1$, while the $d_s^\dagger(\vec{k})$ create particles with energy-momentum k^μ and charge -1. The operators $b_s(\vec{k})$ and $d_s(\vec{k})$ annihilate these respective particles. But for a further clarification of the properties of the particles that $b_s^\dagger(\vec{k})$ and $d_s^\dagger(\vec{k})$ create let's consider the action of our charges, $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$ and Q on b, $b^\dagger$, d, and $d^\dagger$ by Fourier transforming Ψ and $\overline{\Psi}$. Since

$$[\mathcal{P}^\mu, \Psi(x)] = -i\partial^\mu \Psi(x)$$

$$[\mathcal{P}^\mu, \overline{\Psi}(x)] = -i\partial^\mu \overline{\Psi}(x)$$

$$[Q, \Psi(x)] = -\Psi(x)$$

$$[Q, \overline{\Psi}(x)] = +\overline{\Psi}(x)$$

$$[\mathcal{M}^{\mu\nu}, \Psi(x)] = -i[(x^\mu\partial^\nu - x^\nu\partial^\mu)\Psi(x) - \frac{i}{2}\sigma^{\mu\nu}\Psi(x)]$$

$$[\mathcal{M}^{\mu\nu}, \overline{\Psi}(x)] = -i[(x^\mu\partial^\nu - x^\nu\partial^\mu)\overline{\Psi}(x) + \frac{i}{2}\overline{\Psi}(x)\sigma^{\mu\nu}], \tag{2.3.183}$$

having used

$$\gamma^0 \sigma^{\mu\nu\dagger} \gamma^0 = \sigma^{\mu\nu}, \tag{2.3.184}$$

we find

$$[\mathcal{P}^\mu, b_r(\vec{k})] = -k^\mu b_r(\vec{k})$$

$$[\mathcal{P}^\mu, b_r^\dagger(\vec{k})] = +k^\mu b_r^\dagger(\vec{k})$$

$$[\mathcal{P}^\mu, d_r(\vec{k})] = -k^\mu d_r(\vec{k})$$

$$[\mathcal{P}^\mu, d_r^\dagger(\vec{k})] = +k^\mu d_r^\dagger(\vec{k})$$

$$[Q, b_r(\vec{k})] = -b_r(\vec{k})$$

$$[Q, b_r^\dagger(\vec{k})] = +b_r^\dagger(\vec{k})$$

$$[Q, d_r(\vec{k})] = +d_r(\vec{k})$$

$$[Q, d_r^\dagger(\vec{k})] = -d_r^\dagger(\vec{k})$$

$$[\mathcal{M}^{\mu\nu}, b_r(\vec{k})] = \int d^3x e^{ikx} \overline{u}^{(s)}(\vec{k})\gamma^0 \left[-i(x^\mu \overrightarrow{\partial^\nu} - x^\nu \overrightarrow{\partial^\mu}) - \frac{1}{2}\sigma^{\mu\nu}\right] \Psi(x)$$

$$[\mathcal{M}^{\mu\nu}, b_r^\dagger(\vec{k})] = \int d^3x \left[-i(x^\mu \overrightarrow{\partial^\nu} - x^\nu \overrightarrow{\partial^\mu})\overline{\Psi} + \overline{\Psi}\frac{1}{2}\sigma^{\mu\nu}\right]\gamma^0 u^{(s)}(\vec{k})e^{-ikx}$$

$$[\mathcal{M}^{\mu\nu}, d_r(\vec{k})] = \int d^3x \left[-i(x^\mu \overrightarrow{\partial^\nu} - x^\nu \overrightarrow{\partial^\mu})\overline{\Psi} + \overline{\Psi}\frac{1}{2}\sigma^{\mu\nu}\right]\gamma^0 v^{(s)}(\vec{k})e^{ikx}$$

$$[\mathcal{M}^{\mu\nu}, d_r^\dagger(\vec{k})] = \int d^3x e^{-ikx}\overline{v}^{(s)}(\vec{k})\gamma^0 \left[-i(x^\mu \overrightarrow{\partial^\nu} - x^\nu \overrightarrow{\partial^\mu}) - \frac{1}{2}\sigma^{\mu\nu}\right]\Psi(x). \quad (2.3.185)$$

The angular momentum terms require further analysis. Rather than determining the above in complete generality let's discuss what operators we desire in order to label the states. As usual the states of the field theory will be labeled by the eigenvalues of a complete set of commuting observables. In general there are many choices for this complete set and we must choose one. We would like our states to be translationally invariant hence, the operators should commute with $\mathcal{P}^\mu$. Furthermore, the states should have definite mass. Thus, $\mathcal{P}^2$ and $\mathcal{P}^\mu$ are two of the commuting observables. Actually only $\vec{\mathcal{P}}$ is needed since once $\mathcal{P}^2$ and $\vec{\mathcal{P}}$ are given, $\mathcal{P}^0$ is determined due to the constraint that $\mathcal{P}^0 \geq 0$ and $\mathcal{P}^2 \geq 0$. In fact since $\mathcal{P}^2$ commutes with $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$, and Q it will label the irreducible representations of the algebra. Besides $\mathcal{P}^2$ there is one other invariant that one can make out of $\mathcal{P}^\mu$ and $\mathcal{M}^{\mu\nu}$ by using the Pauli-Lubanski vector

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{M}^{\nu\rho}\mathcal{P}^\sigma. \quad (2.3.186)$$

Note that W_μ has the properties

$$\mathcal{P}^\mu W_\mu = 0$$

$$[\mathcal{P}^\mu, W^\nu] = 0$$

$$[\mathcal{M}^{\mu\nu}, W^\lambda] = i[g^{\lambda\nu}W^\mu - g^{\lambda\mu}W^\nu]. \quad (2.3.187)$$

Hence $W^2 = W_\mu W^\mu$ is a scalar and totally invariant i.e. commutes with $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$, and Q. Writing it out we have

$$W^2 = \frac{1}{2}\mathcal{M}^{\mu\nu}\mathcal{M}_{\mu\nu}\mathcal{P}^2 - \mathcal{M}_{\mu\rho}\mathcal{M}^{\nu\rho}\mathcal{P}^\mu\mathcal{P}_\nu. \quad (2.3.188)$$

Recalling that $\mathcal{J}_i = \epsilon_{ijk}\mathcal{M}^{jk}$ and $\mathcal{K}_i = \mathcal{M}^{0i}$ we can write the Pauli-Lubanski vector as

$$W_0 = -\vec{\mathcal{J}} \cdot \vec{\mathcal{P}}$$

$$\vec{W} = \vec{\mathcal{K}} \times \vec{\mathcal{P}} + \vec{\mathcal{J}}\mathcal{P}^0. \tag{2.3.189}$$

Since $[W^2, \mathcal{P}^\mu] = 0 = [W^2, \mathcal{M}^{\mu\nu}] = [W^2, Q]$, $\mathcal{P}^2$ and W^2, when acting on each irreducible representation of the $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$, and Q algebra will be proportional to the identity (Schur's lemma: $\mathcal{P}^2$ and W^2 commute with all operators $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$, and Q and are therefore proportional to the identity). Thus, the eigenvalues of $\mathcal{P}^2$ and W^2 will label the different irreducible representation states. Since we desire translational invariance, $\mathcal{P}^\mu$ that is, $\vec{\mathcal{P}}$, will label the states within each irreducible representation. Also Q, since it commutes with $\vec{\mathcal{P}}$, will be needed to uniquely label the states. $\vec{\mathcal{P}}$ and Q for each $\mathcal{P}^2$ and W^2 eigenvalue will not be a complete set of commuting observables, there will still be spin up and down degeneracy of the states. Hence, we need an operator that commutes with $(\vec{\mathcal{P}}, Q)$ that specifies the spin. Now, $[\mathcal{M}^{12}, \mathcal{P}^1] = i\mathcal{P}^2$ is directly not an acceptible choice and consequently, the $\mathcal{M}^{\mu\nu}$operators are out since they are not translationally invariant. But recall the Pauli-Lubanski vector

$$[W^\mu, \mathcal{P}^\nu] = 0 = [W^\mu, Q]. \tag{2.3.190}$$

So W^μ commutes with $\mathcal{P}^\mu$ and Q but

$$[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma} W_\rho \mathcal{P}_\sigma. \tag{2.3.191}$$

