§2.2 THE SPIN ZERO SCALAR FIELD

We now turn to applying our quantization procedure to various free fields. As we will see all goes smoothly for spin zero fields but we will require some change in the CCR when we apply the rules to spin 1/2 fields; we will need CAR. Further modification will be required when we try to quantize spin 1 fields in a manifestly Lorentz covariant manner due to gauge invariance.

First let's consider the case of a free scalar, spin zero, mass m, Hermitian field. As we have seen earlier it is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 \tag{2.2.1}$$

with the Euler-Lagrange equation $(\partial^2 + m^2)\Phi(x) = 0$. The canonical quantization rules state

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi} \text{ and } \Pi^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi} = \partial^\mu\Phi. \tag{2.2.2}$$

So that the Hamiltonian density becomes

$$\mathcal{H} = \Pi\dot{\Phi} - \mathcal{L} = \dot{\Phi}\dot{\Phi} - \mathcal{L}$$

$$\mathcal{H} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi + \frac{1}{2}m^2\Phi^2, \tag{2.2.3}$$

The equal time commutation relations are

$$[\Pi(\vec{x},t),\Phi(\vec{y},t)] = \left[\dot{\Phi}(\vec{x},t),\Phi(\vec{y},t)\right] = -i\delta^3(\vec{x}-\vec{y})$$

$$[\Phi(\vec{x},t),\Phi(\vec{y},t)] = 0 = \left[\dot{\Phi}(\vec{x},t),\dot{\Phi}(\vec{y},t)\right] \tag{2.2.4}$$

The Euler-Lagrange equations are then equivalent to the Heisenberg equations of motion

$$[H,\Phi(x)] = -i\dot{\Phi}(x)$$

$$\left[H,\dot{\Phi}(x)\right] = -i\ddot{\Phi}(x). \tag{2.2.5}$$

We can construct the energy-momentum tensor and angular momentum tensor as defined earlier according to our classical Noether's theorem

$$T^{\mu\nu} \equiv \left(\partial^\nu\Phi\frac{\partial}{\partial\partial_\mu\Phi}\right)\mathcal{L} - g^{\mu\nu}\mathcal{L}$$

$$T^{\mu\nu} = \frac{1}{2}\left(\partial^\mu \Phi \partial^\nu \Phi + \partial^\nu \Phi \partial^\mu \Phi\right) - g^{\mu\nu}\mathcal{L}, \tag{2.2.6}$$

where we have written $T^{\mu\nu}$ in symmetric form. We can check explicitly that $T^{\mu\nu}$ is conserved by using the Euler-Lagrange equations of Φ

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \frac{1}{2}\partial^2\Phi\partial^\nu\Phi + \frac{1}{2}\partial^\mu\Phi\partial_\mu\partial^\nu\Phi \\ &+ \frac{1}{2}\partial^\mu\partial^\nu\Phi\partial_\mu\Phi + \frac{1}{2}\partial^\nu\Phi\partial^2\Phi - \partial^\nu\mathcal{L} \\ &= -\frac{m^2}{2}\Phi\partial^\nu\Phi - \frac{m^2}{2}\partial^\nu\Phi\Phi + \frac{1}{2}\partial^\nu\left(\partial_\mu\Phi\partial^\mu\Phi\right) - \partial^\nu\mathcal{L} \\ &= \partial^\nu\left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2\right) - \partial^\nu\mathcal{L} = 0 \end{aligned} . \tag{2.2.7}$$

Even though $T^{\mu\nu} = T^{\nu\mu}$ there are still Belinfante improvement terms that one can add

$$G^{\rho\mu\nu} = b\left(g^{\mu\nu}\partial^\rho - g^{\rho\nu}\partial^\mu\right)\Phi^2. \tag{2.2.8}$$

So

$$G^{\rho\mu\nu} = -G^{\mu\rho\nu} \tag{2.2.9}$$

and

$$\partial_\rho G^{\rho\mu\nu} = b\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\Phi^2 \tag{2.2.10}$$

Now the improved energy momentum tensor is given by

$$\Theta_I^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\rho\mu\nu}. \tag{2.2.11}$$

Usually b is chosen so that $\Theta_{I\lambda}^{\lambda} = m^2\Phi^2$ but this is useful only when one considers conformal symmetries in addition to Poincare' symmetries. The additional current for scale transformations is defined by

$$\begin{aligned} D^\mu &= x_\nu \Theta_I^{\mu\nu} \\ \partial_\mu D^\mu &= \Theta_{I\lambda}^{\lambda}. \end{aligned} \tag{2.2.12}$$