Thus, only one component of W^μ can be chosen since the different components do not commute amongst themselves. Let's choose W_3. Our complete set of commuting observables will be $(\mathcal{P}^2, W^2, \vec{\mathcal{P}}, W_3, Q)$ and their eigenvalues will label the states. (Actually we should include Q with $\mathcal{P}^2$ and W^2 as labeling the irreducible representation since Q commutes with all the other operators.) The way we have expanded the fields, in fact, has been in terms of the eigenfunctions of these operators so that the creation and annihilation operators will commute with these operators to give eigenvalues. More specifically, we have seen this for all the operators so far, for instance,

$$[\mathcal{P}^\mu, b_r^\dagger(\vec{k})] = k^\mu b_r^\dagger(\vec{k}), \tag{2.3.192}$$

except W_3 and W^2. Since W_3 is a vector operator and W^2 is a scalar operator we can always transform them to the rest frame. Since $[\mathcal{P}^\mu, b_r^\dagger(\vec{k})] = k^\mu_{rest} b_r^\dagger(\vec{k})$ with $k^\mu_{rest} = (m, 0, 0, 0)$ in the rest frame we find $W^\mu = (0, m\vec{\mathcal{J}})$ in the rest frame. Thus, $\vec{\mathcal{J}} = \frac{\vec{W}}{m}$ is just the spin operator for the particle at rest. While $\frac{W^2}{-m^2} = \vec{\mathcal{J}} \cdot \vec{\mathcal{J}}$ in the rest frame. As we can check explicitly the eigenvalue of $\vec{\mathcal{J}} \cdot \vec{\mathcal{J}} = \frac{1}{2}(1 + \frac{1}{2})$ for spin $\frac{1}{2}$ single particle states. Since W^2 is a Lorentz invariant so are its eigenvalues and $\frac{W^2}{-m^2} = s(s+1)$ in all frames. In the rest frame the eigenvalue of $\frac{W_3}{m} = \mathcal{J}_3$ are just the spin projections on the z axis, that is, $\pm\frac{1}{2}$ as we will see for a spin $\frac{1}{2}$ particle. In general, for a spin s particle the projection of the spin on the z axis may take on the values $s, s-1, \cdots, -s+1, -s$. So we see that we need to consider the commutator of $\vec{\mathcal{J}}$ with the operators b, d, $b^\dagger$, and $d^\dagger$ at rest in order to find what states are created and annihilated. Hence, we desire

$$\mathcal{J}_i = \frac{1}{2}\epsilon_{ijk}\mathcal{M}^{jk}. \tag{2.3.193}$$

Integrating by parts we find

$$[\mathcal{M}^{jk}, b_r(\vec{k})] = -i(k^j\partial_k^k - k^k\partial_k^j)b_r(\vec{k}) - \frac{1}{2}\int d^3x e^{ikx}\overline{u}^{(r)}(\vec{k})\gamma^0\sigma^{jk}\Psi(x)$$

$$[\mathcal{M}^{jk}, b_r^\dagger(\vec{k})] = -i(k^j\partial_k^k - k^k\partial_k^j)b_r^\dagger(\vec{k}) + \frac{1}{2}\int d^3x\overline{\Psi}(x)\sigma^{jk}\gamma^0 u^{(r)}(\vec{k})e^{-ikx}$$

$$[\mathcal{M}^{jk}, d_r(\vec{k})] = -i(k^j\partial_k^k - k^k\partial_k^j)d_r(\vec{k}) + \frac{1}{2}\int d^3x\overline{\Psi}(x)\sigma^{jk}\gamma^0 v^{(r)}(\vec{k})e^{ikx}$$

$$[\mathcal{M}^{jk}, d_r^\dagger(\vec{k})] = -i(k^j\partial_k^k - k^k\partial_k^j)d_r^\dagger(\vec{k}) - \frac{1}{2}\int d^3x e^{-ikx}\overline{v}^{(r)}(\vec{k})\gamma^0\sigma^{jk}\Psi(x). \tag{2.3.194}$$

Defining the 4×4 spin matrix

$$\Sigma^i \equiv \frac{1}{2}\epsilon_{ijk}\frac{1}{2}\sigma^{jk} \tag{2.3.195}$$

we have, at zero momentum where the angular momentum terms

$$L_i = \frac{1}{2}\epsilon_{ijk}i(k^j\partial_k^k - k^k\partial_k^j)$$

vanish since $\vec{k} = 0$ and recalling $\gamma^0 u^{(r)}(0) = +u^{(r)}(0)$ and $\gamma^0 v^{(r)}(0) = -v^{(r)}(0)$, that

$$[\vec{\mathcal{J}}, b_r(0)] = -\int d^3x u^{(r)^\dagger}(0)\vec{\Sigma}\Psi(x)$$

$$[\vec{\mathcal{J}}, b_r^\dagger(0)] = \int d^3x\overline{\Psi}(x)\vec{\Sigma}u^{(r)}(0)$$

$$[\vec{\mathcal{J}}, d_r(0)] = -\int d^3x \overline{\Psi}(x) \vec{\Sigma} v^{(r)}(0)$$
$$[\vec{\mathcal{J}}, d_r^\dagger(0)] = -\int d^3x v^{(r)^\dagger}(0) \vec{\Sigma} \Psi(x). \tag{2.3.196}$$

We see that due to our choice of spinors u and v we have

$$\Sigma_3 u^{(r)}(0) = (-1)^{(r+1)} \frac{1}{2} u^{(r)}(0)$$
$$\Sigma_3 v^{(r)}(0) = (-1)^{(r+1)} \frac{1}{2} v^{(r)}(0) \tag{2.3.197}$$

and using

$$\vec{\Sigma}^\dagger = \vec{\Sigma} \tag{2.3.198}$$

we find

$$[\mathcal{J}_3, b_r(0)] = -\frac{1}{2}(-1)^{(r+1)} b_r(0)$$
$$[\mathcal{J}_3, b_r^\dagger(0)] = +\frac{1}{2}(-1)^{(r+1)} b_r^\dagger(0)$$
$$[\mathcal{J}_3, d_r(0)] = +\frac{1}{2}(-1)^{(r+1)} d_r(0)$$
$$[\mathcal{J}_3, d_r^\dagger(0)] = -\frac{1}{2}(-1)^{(r+1)} d_r^\dagger(0). \tag{2.3.199}$$

We are now in a position to interpret the state content of our spin $\frac{1}{2}$ field theory. Summarizing the commutators of the creation and annihilation operators with the CSCO we have

$$[\mathcal{P}^\mu, b_r(\vec{k})] = -k^\mu b_r(\vec{k})$$
$$[\mathcal{P}^\mu, b_r^\dagger(\vec{k})] = +k^\mu b_r^\dagger(\vec{k})$$
$$[\mathcal{P}^\mu, d_r(\vec{k})] = -k^\mu d_r(\vec{k})$$
$$[\mathcal{P}^\mu, d_r^\dagger(\vec{k})] = +k^\mu d_r^\dagger(\vec{k})$$
$$[Q, b_r(\vec{k})] = -b_r(\vec{k})$$
$$[Q, b_r^\dagger(\vec{k})] = +b_r^\dagger(\vec{k})$$
$$[Q, d_r(\vec{k})] = +d_r(\vec{k})$$
$$[Q, d_r^\dagger(\vec{k})] = -d_r^\dagger(\vec{k})$$

$$
\begin{aligned}
[\mathcal{J}_3, b_r(0)] &= -\frac{1}{2}(-1)^{(r+1)} b_r(0) \\
[\mathcal{J}_3, b_r^\dagger(0)] &= +\frac{1}{2}(-1)^{(r+1)} b_r^\dagger(0) \\
[\mathcal{J}_3, d_r(0)] &= +\frac{1}{2}(-1)^{(r+1)} d_r(0) \\
[\mathcal{J}_3, d_r^\dagger(0)] &= -\frac{1}{2}(-1)^{(r+1)} d_r^\dagger(0).
\end{aligned} \tag{2.3.200}
$$

As in the scalar case, we assume that we are given an eigenstate of $\mathcal{P}^\mu$, $|p>$, where $\mathcal{P}^\mu|p>= p^\mu|p>$. We see that

$$
\begin{aligned}
\mathcal{P}^\mu[b_r^\dagger(\vec{k})|p>] &= (p^\mu + k^\mu)[b_r^\dagger(\vec{k})|p>] \\
\mathcal{P}^\mu[d_r^\dagger(\vec{k})|p>] &= (p^\mu + k^\mu)[d_r^\dagger(\vec{k})|p>]
\end{aligned} \tag{2.3.201}
$$

while

$$
\begin{aligned}
\mathcal{P}^\mu[b_r(\vec{k})|p>] &= (p^\mu - k^\mu)[b_r(\vec{k})|p>] \\
\mathcal{P}^\mu[d_r(\vec{k})|p>] &= (p^\mu - k^\mu)[d_r(\vec{k})|p>].
\end{aligned} \tag{2.3.202}
$$

Repeating the application of annihilation operators on the state leads eventually to a state for which

$$
H[b_{r_1}(\vec{k}_1)\cdots d_{r_n}(\vec{k}_n)|p>] = (p^0 - \omega_{k_1} - \cdots - \omega_{k_n})[b_{r_1}(\vec{k}_1)\cdots d_{r_n}(\vec{k}_n)|p>] \tag{2.3.203}
$$

goes negative. But H is a non-negative operator, thus, there must be a lowest energy state, $|0>$, such that

$$
\begin{aligned}
b_r(\vec{k})|0>&= 0 \\
d_r(\vec{k})|0>&= 0
\end{aligned} \tag{2.3.204}
$$

for all $\vec{k}$ and $r = 1, 2$. This state is called the ground state, vacuum state or no particle state and we choose its normalization such that $< 0|0 >= 1$. Since $\mathcal{P}^\mu$ is defined to be normal ordered we have that the vacuum is the zero energy-momentum state, $\mathcal{P}^\mu|0>= 0$. The vacuum state has zero charge, $Q|0>= 0$, and no intrinsic spin, $\mathcal{M}^{\mu\nu}|0>= 0$, as well. The single particle states are given by the momentum value $\vec{k}$, the spin $\pm\frac{1}{2}$, and the charge ±1 eigenvalues. Consequently, there are four states for each value of $\vec{k}$. We have

$$
|\vec{k}, \frac{1}{2}(-1)^{r+1}, +>= b_r^\dagger(\vec{k})|0>
$$

$$|\vec{k}, \frac{1}{2}(-1)^r, - >= d_r^\dagger(\vec{k})|0> . \qquad (2.3.205)$$

Using the commutators in equation (2.3.200), we have

$$\vec{\mathcal{P}}|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >= \vec{k}|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >$$

$$H|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >= \omega_k|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >$$

$$Q|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >= \pm|\vec{k}, \frac{1}{2}(-1)^{r+1}, \pm >$$

$$\mathcal{J}_3|\vec{0}, \frac{1}{2}(-1)^{r+1}, + >= \frac{1}{2}(-1)^{r+1}|\vec{0}, \frac{1}{2}(-1)^{r+1}, + >$$

$$\mathcal{J}_3|\vec{0}, \frac{1}{2}(-1)^{r}, - >= \frac{1}{2}(-1)^{r}|\vec{0}, \frac{1}{2}(-1)^{r}, - > . \qquad (2.3.206)$$

Thus, there are four states with momentum $\vec{k}$, two having charge $+1$ which are conventionally called the particle state, and two having charge -1 which are conventionally refered to as the antiparticle state. The particle has two spin states, at rest the spin being $\pm\frac{1}{2}$ when projected on the z axis

$$\mathcal{J}_3|\vec{0}, \pm\frac{1}{2}, + >= \pm\frac{1}{2}|\vec{0}, \pm\frac{1}{2}, + > . \qquad (2.3.207)$$

Correspondingly, the antiparticle has at rest two spin states $\pm\frac{1}{2}$ when projected on the z axis

$$\mathcal{J}_3|\vec{0}, \pm\frac{1}{2}, - >= \pm\frac{1}{2}|\vec{0}, \pm\frac{1}{2}, - > . \qquad (2.3.208)$$

For every particle state there is a corresponding oppositely charged antiparticle state. We see that $b_1^\dagger(\vec{k})$ and $b_1(\vec{k})$ create and annihilate, respectively, particles with momentum $\vec{k}$, energy ω_k, charge $+1$, and in the rest frame spin projected on the z axis of $+\frac{1}{2}$. Similarly, $b_2^\dagger(\vec{k})$ creates a particle with momentum $\vec{k}$, energy ω_k, charge $+1$, and in the rest frame spin projected on the z axis of $-\frac{1}{2}$ while $b_2(\vec{k})$ annihilates such a particle. ¿From above we also see that $d_2^\dagger(\vec{k})$ creates an antiparticle with momentum $\vec{k}$, energy ω_k, charge -1, and in the rest frame spin projected on the z axis of $+\frac{1}{2}$ while $d_2(\vec{k})$ annihilates such an antiparticle. $d_1^\dagger(\vec{k})$ creates an antiparticle with momentum $\vec{k}$, energy ω_k, charge -1, and in the rest frame spin projected on the z axis of $-\frac{1}{2}$, and $d_1(\vec{k})$ annihilates such an antiparticle.

The inner products of the one particle states are found by using the anticommutators of the creation and annihilation operators and $<0|0>=1$

$$\begin{aligned} < \vec{k}, \frac{(-1)^{r+1}}{2}, +|\vec{k}', \frac{(-1)^{r'+1}}{2}, + > &= < 0| \left\{ b_r(\vec{k}), b^\dagger_{r'}(\vec{k}') \right\} |0 > \\ &= (2\pi)^3 (2\omega_k) \delta_{rr'} \delta^3(\vec{k} - \vec{k}') \end{aligned} \tag{2.3.209}$$

and

$$\begin{aligned} < \vec{k}, \frac{(-1)^{r}}{2}, -|\vec{k}', \frac{(-1)^{r'}}{2}, - > &= < 0| \left\{ d_r(\vec{k}), d^\dagger_{r'}(\vec{k}') \right\} |0 > \\ &= (2\pi)^3 (2\omega_k) \delta_{rr'} \delta^3(\vec{k} - \vec{k}') \end{aligned} \tag{2.3.210}$$

while

$$< \vec{k}, \frac{(-1)^{r+1}}{2}, +|\vec{k}', \frac{(-1)^{r'}}{2}, - > = 0. \tag{2.3.211}$$

Hence, the resolution of the identity in the one particle subspace is

$$\begin{aligned} \mathbf{1} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \Big[&|\vec{k}, \frac{(-1)^{r+1}}{2}, + >< \vec{k}, \frac{(-1)^{r+1}}{2}, +| \\ &+ |\vec{k}, \frac{(-1)^{r}}{2}, - >< \vec{k}, \frac{(-1)^{r}}{2}, -| \Big] . \end{aligned} \tag{2.3.212}$$

The multi-particle and multi-antiparticle states are made from multiple application of the creation operators

$$|(\vec{k}_1, +\frac{1}{2}, +), \cdots, (k_n, +\frac{1}{2}, -) >= b^\dagger_1(\vec{k}_1) \cdots d^\dagger_2(\vec{k}_n)|0 > . \tag{2.3.213}$$