Now

$$\begin{aligned} \Theta_{I\lambda}^{\lambda} &= T_\lambda^\lambda + 3b\partial^2\Phi^2 \\ &= \partial_\lambda\Phi\partial^\lambda\Phi - 2\partial_\lambda\Phi\partial^\lambda\Phi + 2m^2\Phi^2 + 3b\partial^2\Phi^2 \\ &= 2m^2\Phi^2 - \partial_\lambda\left(\Phi\partial^\lambda\Phi\right) + \Phi\partial^2\Phi + 3b\partial^2\Phi^2 \\ &= -\frac{1}{2}\partial^2\Phi^2 + 3b\partial^2\Phi^2 + m^2\Phi^2 + \Phi\left(\partial^2 + m^2\right)\Phi - \frac{1}{2}\partial_\lambda[\partial^\lambda\Phi, \Phi] \\ &= (3b - \frac{1}{2})\partial^2\Phi^2 + m^2\Phi^2 \end{aligned} . \tag{2.2.13}$$

Hence we choose

$$b = \frac{1}{6}. \tag{2.2.14}$$

So we have

$$\begin{aligned}\Theta_I^{\mu\nu} &= T^{\mu\nu} + \frac{1}{6}\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\Phi^2 \\ \partial_\mu \Theta_I^{\mu\nu} &= 0 \\ \Theta_I^{\mu\nu} &= \Theta_I^{\nu\mu} \\ \Theta_{I\lambda}^{\lambda} &= m^2\Phi^2.\end{aligned} \tag{2.2.15}$$

The energy momentum operator is given by either $T^{\mu\nu}$ or $\Theta^{\mu\nu}$ as

$$\begin{aligned}\mathcal{P}^\mu &= \int d^3x T^{0\mu} \\ &= \int d^3x \left[\frac{1}{2}\left(\dot{\Phi}\partial^\nu\Phi + \partial^\nu\Phi\dot{\Phi}\right) - g^{0\mu}\mathcal{L}\right]\end{aligned}$$

$$\mathcal{P}^\mu = \int d^3x \left[\mathcal{H}, \frac{1}{2}\left(\Pi\partial^i\Phi + \partial^i\Phi\Pi\right)\right]. \tag{2.2.16}$$

Now using the ETCR equations (2.2.4) we find

$$\begin{aligned}\left[T^{0\nu}(\vec{y},t), \Phi(\vec{x},t)\right] &= \left[\frac{1}{2}\left(\dot{\Phi}(\vec{y},t)\partial^\nu\Phi(\vec{y},t) + \partial^\nu\Phi(\vec{y},t)\dot{\Phi}(\vec{y},t)\right)\right. \\ &\quad \left. -g^{0\nu}\frac{1}{2}\dot{\Phi}(\vec{y},t)^2, \Phi(\vec{x},t)\right] \\ &= -i\dot{\Phi}(x)\delta^3(\vec{x}-\vec{y}), \qquad \nu = 0 \\ &= -i\partial^i\Phi(x)\delta^3(\vec{x}-\vec{y}), \qquad \nu = 1,2,3.\end{aligned}$$

that is

$$\delta(x^0 - y^0)\left[T^{0\nu}(y), \Phi(x)\right] = -i\partial^\nu\Phi(x)\delta^4(x-y). \tag{2.2.17}$$

So

$$\left[\mathcal{P}^\mu, \Phi(x)\right] = -i\partial^\mu\Phi(x) \tag{2.2.18}$$

as desired. Further the angular momentum tensor is

$$M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$$

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{2.2.19}$$

since

$$T^{\mu\nu} = T^{\nu\mu}.$$

The angular momentum operator is then

$$\begin{aligned} \mathcal{M}^{\mu\nu} &= \int d^3x M^{0\mu\nu} \\ &= \int d^3x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu} \right) \end{aligned} . \tag{2.2.19$'$}$$

So we have using (2.2.17)

$$[\mathcal{M}^{\mu\nu}, \Phi(x)] = -i \left(x^\mu \partial^\nu - x^\nu \partial^\mu \right) \Phi(x). \tag{2.2.20}$$

Given these quantum field theoretic properties for the field operators we would like to further interpret the physical system being described by them. To do this let's go over to momentum space and expand the field operator

$$\Phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\Phi}(k). \tag{2.2.21}$$