Recall that since $b^\dagger_r$ and $d^\dagger_r$ anticommute the fermion particle states are antisymmetric under the interchange of particles. The operators $\mathcal{P}^\mu$, Q, and $\mathcal{J}_3$ acting on the states give the sum of the individual particle eigenvalues

$$H|(\vec{k}_1, +\frac{1}{2}, +), \cdots, (\vec{k}_n, +\frac{1}{2}, -) >= \left(\sum_{i=1}^{n} \omega_{k_i} \right) |(\vec{k}_1, +\frac{1}{2}, +), \cdots, (\vec{k}_n, +\frac{1}{2}, -) >$$

$$\vec{\mathcal{P}}|(\vec{k}_1, +\frac{1}{2}, +), \cdots, (\vec{k}_n, +\frac{1}{2}, -) >= \left(\sum_{i=1}^{n} \vec{k}_i \right) |(\vec{k}_1, +\frac{1}{2}, +), \cdots, (\vec{k}_n, +\frac{1}{2}, -) >$$

$$Q|(\vec{k}_1,+\frac{1}{2},+),\cdots,(\vec{k}_n,+\frac{1}{2},-)>=(N_+-N_-)|(\vec{k}_1,+\frac{1}{2},+),\cdots,(\vec{k}_n,+\frac{1}{2},-)>$$

$$\mathcal{J}_3|(\vec{k}_1,+\frac{1}{2},+),\cdots,(\vec{k}_n,+\frac{1}{2},-)>$$

$$=\frac{1}{2}(N_+^{\uparrow}-N_+^{\downarrow}+N_-^{\uparrow}-N_-^{\downarrow})|(\vec{k}_1,+\frac{1}{2},+),\cdots,(\vec{k}_n,+\frac{1}{2},-)> \tag{2.3.214}$$

where $N_\pm = N_\pm^{\uparrow} + N_\pm^{\downarrow}$ with $N_\pm^{\uparrow}$ and $N_\pm^{\downarrow}$ the number of particles with charge ± 1 and spin $+\frac{1}{2}$ ($\uparrow$) or $-\frac{1}{2}$ ($\downarrow$), respectively. The number operator for the number of particles with charge $+1$ and spin $+\frac{1}{2}$ is

$$N_+^{\uparrow} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} b_1^\dagger(\vec{k}) b_1(\vec{k}) \tag{2.3.215}$$

while for particles with charge $+1$ and spin $-\frac{1}{2}$ the number operator is

$$N_+^{\downarrow} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} b_2^\dagger(\vec{k}) b_2(\vec{k}). \tag{2.3.216}$$

The number operator for the number of antiparticles with charge -1 and spin $+\frac{1}{2}$ is

$$N_-^{\uparrow} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} d_2^\dagger(\vec{k}) d_2(\vec{k}) \tag{2.3.217}$$

and for antiparticles with charge -1 and spin $-\frac{1}{2}$ the number operator is

$$N_-^{\downarrow} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} d_1^\dagger(\vec{k}) d_1(\vec{k}). \tag{2.3.218}$$

The total number operator being

$$N = N_+^{\uparrow} + N_+^{\downarrow} + N_-^{\uparrow} + N_-^{\downarrow}. \tag{2.3.219}$$

Finally, we can calculate the covariant anticommutator of the fields

$$\{\Psi(x), \overline{\Psi}(y)\} = \left\{\Psi^+(x), \overline{\Psi}^-(y)\right\} + \left\{\Psi^-(x), \overline{\Psi}^+(y)\right\}. \tag{2.3.220}$$

Now as usual we Fourier transform the field operators and use the CAR

$$\begin{aligned}\left\{\Psi^+(x), \overline{\Psi}^-(y)\right\} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} \sum_{r,s=1}^{2} \left\{b_r(\vec{k}), b_s^\dagger(\vec{l})\right\} \\ &\quad \times u^{(r)}(\vec{k}) e^{-ikx} \overline{u}^{(s)}(\vec{l}) e^{+ily} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} u^{(r)}(\vec{k}) \overline{u}^{(r)}(\vec{k}) e^{-ik(x-y)} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (\not{k} + m) e^{-ik(x-y)} \\ &= (i\not{\partial}_x + m) \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)}.\end{aligned} \tag{2.3.221}$$

But recall

$$i\Delta^+(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \tag{2.3.222}$$

so

$$\begin{aligned}\left\{\Psi^+(x), \overline{\Psi}^-(y)\right\} &= (i\not{\partial}_x + m) i\Delta^+(x-y) \\ &\equiv iS^+(x-y).\end{aligned} \tag{2.3.223}$$

Similarly

$$\begin{aligned}\left\{\Psi^-(x), \overline{\Psi}^+(y)\right\} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} v^{(r)}(\vec{k}) \overline{v}^{(r)}(\vec{k}) e^{+ik(x-y)} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (\not{k} - m) e^{+ik(x-y)} \\ &= -(i\not{\partial}_x + m) \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+ik(x-y)},\end{aligned} \tag{2.3.224}$$

but

$$i\Delta^-(x-y) \equiv -\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+ik(x-y)}, \tag{2.3.225}$$

so

$$\begin{aligned}\left\{\Psi^-(x), \overline{\Psi}^+(y)\right\} &= (i\not{\partial}_x + m) i\Delta^-(x-y) \\ &\equiv iS^-(x-y).\end{aligned} \tag{2.3.226}$$

The covariant commutator becomes

$$\begin{aligned}\{\Psi(x), \overline{\Psi}(y)\} &= i(i\partial\!\!\!/_x + m)(\Delta^+(x-y) + \Delta^-(x-y)) \\ &= i(i\partial\!\!\!/_x + m)\Delta(x-y) \\ &\equiv iS(x-y).\end{aligned} \tag{2.3.227}$$

The Lorentz covariance of $S(x-y)$ can be checked directly since $\Delta(x-y)$ is invariant. Also since $\Delta(x-y) = 0$ for $(x-y)^2 < 0$, we find

$$\{\Psi(x), \overline{\Psi}(y)\} = 0 \tag{2.3.228}$$

for $(x-y)^2 < 0$. This is the statement of the microcausality principle for fermions. Since the observables are bilinear in Ψ and $\overline{\Psi}$, it guarantees that the observables commute at space-like separations.

In addition, we see that $S(x)$ obeys the Dirac equation

$$(i\partial\!\!\!/_x - m)S(x) = -(\partial^2 + m^2)\Delta(x) = 0 \tag{2.3.229}$$

with $S(\vec{x}, 0) = i\gamma^0\delta^3(\vec{x})$ as the initial condition. Since we can represent Δ as a contour integral

$$i\Delta(x) = \int_C \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - m^2} \tag{2.3.230}$$

where the contour in the complex k^0-plane is $C = C_+ + C_-$ and is given in Figure 2.3.1,

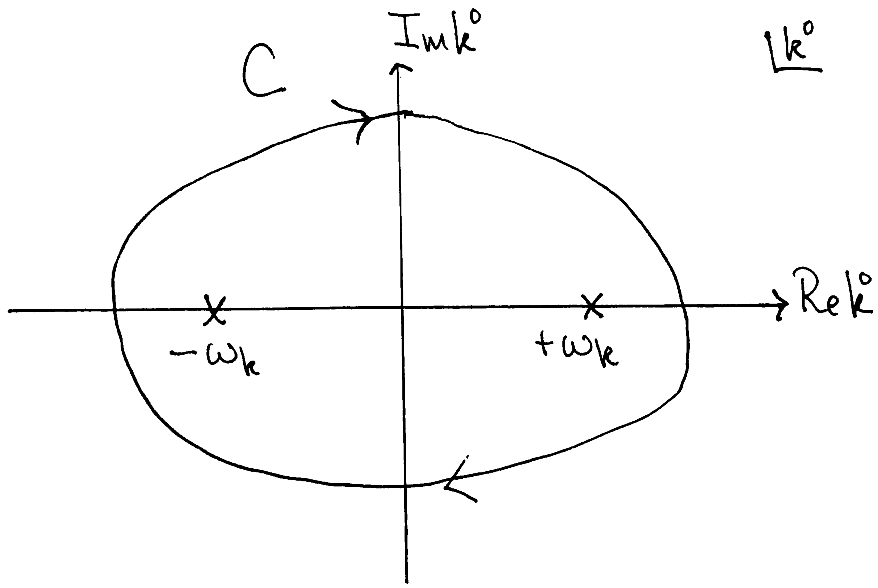

Figure 2.3.1

we find the contour integral for S

$$iS(x) = \int_C \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i(\not k + m)}{k^2 - m^2}. \tag{2.3.231}$$

Since $k^2 - m^2 = (\not k + m)(\not k - m)$ we often symbolically write

$$\frac{1}{\not k - m} \equiv \frac{(\not k + m)}{k^2 - m^2} \tag{2.3.232}$$

so that

$$iS(x) = \int_C \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{\not k - m}. \tag{2.3.233}$$

Similarly

$$\begin{aligned} iS^+(x) &= \int_{C^+} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{\not k - m} \\ iS^-(x) &= \int_{C^-} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{\not k - m}. \end{aligned} \tag{2.3.234}$$

Also we have the trivial covariant anticommutators

$$\begin{aligned} \{\Psi(x), \Psi(y)\} &= 0 \\ \left\{\overline{\Psi}(x), \overline{\Psi}(y)\right\} &= 0. \end{aligned} \tag{2.3.235}$$

Besides the covariant anti-commutators, we can evaluate the vacuum expectation values of fermion fields and the vacuum expectation values of the time ordered products of the fermion fields. The vacuum expectation value of $\Psi(x)$ and $\overline{\Psi}(y)$, $< 0|\Psi(x)\overline{\Psi}(y)|0 >$, is simply $iS^+(x-y)$

$$\begin{aligned} < 0|\Psi(x)\overline{\Psi}(y)|0 > &=< 0|\Psi^+(x)\overline{\Psi}^-(y)|0 > \\ &=< 0| \left\{\Psi^+(x), \overline{\Psi}^-(y)\right\} |0 > \\ &= iS^+(x-y). \end{aligned} \tag{2.3.236}$$

The general field product vacuum expectation value, the Wightman function, is defined as

$$\begin{aligned} &W^{(n,\overline{n})}(x_1, \cdots, x_n, y_1, \cdots, y_{\overline{n}}) \\ &\equiv< 0|\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 > \end{aligned}$$

$$=< 0| \left[\Psi^+(x_1), \Psi(x_2) \cdots \Psi(x_n)\overline{\Psi}(y_1) \cdots \overline{\Psi}(y_{\overline{n}})\right]_{\pm} |0 > \qquad (2.3.237)$$

where we use an anticommutator if $n + \bar{n} - 1$ is odd and a commutator if $n + \bar{n} - 1$ is even. Using $[A, BC] = \{A, B\}C - B\{A, C\}$ and $\{A, BC\} = \{A, B\}C - B[A, C]$ we have

$$W^{(n,\bar{n})}(x_1, \cdots, x_n, y_1, \cdots, y_{\bar{n}})$$

$$= \sum_{i=1}^{\overline{n}} (-1)^{n-1+i-1} \left\{\Psi^+(x_1), \overline{\Psi}^-(y_i)\right\}$$

$$\times < 0|\Psi(x_2) \cdots \Psi(x_n)\overline{\Psi}(y_1) \cdots \cancel{\overline{\Psi}(y_i)} \cdots \overline{\Psi}(y_{\overline{n}})|0 > . \qquad (2.3.238)$$

Note that

$$< 0|\overline{\Psi}(y)\Psi(x)|0 >=< 0| \left\{\overline{\Psi}^+(y), \Psi^-(x)\right\} |0 >= iS^-(x - y). \qquad (2.3.239)$$

Thus, $W^{(n,\overline{n})}$ reduces to S^+ times $W^{(n-1,\overline{n}-1)}$ and continuing the procedure we find

$$< 0|\Psi(x_1) \cdots \Psi(x_n)\overline{\Psi}(y_1) \cdots \overline{\Psi}(y_{\overline{n}})|0 >= 0 \ , \quad \text{if } n \neq \bar{n}$$

and

$$< 0|\Psi(x_1) \cdots \Psi(x_n)\overline{\Psi}(y_1) \cdots \overline{\Psi}(y_{\overline{n}})|0 >$$

$$= \sum_{(1,\cdots,n) \xrightarrow{P} (i_1,\cdots,i_n)} (-1)^{\frac{n(n-1)}{2}} (-1)^{|P|} < 0|\Psi(x_1)\overline{\Psi}(y_{i_1})|0 > \cdots$$

$$\cdots < 0|\Psi(x_n)\overline{\Psi}(y_{i_n})|0 > , \quad \text{if } n = \overline{n}, \qquad (2.3.240)$$

where $(-1)^{|P|}$ is the parity of the transformation of $(1, 2, \cdots, n)$ into $(i_1, i_2, \cdots, i_n)$. That is we have simply

$$< 0|\Psi(x_1) \cdots \Psi(x_n)\overline{\Psi}(y_1) \cdots \overline{\Psi}(y_{\overline{n}})|0 >$$

$$= \delta_{n\overline{n}} \sum_P (-1)^P \prod_{a=1}^{n} < 0|\Psi(x_a)\overline{\Psi}(y_{i_a})|0 > \qquad (2.3.241)$$

with $(-1)^P$ equal to the parity of the fermion field order

$$x_1 y_{i_1} x_2 y_{i_2} \cdots x_n y_{i_n} \xrightarrow{P} x_1 \cdots x_n y_1 \cdots y_n. \qquad (2.3.242)$$

This can be conveniently written in a form reminiscent of the Slater determinant

$$
< 0|\Psi(x_1)\cdots\Psi(x_n)\bar{\Psi}(y_1)\cdots\bar{\Psi}(y_{\bar{n}})|0 >
$$

$$
= \delta_{n\bar{n}}(-1)^M \begin{vmatrix} iS^+(x_1-y_1) & iS^+(x_1-y_2) & \dots & iS^+(x_1-y_n) \\ iS^+(x_2-y_1) & iS^+(x_2-y_2) & \dots & iS^+(x_2-y_n) \\ \vdots & \vdots & & \vdots \\ iS^+(x_n-y_1) & iS^+(x_n-y_2) & \dots & iS^+(x_n-y_n) \end{vmatrix} \tag{2.3.243}
$$

where

$$
M = \begin{cases} \frac{n}{2}, & \text{if } n = \text{even} \\ \frac{n-1}{2}, & \text{if } n = \text{odd}. \end{cases} \tag{2.3.244}
$$

Besides the Wightman functions we will make use of the time ordered functions, that is, the vacuum expectation value of the time ordered products of the fermi fields. Since the fermi fields anticommute we will define the time ordered product to have a sign depending upon the permutation of the operators

$$
T\Psi(x) = \Psi(x)
$$

$$
T\overline{\Psi}(x) = \overline{\Psi}(x)
$$

$$
T\Psi(x)\overline{\Psi}(x) = \theta(x^0-y^0)\Psi(x)\overline{\Psi}(y) - \theta(y^0-x^0)\overline{\Psi}(y)\Psi(x), \tag{2.3.245}
$$

for instance. In order to make the notation concise, let $\tilde{\Psi}$ stand for either Ψ or $\overline{\Psi}$. Then we define in general

$$
T\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n)
$$

$$
\equiv \sum_P (-1)^P \theta(x^0_{i_1}-x^0_{i_2})\theta(x^0_{i_2}-x^0_{i_3})\cdots\theta(x^0_{i_{n-1}}-x^0_{i_n})\tilde{\Psi}(x_{i_1})\cdots\tilde{\Psi}(x_{i_n}) \tag{2.3.246}
$$

where P permutes $(1,2,\cdots,n)$ into $(i_1,i_2,\cdots,i_n)$ and

$$
(-1)^P = \begin{cases} +1, & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd}. \end{cases} \tag{2.3.247}
$$

For instance,

$$
T\Psi(x)\overline{\Psi}(y) = -T\overline{\Psi}(y)\Psi(x) \tag{2.3.248}
$$

and in general,

$$
T\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_n) = (-1)^P T\tilde{\Psi}(x_{i_1})\cdots\tilde{\Psi}(x_{i_n}). \tag{2.3.249}
$$