$\Phi(x)$ obeys the Klein-Gordon equation $(\partial^2 + m^2)\Phi(x) = 0$ thus we should be expanding $\Phi(x)$ in terms of solutions to the K.G. equation

$$(\partial^2 - m^2)\Phi(x) = 0 = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (-k^2 + m^2) \tilde{\Phi}(k) \tag{2.2.22}$$

this implies

$$\tilde{\Phi}(k) = (2\pi)\delta(k^2 - m^2) a(\vec{k}, k^0) \tag{2.2.23}$$

with $a(\vec{k}, k^0)$ an operator coefficient, and hence

$$\Phi(x) = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) a(\vec{k}, k^0) e^{-ikx}. \tag{2.2.24}$$

As before we use

$$\delta(k^2 - m^2) = \frac{1}{2\omega_k} \left[\theta(k^0)\delta(k^0 - \omega_k) + \theta(-k^0)\delta(k^0 + \omega_k) \right] \tag{2.2.25}$$

with

$$\omega_k = +\sqrt{\vec{k}^2 + m^2} \tag{2.2.26}$$

so that

$$\begin{aligned}
\Phi(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}, +\omega_k) e^{-i\omega_k x^0 + i\vec{k}\cdot\vec{x}} + a(\vec{k}, -\omega_k) e^{+i\omega_k x^0 + i\vec{k}\cdot\vec{x}} \right] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}, +\omega_k) e^{-ikx} + a(-\vec{k}, -\omega_k) e^{+ikx} \right]
\end{aligned} , \tag{2.2.27}$$

where the second equality was obtained by letting $\vec{k} \to -\vec{k}$ in the second integral. But the field is Hermitian, $\Phi(x) = \Phi^\dagger(x)$, implying $a^\dagger(\vec{k}, +\omega_k) = a(-\vec{k}, -\omega_k)$, thus defining the operators

$$\begin{aligned}
a(\vec{k}) &\equiv a(\vec{k}, +\omega_k) \\
a^\dagger(\vec{k}) &= a^\dagger(\vec{k}, +\omega_k) = a(-\vec{k}, -\omega_k)
\end{aligned} \tag{2.2.28}$$

we have the hermitian form for the Fourier transform

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right] \tag{2.2.29}$$

where it is understood that $k^2 = m^2$ here, that is $k^0 = \omega_k$. We have expanded $\Phi(x)$ in terms of the positive frequency $u_{\vec{k}}(x) = e^{-ikx}$ and negative frequency solutions $v_{\vec{k}}(x) = u^*_{\vec{k}}(x) = e^{ikx}$ of the Klein-Gordon equation

$$\begin{aligned}
(\partial^2 + m^2) u_{\vec{k}}(x) &= 0 \\
(\partial^2 + m^2) v_{\vec{k}}(x) &= 0.
\end{aligned} \tag{2.2.30}$$

These have the orthogonality property

$$\begin{aligned}
(u_{\vec{k}}, u_{\vec{k}'}) &= i \int d^3x u^*_{\vec{k}}(x) \overleftrightarrow{\partial_0} u_{\vec{l}}(x) \\
&= \int d^3x (\omega_l + \omega_k) e^{-i(\omega_l - \omega_k)x^0} e^{i(\vec{l} - \vec{k})\cdot\vec{x}} \\
&= (2\pi)^3 2\omega_k
\end{aligned} \tag{2.2.31}$$

where

$$f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f) g \tag{2.2.32}$$

and the completeness relation

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} u_{\vec{k}}(x) u^*_{\vec{k}}(y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} = i \bigtriangleup^+ (x-y) \tag{2.2.33}$$

which will be discussed in further detail later. Thus

$$\Phi(x) = \Phi^+(x) + \Phi^-(x) \tag{2.2.34}$$

where

$$\Phi^+(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} a(\vec{k}) e^{-ikx} \tag{2.2.35}$$

$$\Phi^-(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger(\vec{k}) e^{ikx} \tag{2.2.36}$$

are the positive and negative frequency components respectively. Inverting the Fourier transform we find

$$a(\vec{k}) = i \int d^3x e^{ikx} \overleftrightarrow{\partial_0} \Phi(x) \tag{2.2.37}$$

$$a^\dagger(\vec{k}) = i \int d^3x \Phi(x) \overleftrightarrow{\partial_0} e^{-ikx}. \tag{2.2.38}$$