Now we define the vacuum expectation value of these products as the Green functions, τ-functions, time ordered functions, or $(n,\overline{n})$ point functions

$$\begin{aligned} G^{(n,\overline{n})}(x_1,\cdots,x_n,y_1,\cdots,y_{\overline{n}}) &\equiv < 0|T\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0> \\ &= \sum_P (-1)^P \theta(x^0_{i_1}-x^0_{i_2})\cdots\theta(x^0_{i_{n+\overline{n}-1}}-x^0_{i_{n+\overline{n}}}) \\ &\times < 0|\tilde{\Psi}(x_{i_1})\cdots\tilde{\Psi}(x_{i_{n+\overline{n}}})|0> \end{aligned} \tag{2.3.250}$$

where here

$$\tilde{\Psi}(x_i) = \begin{cases} \Psi(x_i) & \text{if } i \le n \\ \overline{\Psi}(y_{i-n}) & \text{if} i > n \end{cases} \tag{2.3.251}$$

and

$$x_i = \begin{cases} x_i & \text{if } i \le n \\ y_{i-n} & \text{if } i > n. \end{cases} \tag{2.3.252}$$

$W^{(n,\overline{n})}$ vanishes unless $n = \overline{n}$, we find the same for $G^{(n,\overline{n})}$,

$$G^{(n,\overline{n})} = 0 \qquad \text{if } n \neq \overline{n}. \tag{2.3.253}$$

We evaluate $G^{(n,\bar{n})}$ by using the results for $W^{(n,\bar{n})}$ and the properties of the θ functions. First, the two point function

$$\begin{aligned} < 0|T\Psi(x)\overline{\Psi}(y)|0> &= \theta(x^0-y^0) < 0|\Psi(x)\overline{\Psi}(y)|0> \\ &- \theta(y^0-x^0) < 0|\overline{\Psi}(y)\Psi(x)|0> \\ &= \theta(x^0-y^0)iS^+(x-y) - \theta(y^0-x^0)iS^-(x-y). \end{aligned} \tag{2.3.254}$$

Since this combination occurs frequently we define it as

$$< 0|T\Psi(x)\bar{\Psi}(y)|0> \equiv S_F(x-y) \tag{2.3.255}$$

and call it the fermion Feynman propagator,

$$\begin{aligned} S_F(x) &\equiv \theta(x^0)iS^+(x) - \theta(-x^0)iS^-(x) \\ &= \theta(x^0)(i\partial\!\!\!/_x+m)i\Delta^+(x) - \theta(-x^0)(i\partial\!\!\!/+m)i\Delta^-(x). \end{aligned} \tag{2.3.256}$$

When we bring the ∂^0_x through the $\theta(x^0)$ and $\theta(-x^0)$ we pick up $\delta(x^0)$ so

$$S_F(x) = (i\partial\!\!\!/_x+m)\left[\theta(x^0)i\Delta^+(x) - \theta(-x^0)i\Delta^-(x)\right] + \gamma^0\delta(x^0)\left[\Delta^+(x)+\Delta^-(x)\right], \tag{2.3.257}$$

but

$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x) \tag{2.3.258}$$

and by equation (2.2.145)

$$\delta(x^0)\Delta(x) = 0. \tag{2.3.259}$$

Hence,

$$S_F(x) = (i\partial\!\!\!/_x + m)\left[\theta(x^0)i\Delta^+(x) - \theta(-x^0)i\Delta^-(x)\right]. \tag{2.3.260}$$

Recall that the scalar Feynman propagator is given by

$$\Delta_F(x) \equiv \left[\theta(x^0)i\Delta^+(x) - \theta(-x^0)i\Delta^-(x)\right] \tag{2.3.261}$$

so that

$$S_F(x) = (i\partial\!\!\!/_x + m)\Delta_F(x). \tag{2.3.262}$$

Thus, we finally secure

$$\begin{aligned} < 0|T\Psi(x)\overline{\Psi}(y)|0 > &= S_F(x-y) \\ &= (i\partial\!\!\!/_x + m)\Delta_F(x-y). \end{aligned} \tag{2.3.263}$$

Recalling the integral representation we had for Δ_F, equation (2.2.201),

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

we find

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon}. \tag{2.3.264}$$

Again we symbolically write

$$\frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon} \equiv \frac{i}{k\!\!\!/ - m} \tag{2.3.265}$$

so that

$$\begin{aligned} < 0|T\Psi(x)\overline{\Psi}(y)|0 > &= S_F(x-y) \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{(k\!\!\!/ - m)}. \end{aligned} \tag{2.3.266}$$

Returning to the $(n, \bar{n})$-point function we can show that the functions factorize into products of propagators

$$< 0|T\Psi(x)\overline{\Psi}(y)|0 >= S_F(x-y)$$

for instance

$$< 0|T\Psi(x_1)\Psi(x_2)\overline{\Psi}(y_1)\overline{\Psi}(y_2)|0 >=< 0|T\Psi(x_1)\overline{\Psi}(y_2)|0 >< 0|T\Psi(x_2)\overline{\Psi}(y_1)|0 >$$
$$- < 0|T\Psi(x_1)\overline{\Psi}(y_1)|0 >< 0|T\Psi(x_2)\overline{\Psi}(y_2)|0 > . \qquad (2.3.267)$$

In general we find

$$< 0|T\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 >= \delta_{n\overline{n}} \sum_{(1,\cdots,n)\overset{P}{\rightarrow}(i_1,\cdots,i_n)} (-1)^P$$
$$\times \prod_{a=1}^{n} < 0|T\Psi(x_a)\overline{\Psi}(y_{i_a})|0 > \qquad (2.3.268)$$

where $(-1)^P$ equals the parity of the fermion order due to the permutation P

$$x_1 y_{i_1} x_2 y_{i_2} \cdots x_n y_{i_n} \overset{P}{\longrightarrow} x_1 \cdots x_n y_1 \cdots y_n. \qquad (2.3.269)$$

Again we are able to write this as a determinant

$$< 0|T\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 >$$
$$= \delta_{n\bar{n}}(-1)^M \begin{vmatrix} iS_F(x_1-y_1) & iS_F(x_1-y_2) & \ldots & iS_F(x_1-y_n) \\ iS_F(x_2-y_1) & iS_F(x_2-y_2) & \ldots & iS_F(x_2-y_n) \\ \vdots & \vdots & & \vdots \\ iS_F(x_n-y_1) & iS_F(x_n-y_2) & \ldots & iS_F(x_n-y_n) \end{vmatrix} \qquad (2.3.270)$$

where

$$M = \begin{cases} \frac{n}{2} & \text{if } n = \text{even} \\ \frac{n-1}{2} & \text{if } n = \text{odd}. \end{cases} \qquad (2.3.271)$$

As before, the time ordered function of free fields is just the sum over all possible chronological pairings of the coordinates of the product of the associated two point functions. Since $< 0|T\Psi(x_1)\Psi(x_2)|0 >=< 0|T\overline{\Psi}(y_1)\overline{\Psi}(y_2)|0 >= 0$ chronological pairings occur only between $\{x_i\}$ and $\{y_j\}$. As in the scalar case we can represent this process by writing lines joining the chronologically paired fields for a particular term in the sum. These are then said to be contracted. For example, one possible term in the sum over all possible products of contractions is given by

$$
\begin{aligned}
&< 0|T\Psi(x_1)\Psi(x_2)\overline{\Psi}(y_1)\overline{\Psi}(y_2)|0 > \\
&= - < 0|T\Psi(x_1)\overline{\Psi}(y_1)|0 >< 0|T\Psi(x_2)\overline{\Psi}(y_2)|0 > . \qquad (2.3.272)
\end{aligned}
$$

The minus sign comes from the odd interchange of y_1 and x_2.