So writing out the time derivatives, we have

$$a(\vec{k}) = i \int d^3x e^{ikx} \left[\dot{\Phi}(x) - i\omega_k \Phi(x)\right] \tag{2.2.39}$$

$$a^\dagger(\vec{k}) = i \int d^3x e^{-ikx} \left[-i\omega_k \Phi(x) - \dot{\Phi}(x)\right]. \tag{2.2.40}$$

Note, since $k^2 = m^2$ the above integrals are indeed independent of time

$$\begin{aligned}
\dot{a}(\vec{k}) &= i \int d^3x e^{ikx} \left[i\omega_k \dot{\Phi} + \omega_k^2 \Phi + \ddot{\Phi} - i\omega_k \dot{\Phi}\right] \\
&= i \int d^3x e^{ikx} \left[\vec{k}^2 \Phi + m^2 \Phi + \ddot{\Phi}\right] \\
&= i \int d^3x e^{ikx} \left[- \bigtriangledown^2 \Phi + \ddot{\Phi} + m^2 \Phi\right] \\
&= i \int d^3x e^{ikx} \left(\partial^2 + m^2\right) \Phi = 0
\end{aligned} \quad . \tag{2.2.41}$$

We can evaluate the commutators for $a(\vec{k})$ and $a^\dagger(\vec{k})$ by using the equal time commutation relations equation (2.2.4) for $\Phi(x)$ and $\dot{\Phi}(x)$. We secure

$$\left[a(\vec{k}), a^\dagger(\vec{l}\,)\right] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}\,)$$

$$\left[a(\vec{k}), a(\vec{l}\,)\right] = 0 = \left[a^\dagger(\vec{k}), a^\dagger(\vec{l}\,)\right]. \tag{2.2.42}$$

We obtain the "harmonic oscillator" creation operator $a^\dagger(\vec{k})$ and annihilation operator $a(\vec{k})$ commutation relations.

We can also apply the energy-momentum commutation relation

$$[\mathcal{P}^\mu, \Phi(x)] = -i\partial^\mu \Phi(x) \tag{2.2.43}$$

to $a(\vec{k})$and $a^\dagger(\vec{k})$ to find

$$\left[\mathcal{P}^\mu, a(\vec{k})\right] = -k^\mu a(\vec{k})$$

$$\left[\mathcal{P}^\mu, a^\dagger(\vec{k})\right] = +k^\mu a^\dagger(\vec{k}). \tag{2.2.44}$$

Thus, if $|p>$ is an eigenstate of energy and momentum

$$\mathcal{P}^\mu |p>= p^\mu |p>, \tag{2.2.45}$$

then

$$\begin{aligned} \mathcal{P}^\mu a(\vec{k})|p> &= \left[\mathcal{P}^\mu, a(\vec{k})\right]|p> + a(\vec{k})\mathcal{P}^\mu |p> \\ &= (-k^\mu + p^\mu) a(\vec{k})|p> \end{aligned}. \tag{2.2.46}$$

Hence

$$\mathcal{P}^\mu \left[a(\vec{k})|p>\right] = (p^\mu - k^\mu)\left[a(\vec{k})|p>\right] \tag{2.2.47}$$

and similarly

$$\mathcal{P}^\mu \left[a^\dagger(\vec{k})|p>\right] = (p^\mu + k^\mu)\left[a^\dagger(\vec{k})|p>\right], \tag{2.2.48}$$

thus $a(\vec{k})|p>$ is an eigenstate of $\mathcal{P}^\mu$ with energy-momentum $(p-k)^\mu$ and $a^\dagger(\vec{k})|p>$ is an eigenstate of $\mathcal{P}^\mu$ with energy-momentum $(p+k)^\mu$. Now we could keep acting on $|p>$ with annihilation operators, then

$$\mathcal{P}^\mu a(\vec{k}_1)\cdots a(\vec{k}_N)|p>= \left(p - \sum_{i=1}^N k_i\right)^\mu a(\vec{k}_1)\cdots a(\vec{k}_N)|p> . \tag{2.2.49}$$

Eventually $\mathcal{P}^0 = H$ will go negative!! However,

$$H = \int d^3x \frac{1}{2}\left[\dot{\Phi}^2 + \vec{\bigtriangledown}\Phi \cdot \vec{\bigtriangledown}\Phi + m^2\Phi^2\right] \tag{2.2.50}$$

is a positive operator. This implies that there is a lowest energy state so that, for some number N, we must have

$$a(\vec{k})\left[a(\vec{k}_1)\cdots a(\vec{k}_N)|p>\right] = 0. \tag{2.2.51}$$

Denoting this lowest energy state, called the ground state or vacuum, by $|0>$ and its energy by E_0 and noting that its momentum is $\vec{0}$, we have for equation (2.2.51)

$$a(\vec{k})|0>= 0 \tag{2.2.52}$$

for all $\vec{k}$. $a^\dagger(\vec{k})|0>$ then has the energy and momentum of a single particle relative to that of $|0>$,

$$\begin{aligned} H\left[a^\dagger(\vec{k})|0>\right] &= (E_0 + \omega_k)\left[a^\dagger(\vec{k})|0>\right] \\ \vec{P}\left[a^\dagger(\vec{k})|0>\right] &= \vec{k}\left[a^\dagger(\vec{k})|0>\right] \end{aligned} \tag{2.2.53}$$

with $\omega_k^2 = \vec{k}^2 + m^2$, the relativistic relation. Thus we interpret the state $|\vec{k}>\equiv \left[a^\dagger(\vec{k})|0>\right]$ as a single particle state with momentum $\vec{k}$ and mass m, hence energy ω_k. Continuing,

$$|\vec{k}_1, \vec{k}_2>= a^\dagger(k_1)a^\dagger(k_2)|0> \tag{2.2.54}$$

has energy

$$H|\vec{k}_1, \vec{k}_2>= (E_0 + \omega_{k_1} + \omega_{k_2})|\vec{k}_1, \vec{k}_2> \tag{2.2.55}$$

and momentum

$$\vec{P}|\vec{k}_1, \vec{k}_2>= (\vec{k}_1 + \vec{k}_2)|\vec{k}_1, \vec{k}_2> \tag{2.2.56}$$

and represents states with two noninteracting particles each with momentum $\vec{k}_i$ and energy ω_{k_i}. Continuing further, the N-particle states are given by

$$|\vec{k}_1, \cdots, \vec{k}_N>= a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_N)|0> \tag{2.2.57}$$

with

$$H|\vec{k}_1, \cdots, \vec{k}_N>= \left(E_0 + \sum_{i=1}^N \omega_{k_i}\right)|\vec{k}_1, \cdots, \vec{k}_N> \tag{2.2.58}$$

$$\vec{P}|\vec{k}_1, \cdots, \vec{k}_N >= \left(\sum_{i=1}^{N} \vec{k}_i \right) |\vec{k}_1, \cdots, \vec{k}_N > . \tag{2.2.59}$$

Thus, we can define the number density operator as

$$\mathcal{N}(\vec{k}) = \frac{1}{(2\pi)^3 2\omega_k} a^\dagger(\vec{k}) a(\vec{k}) \tag{2.2.60}$$

with

$$\left[\mathcal{N}(\vec{k}), a(\vec{k}') \right] = -\delta^3(\vec{k} - \vec{k}') a(\vec{k}) \tag{2.2.61}$$

$$\left[\mathcal{N}(\vec{k}), a^\dagger(\vec{k}') \right] = +\delta(\vec{k} - \vec{k}') a^\dagger(\vec{k}). \tag{2.2.62}$$

So operating on the N-particle state gives

$$\mathcal{N}(\vec{k})|\vec{k}_1, \cdots, \vec{k}_N >=$$

$$\left[\delta^3(\vec{k} - \vec{k}_1) + \cdots + \delta^3(\vec{k} - \vec{k}_N) \right] |\vec{k}_1, \cdots, \vec{k}_N >, \tag{2.2.63}$$

$\mathcal{N}(\vec{k})$ equals the number of particles per unit volume in momentum space with momentum $\vec{k}$. Therefore,

$$\mathcal{N}(\vec{k}) d^3k \tag{2.2.64}$$

is equal to the number of particles with momentum differentially close to $\vec{k}$. Hence, the total number operator is given by

$$N_\infty = \int d^3k \mathcal{N}(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger(\vec{k}) a(\vec{k}) \tag{2.2.65}$$

and

$$N_\infty |\vec{k}_1, \cdots, \vec{k}_N >= N|\vec{k}_1, \cdots, \vec{k}_N > . \tag{2.2.66}$$

So $a^\dagger(\vec{k})$ increases the number of particles by creating a particle with momentum $\vec{k}$ and energy ω_k while $a(\vec{k})$ decreases the number of particles by annihilating a particle with momentum $\vec{k}$ and energy ω_k.