The proof of Wick's Theorem for fermion field time ordered functions follows from the lemma

$$
\begin{aligned}
&< 0|T\Psi(x)\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 > \\
&= \sum_{j=1}^{\overline{n}}(-1)^{n+j-1} < 0|T\Psi(x)\overline{\Psi}(y_j)|0 > \\
&\times < 0|T\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\cancel{\overline{\Psi}(y_j)}\cdots\overline{\Psi}(y_{\overline{n}})|0 > . \qquad (2.3.273)
\end{aligned}
$$

The proof of this follows the same lines as that for the scalar fields. In fact we can convert our fermion fields into "boson-like" fields by multiplying each field by its own Grassmann c-number

$$
\begin{aligned}
\Psi_a(x_1) &\longrightarrow \overline{\eta}_a(1)\Psi_a(x_1) \equiv \Phi(x_1) \\
\overline{\Psi}_a(y_1) &\longrightarrow \overline{\Psi}_a(y_1)\eta_a(1) \equiv \Phi^\dagger(y_1). \qquad (2.3.274)
\end{aligned}
$$

Then the Wightman and Green functions of the Φ and $\Phi^\dagger$ fields have the same properties as the scalar fields of our previous proof on pages 140-142, that is, with $\tilde{\Phi}$ either Φ or $\Phi^\dagger$ we have analogous to equations (2.2.207) and (2.2.208) for the particular time ordering $x_1^0 > x_2^0 > \cdots > x_j^0 > x^0 > x_{j+1}^0 > \cdots > x_m$

$$
\begin{aligned}
&< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_1)\cdots\tilde{\Phi}(x_m)|0 > \\
&=< 0|\tilde{\Phi}(x_1)\cdots\tilde{\Phi}(x_j)\tilde{\Phi}(x)\tilde{\Phi}(x_{j+1})\cdots\tilde{\Phi}(x_m)|0 > \\
&= \sum_{k=1}^{j}\left[\tilde{\Phi}(x_k),\tilde{\Phi}^-(x)\right] < 0|\tilde{\Phi}(x_1)\cdots\cancel{\tilde{\Phi}(x_k)}\cdots\cancel{\tilde{\Phi}(x)}\cdots\tilde{\Phi}(x_m)|0 >
\end{aligned}
$$

$$+\sum_{k=j+1}^{m}\left[\tilde{\Phi}^{+}(x),\tilde{\Phi}(x_k)\right] < 0|\tilde{\Phi}(x_1)\cdots\not{\tilde{\Phi}}(x)\cdots\not{\tilde{\Phi}}(x_k)\cdots\tilde{\Phi}(x_m)|0> \qquad (2.3.275)$$

where now we have, for instance,

$$\begin{aligned} < 0|\Phi(x)\Phi^{\dagger}(y)|0> &=< 0|[\Phi^{+}(x),\Phi^{\dagger^{-}}(y)]|0> \\ &=< 0|\Phi^{+}(x)\Phi^{\dagger-}(y)|0> - < 0|\Phi^{\dagger-}(y)\Phi^{+}(x)|0> \\ &= \overline{\eta}_a(x)\eta_b(y) < 0|\Psi_a^{+}(x)\overline{\Psi}_b^{-}(y)|0> \\ &+ \overline{\eta}_a(x)\eta_b(y) < 0|\overline{\Psi}_b^{-}(y)\Psi_a^{+}(x)|0> \\ &= \bar{\eta}_a(x)\eta_b(y)\left\{\Psi_a^{+}(x),\bar{\Psi}_b^{-}(y)\right\} \\ &= \bar{\eta}_a(x)\eta_b(y) < 0|\Psi_a(x)\bar{\Psi}_b(y)|0> . \end{aligned} \qquad (2.3.276)$$

So we have converted the commutator to an anticommutator

$$\begin{aligned} < 0|\Phi(x)\Phi^{\dagger}(y)|0> &= \overline{\eta}_a(x)\eta_b(y) < 0|\{\Psi_a^{+}(x),\overline{\Psi}_b^{-}(y)\}|0> \\ &= i\overline{\eta}(x)S^{+}(x-y)\eta(y). \end{aligned} \qquad (2.3.277)$$

Furthermore since the remaining fields are still in chronological order they are time ordered functions. As well since $x^0 > x_k^0$ for $k > j$, the anti-commutator is equal to the Feynman propagator

$$\left[\tilde{\Phi}^{+}(x),\tilde{\Phi}(x_k)\right] =< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_k)|0>, \qquad (2.3.278)$$

for $x^0 > x_k^0$. Similarly since $x^0 < x_k^0$ for $k \leq j$, the commutator is again just the Feynman propagator for that ordering

$$\left[\tilde{\Phi}(x_k),\tilde{\Phi}^{-}(x)\right] =< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_k)|0>, \qquad (2.3.279)$$

for $x^0 < x_k^0$. Thus we obtain, as in equations (2.2.211) and (2.2.212), for any chronological ordering the free field identity

$$< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_1)\tilde{\Phi}(x_2)\cdots\tilde{\Phi}(x_m)|0>$$

$$=\sum_{j=1}^{m} < 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_j)|0>< 0|T\tilde{\Phi}(x_1)\cdots\not{\tilde{\Phi}}(x_j)\cdots\tilde{\Phi}(x_m)|0> . \qquad (2.3.280)$$

For the above example

$$< 0|T\Phi(x)\Phi^\dagger(x_k)|0 >= \bar{\eta}_a(x)\eta_b(k) < 0|T\Psi_a(x)\bar{\Psi}_b(x_k)|0 >$$

$$= \bar{\eta}_a(x)\eta_b(k)S_{Fab}(x-x_k). \tag{2.3.281}$$

Next we pull out the Grassmann c-numbers from equation (2.3.280) in the same order from each term to obtain the identity for Fermi fields

$$< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_1)\cdots\tilde{\Phi}(x_m)|0 >$$

$$= (-1)^{\overline{n}+\frac{m(m+1)}{2}}\tilde{\eta}(x)\tilde{\eta}(x_1)\cdots\tilde{\eta}(x_m) < 0|T\tilde{\Psi}(x)\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_m)|0 > . \tag{2.3.282}$$

While

$$\begin{aligned}
&< 0|T\tilde{\Phi}(x)\tilde{\Phi}(x_j)|0 >< 0|T\tilde{\Phi}(x_1)\cdots\cancel{\tilde{\Phi}(x_j)}\cdots\tilde{\Phi}(x_m)|0 > \\
&= (-1)^{\overline{n}}(-1)^{\frac{(m-2)(m-1)}{2}}(-1)^1\tilde{\eta}(x)\cancel{\tilde{\eta}(x_j)}\tilde{\eta}(x_1)\cdots\tilde{\eta}(x_m) \\
&\times < 0|T\tilde{\Psi}(x)\tilde{\Psi}(x_j)|0 >< 0|T\tilde{\Psi}(x_1)\cdots\cancel{\tilde{\Psi}(x_j)}\cdots\tilde{\Psi}(x_m)|0 > \\
&= (-1)^{\overline{n}+1}(-1)^{\frac{(m-2)(m-1)}{2}}(-1)^{j-1}\tilde{\eta}(x)\tilde{\eta}(x_1)\cdots\tilde{\eta}(x_m) \\
&\times < 0|T\tilde{\Psi}(x)\tilde{\Psi}(x_j)|0 >< 0|T\tilde{\Psi}(x_1)\cdots\cancel{\tilde{\Psi}(x_j)}\cdots\tilde{\Psi}(x_m)|0 > .
\end{aligned} \tag{2.3.283}$$

Hence, we obtain our desired result

$$< 0|T\tilde{\Psi}(x)\tilde{\Psi}(x_1)\cdots\tilde{\Psi}(x_m)|0 >$$

$$= \sum_{j=1}^{m}(-1)^{j-1} < 0|T\tilde{\Psi}(x)\tilde{\Psi}(x_j)|0 >< 0|T\tilde{\Psi}(x_1)\cdots\cancel{\tilde{\Psi}(x_j)}\cdots\tilde{\Psi}(x_m)|0 > . \tag{2.3.284}$$

Applying this lemma to the right hand side repeatedly we finally obtain Wick's Theorem for the Green functions, equation (2.3.268).