For further insight into the vacuum energy E_0 let's Fourier transform our expression for the energy momentum operator $\mathcal{P}^\mu$

$$\mathcal{P}^0 = H = \int d^3x \left[\frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi + \frac{1}{2} m^2 \Phi^2 \right]$$

$$\vec{\mathcal{P}} = -\int d^3x \frac{1}{2}\left[\dot{\Phi}\vec{\bigtriangledown}\Phi + \vec{\bigtriangledown}\Phi\dot{\Phi}\right]. \tag{2.2.67}$$

First the momentum operator

$$\begin{aligned}
\vec{\mathcal{P}} &= -\int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3l}{(2\pi)^3 2\omega_l}\frac{1}{2}\left[-i\omega_k a(\vec{k})e^{-ikx} + i\omega_k a^\dagger(\vec{k})e^{+ikx}\right] \\
&\times \left[i\vec{l}a(\vec{l}\,)e^{-ilx} - i\vec{l}a^\dagger(\vec{l}\,)e^{ilx}\right] + \text{interchange} \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3l}{(2\pi)^3 2\omega_l}\int d^3x \frac{1}{2}\left[-\omega_k \vec{l}a(\vec{k})a(\vec{l}\,)e^{-i(k+l\ )}x\right. \\
&+ \omega_k \vec{l}a(\vec{k})a^\dagger(\vec{l}\,)e^{-i(k-l\ )x} + \omega_k \vec{l}a^\dagger(\vec{k})a(\vec{l}\,)e^{i(k-l\ )x} \\
&\left. - \omega_k \vec{l}a^\dagger(\vec{k})a^\dagger(\vec{l}\,)e^{i(k+l\ )x}\right] + \text{interchange} \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3l}{(2\pi)^3 2\omega_l}\frac{1}{2}\left[(2\pi)^3\delta^3(\vec{k}+\vec{l}\,)e^{-i2\omega_k x^0}(+\omega_k\vec{k})a(\vec{k})a(-\vec{k})\right. \\
&+ (2\pi)^3\delta^3(\vec{k}+\vec{l}\,)e^{i2\omega_k x^0}(\omega_k\vec{k})a^\dagger(\vec{k})a^\dagger(-\vec{k}) \\
&\left. + (2\pi)^3\delta^3(\vec{k}-\vec{l}\,)\left[\omega_k\vec{k}a(\vec{k})a^\dagger(+\vec{k}) + \omega_k\vec{k}a^\dagger(\vec{k})a(\vec{k})\right]\right] + \text{interchange}
\end{aligned} \quad . \tag{2.2.68}$$

Thus we find

$$\begin{aligned}
\vec{\mathcal{P}} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{2}\vec{k}\left[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right] \\
&+ \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{4}\vec{k}\left[e^{-i2\omega_k x^0}\left(a(\vec{k})a(-\vec{k}) + a(-\vec{k})a(\vec{k})\right)\right. \\
&\left. + e^{i2\omega_k x^0}\left(a^\dagger(\vec{k})a^\dagger(-\vec{k}) + a^\dagger(-\vec{k})a^\dagger(\vec{k})\right)\right]
\end{aligned} \quad . \tag{2.2.69}$$

The integrand of the second integral on the right hand side is odd in $\vec{k}$ and hence vanishes, yielding

$$\vec{\mathcal{P}} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{2}\vec{k}\left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right]. \tag{2.2.70}$$

Similarly, in this integral we can rearrange $a(\vec{k})a^\dagger(\vec{k})$ since the commutator is even in $\vec{k}$

$$\begin{aligned}
\frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a(\vec{k})a^\dagger(\vec{k}) &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a^\dagger(\vec{k})a(\vec{k}) \\
&+ \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}\left[a(\vec{k}), a^\dagger(\vec{k})\right]
\end{aligned}$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a^\dagger(\vec{k})a(\vec{k}) + \int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k})$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a^\dagger(\vec{k})a(\vec{k}). \tag{2.2.71}$$

Consequently, we find the usual expression for the number operator of particles with momentum $\vec{k}$ times the momentum $\vec{k}$

$$\vec{\mathcal{P}} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a^\dagger(\vec{k})a(\vec{k}). \tag{2.2.72}$$

Thus we find $\vec{\mathcal{P}}|0>=0$ since $a(\vec{k})|0>=0$, as previously stated.

Next consider the Hamiltonian, after an integration by parts with vanishing surface term,

$$H = \int d^3x \left[\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\Phi(\bigtriangledown^2\Phi - m^2\Phi)\right] \tag{2.2.73}$$

but $\ddot{\Phi} - \bigtriangledown^2\Phi + m^2\Phi = 0$, so

$$H = \int d^3x \left[\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\Phi\ddot{\Phi}\right]. \tag{2.2.74}$$

Expanding the fields, we have

$$\begin{aligned}
H = &\frac{1}{2}\int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3l}{(2\pi)^3 2\omega_k}\big[-\omega_k\omega_l a(\vec{k})a(\vec{l}\,)e^{-i(k+l\ )x} \\
&+ \omega_k\omega_l a(\vec{k})a^\dagger(\vec{l}\,)e^{-i(k-l\ )x} + \omega_k\omega_l a^\dagger(\vec{k})a(\vec{l}\,)e^{i(k-l\ )x} - \omega_k\omega_l a^\dagger(\vec{k})a^\dagger(\vec{l}\,)e^{i(k+l\ )x} \\
&- \left(a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}\right)\left(-\omega_l^2 a(\vec{l}\,)e^{-ilx} - \omega_l^2 a^\dagger(\vec{l}\,)e^{ilx}\right)\big] \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3l}{(2\pi)^3 2\omega_l}(2\pi)^3\Big\{\delta^3(\vec{k}+\vec{l}\,)(-\omega_k^2)\Big(e^{-i2\omega_k x^0}a(\vec{k})a(-\vec{k}) \\
&+ e^{i2\omega_k x^0}a^\dagger(\vec{k})a^\dagger(-\vec{k})\Big) \\
&+ \omega_k^2\delta^3(\vec{k}-\vec{l}\,)\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right) \\
&+ \omega_k^2\Big[\delta^3(\vec{k}+\vec{l}\,)\left(a(\vec{k})a(-\vec{k})e^{-2i\omega_k x^0} + a^\dagger(\vec{k})a^\dagger(-\vec{k})e^{+i2\omega_k x^0}\right) \\
&+ \delta(\vec{k}-\vec{l}\,)\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(+\vec{k})\right)\Big]\Big\} \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{2}\omega_k\left[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k a^\dagger(\vec{k})a(\vec{k}) + \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{2}\omega_k(2\pi)^3 2\omega_k\delta^3(0)
\end{aligned} \tag{2.2.75}$$

That is we finally find

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(\vec{k}) a(\vec{k}) + \int d^3k \frac{1}{2} \omega_k \delta^3(0). \tag{2.2.76}$$

Thus when acting on the lowest energy state $|0>$ defined by $a(\vec{k})|0>=0$, we have

$$H|0>= \left(\int d^3k \frac{1}{2} \omega_k \delta^3(0) \right) |0>= E_0|0>, \tag{2.2.77}$$

hence

$$E_0 = \int d^3k \frac{1}{2} \omega_k \delta^3(0) = \infty = \text{constant}. \tag{2.2.78}$$

One way to look at this is that the vacuum contains an infinite zero point energy from the infinite number of harmonic oscillators in the Hamiltonian and that all measurements are made relative to this vacuum state i.e. only differences in energy are measured so that the infinite constant does not matter. Alternatively, the Hamiltonian is defined only up to a constant, the zero of energy being arbitrary. Thus, we can define another Hamiltonian as $\hat{H} \equiv H - E_0$, the lowest energy state, defined by $a(\vec{k})|0>=0$, having zero energy $\hat{H}|0>=0$.

Physically it is reasonable to define the no particle state, the vacuum, $|0>$, which is the lowest energy state as having zero as its energy and momentum

$$a(\vec{k})|0>= 0 \tag{2.2.79}$$

and

$$\hat{H}|0>= 0 \tag{2.2.80}$$

$$\vec{P}|0>= 0 \tag{2.2.81}$$

so that the vacuum is space-time translation invariant $U(a,1)|0>=|0>$. Mathematically what we are finding is that the formal manipulations of writing a Lagrangian and a Hamiltonian for quantum fields in analogy to classical fields or even quantum mechanical systems with a finite number of degrees of freedom must be modified. The problem arises from the products of field operators at the same space-time point as in the Lagrangian or the Hamiltonian (this follows from the equal time commutation relations

$$[\dot{\Phi}(\vec{x},t), \Phi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y}) \sim \infty$$

for $\vec{x} \to \vec{y}$). For noninteracting fields we can easily eliminate these difficulties by defining the products of the fields more carefully. In the interacting case it is more difficult but possible and comprises the subject of renormalization theory.