As in the scalar case, these product formulae for $W^{(n,\bar{n})}$ and $G^{(n,\bar{n})}$ are special cases of the general reduction of ordinary and time ordered products of free Fermi fields in terms of Wick or Normal products of the free Fermi fields. The reduction identity again being known as Wick's Theorem. Since the proof of Wick's Theorem given in section 2.2 relied only on the linearity of the normal products and the decomposition of the free fields into a sum of positive and negative frequency annihilation and creation operators we can apply the results immediately to our "bosonized" fields

$$\Phi(x) = \bar{\eta}_a(x)\Psi_a(x)$$

$$\Phi^\dagger(x) = \bar{\Psi}_a(x)\eta_a(x). \tag{2.3.285}$$

Factoring the c-number Grassmann variables η and $\bar{\eta}$ from the products we obtain Wick's Theorem for free Fermi fields.

Wick's Theorem:

$$1) \quad \Psi(x_1)\cdots\Psi(x_n)\bar{\Psi}(y_1)\cdots\bar{\Psi}(y_{\bar{n}}) = N\left[\Psi(x_1)\cdots\Psi(x_n)\bar{\Psi}(y_1)\cdots\bar{\Psi}(y_{\bar{n}})\right]$$

$$+ \sum_{1\ pairing} (-1)^{P(i,j)} < 0|\Psi(x_i)\bar{\Psi}(y_j)|0 > N\left[\frac{\Psi(x_1)\cdots\bar{\Psi}(y_{\bar{n}})}{\Psi(x_i)\bar{\Psi}(y_j)}\right]$$

$$+ \sum_{\substack{2\ pairings \\ i_1<i_2}} (-1)^{P(i_1,i_2,j_1,j_2)} < 0|\Psi(x_{i_1})\bar{\Psi}(y_{j_1})|0 >< 0|\Psi(x_{i_2})\bar{\Psi}(y_{j_2})|0 >$$

$$\times N\left[\frac{\Psi(x_1)\cdots\bar{\Psi}(y_{\bar{n}})}{\Psi(x_{i_1})\Psi(x_{i_2})\bar{\Psi}(y_{j_1})\bar{\Psi}(y_{j_2})}\right]$$

$$+ \sum_{3\ pairings} \cdots$$

$$+ \sum_{\substack{P \\ n=\bar{n}}} (-1)^P < 0|\Psi(x_1)\bar{\Psi}(y_{i_1})|0 > \cdots < 0|\Psi(x_n)\bar{\Psi}(y_{i_n})|0 > \tag{2.3.286}$$

$$2) \quad T\Psi(x_1)\cdots\Psi(x_n)\bar{\Psi}(y_1)\cdots\bar{\Psi}(y_{\bar{n}}) = N\left[\Psi(x_1)\cdots\Psi(x_n)\bar{\Psi}(y_1)\cdots\bar{\Psi}(y_{\bar{n}})\right]$$

$$+ \sum_{1\ contraction} (-1)^{P(i,j)} < 0|T\Psi(x_i)\bar{\Psi}(y_j)|0 > N\left[\frac{\Psi(x_1)\cdots\bar{\Psi}(y_{\bar{n}})}{\Psi(x_i)\bar{\Psi}(y_j)}\right]$$

$$+ \sum_{\substack{2\ contractions \\ i_1<i_2}} (-1)^{P(i_1,i_2,j_1,j_2)} < 0|T\Psi(x_{i_1})\bar{\Psi}(y_{j_1})|0 >< 0|T\Psi(x_{i_2})\bar{\Psi}(y_{j_2})|0 >$$

$$\times N\left[\frac{\Psi(x_1)\cdots\bar{\Psi}(y_{\bar{n}})}{\Psi(x_{i_1})\Psi(x_{i_2})\bar{\Psi}(y_{j_1})\bar{\Psi}(y_{j_2})}\right]$$

$$+ \sum_{3\ contractions} \cdots$$

$$+\sum_{\substack{P\\ n=\bar{n}}}(-1)^P <0|T\Psi(x_1)\bar{\Psi}(y_{i_1})|0>\cdots<0|T\Psi(x_n)\bar{\Psi}(y_{i_n})|0>. \tag{2.3.287}$$

Wick's Theorem also applies to the chronological product of normal products as discussed following equation (2.2.217).

Finally, let's check the Green function property of the fermion time ordered functions. Since

$$S_F(x)=(i\partial\!\!\!/_x+m)\Delta_F(x) \tag{2.3.288}$$

and

$$(\partial^2+m^2)\Delta_F(x)=-i\delta^4(x) \tag{2.3.289}$$

we find

$$(i\partial\!\!\!/_x-m)S_F(x-y)=i\delta^4(x-y). \tag{2.3.290}$$

That is

$$(i\partial\!\!\!/_x-m)<0|T\Psi(x)\overline{\Psi}(y)|0>=i\delta^4(x-y). \tag{2.3.291}$$

As usual this follows directly from the ETAR, using

$$T\Psi(x)\overline{\Psi}(y)=\theta(x^0-y^0)\Psi(x)\overline{\Psi}(y)-\theta(y^0-x^0)\overline{\Psi}(y)\Psi(x) \tag{2.3.292}$$

we have

$$\partial_x^0 T\Psi(x)\overline{\Psi}(y)=\theta(x^0-y^0)\dot{\Psi}(x)\overline{\Psi}(y)$$
$$-\theta(y^0-x^0)\overline{\Psi}(y)\dot{\Psi}(x)+\delta(x^0-y^0)\{\Psi(x),\overline{\Psi}(y)\}. \tag{2.3.293}$$

But from the ETAR

$$\delta(x^0-y^0)\{\Psi(x),\overline{\Psi}(y)\}=\gamma^0\delta^4(x-y). \tag{2.3.294}$$

Thus,

$$\partial_x^0 T\Psi(x)\overline{\Psi}(y)=T\dot{\Psi}(x)\overline{\Psi}(y)+\gamma^0\delta^4(x-y) \tag{2.3.295}$$

and

$$(i\partial\!\!\!/_x-m)T\Psi(x)\overline{\Psi}(y)=T(i\partial\!\!\!/_x-m)\Psi(x)\overline{\Psi}(y)+i\delta^4(x-y). \tag{2.3.296}$$

Since $(i\partial\!\!\!/_x-m)\Psi(x)=0$ by the Dirac equation we have

$$(i\partial\!\!\!/_x-m)T\Psi(x)\overline{\Psi}(y)=+i\delta^4(x-y). \tag{2.3.297}$$

Taking the vacuum expectation value and using $<0|0>=1$ we obtain

$$(i\not{\partial}_x - m) < 0|T\Psi(x)\overline{\Psi}(y)|0 >= +i\delta^4(x-y). \qquad (2.3.298)$$

Applying $(i\not{\partial}_y + m)$ on the right we also have

$$< 0|T\Psi(x)\overline{\Psi}(y)|0 > (i \overleftarrow{\not{\partial}}_y + m) = -i\delta^4(x-y). \qquad (2.3.299)$$

Finally, applying the Dirac and adjoint Dirac equations to the lemma for Wick's Theorem for Green functions, equation (2.3.268), we have

$$(i\not{\partial}_x - m) < 0|T\Psi(x)\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 >$$

$$= \sum_{j=1}^{\overline{n}} (-1)^{n+j-1} i\delta^4(x-y_j) < 0|T\Psi(x_1)\cdots \cancel{\overline{\Psi}(y_j)} \cdots \overline{\Psi}(y_{\overline{n}})|0 > \qquad (2.3.300)$$

and

$$< 0|T\overline{\Psi}(y)\Psi(x_1)\cdots\Psi(x_n)\overline{\Psi}(y_1)\cdots\overline{\Psi}(y_{\overline{n}})|0 > (i \overleftarrow{\not{\partial}}_y + m)$$

$$= \sum_{j=1}^{n} (-1)^{j-1} i\delta^4(y_j - x) < 0|T\Psi(x_1)\cdots \cancel{\Psi(x_j)} \cdots \overline{\Psi}(y_{\overline{n}})|0 > . \qquad (2.3.301)$$

We are now ready to consider massless particles with spin 1, or more specifically, the photon.