A well defined product of free fields is given by the normal ordering or Wick ordering of the fields. For example, the normal product denoted $N[\Phi^2(x)]$ or $:\Phi^2(x):$ (or $N[\Phi(x)\Phi(y)]$ for that matter) is defined so that all annihilation operators in the product are to the right of all the creation operators, hence,

$$N[\Phi^2(x)] = \Phi^+(x)\Phi^+(x) + \Phi^-(x)\Phi^-(x) + 2\Phi^-(x)\Phi^+(x) \tag{2.2.81}$$

so that the vacuum expectation value of a normal product vanishes

$$< 0|N[\Phi^2(x)]|0 >= 0. \tag{2.2.82}$$

On the other hand

$$\Phi^2(x) = \Phi^+(x)\Phi^+(x) + \Phi^-(x)\Phi^-(x) + \Phi^-(x)\Phi^+(x) + \Phi^+(x)\Phi^-(x). \tag{2.2.83}$$

Thus, we can relate the two products

$$N[\Phi^2(x)] = \Phi^2(x) + [\Phi^-(x), \Phi^+(x)]. \tag{2.2.84}$$

In terms of the Fourier transform creation and annihilation operators we have

$$N[a(\vec{k})a^\dagger(\vec{k})] = a^\dagger(\vec{k})a(\vec{k}) = N[a^\dagger(\vec{k})a(\vec{k})]. \tag{2.2.85}$$

So for the product at zero momentum we have

$$\int d^3x N[\Phi^2(x)] = \int d^3x \left(\Phi^2(x) + [\Phi^-(x), \Phi^+(x)]\right). \tag{2.2.86}$$

But

$$\begin{aligned}\int d^3x[\Phi^-(x), \Phi^+(x)] &= \int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} e^{i(k-l\ )x}[a^\dagger(\vec{k}), a(\vec{l}\ )] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} (2\pi)^3\delta^3(\vec{k}-\vec{l}\ )[a^\dagger(\vec{k}), a(\vec{l}\ )] \\ &= -\int \frac{d^3k}{(2\pi)^3 2\omega_k} (2\pi)^3\delta^3(0)\end{aligned} \tag{2.2.87}$$

which is just the type of singularity we are finding in the Hamiltonian and this normal ordering can be used to eliminate the singularity in H! Thus, the well defined product of field operators making up the Hamiltonian is

$$\hat{H} = N[H]. \tag{2.2.88}$$

Similarly, the Lagrangian should be defined with normal ordering

$$\hat{\mathcal{L}} = N[\mathcal{L}] \tag{2.2.89}$$

as well as all other composite operators

$$\hat{T}^{\mu\nu} = N[T^{\mu\nu}] \tag{2.2.90}$$

$$\hat{M}^{\mu\nu\rho} = N[M^{\mu\nu\rho}]. \tag{2.2.91}$$

The field equations in the free field case stay the same since this normal ordering of H just corresponds, as we have seen, to the addition of an infinite constant to the naive Hamiltonian. So

$$\left(\partial^2 + m^2\right)\Phi(x) = 0 \tag{2.2.92}$$

still holds. Fourier transforming we now obtain directly that

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k a^\dagger(\vec{k})a(\vec{k}) \tag{2.2.93}$$

$$\hat{\vec{P}} = \vec{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k}a^\dagger(\vec{k})a(\vec{k}) \tag{2.2.94}$$

and $\hat{H}|0>= 0$ and $\hat{\vec{P}}|0>= 0$. All conservation equations stay intact and $\hat{P}^\mu = N[P^\mu]$ still generates the space-time translations.

Following our quantization rules for free fields we must Wick or normal order all products of field operators to make them well defined in the limit of coincident points. (Of course we could have started in momentum space as in the introduction then this would have been the natural order to appear!) The general definition of $N[\Phi(x_1)\cdots\Phi(x_N)]$ is to rearrange all creation operators to the left and all annihilation operators to the right (the order among creation operators is irrelevant since they commute and similarly for the annihilation operators). To summarize the free scalar field case then, we define the dynamics through the normal product of a formal Lagrangian

$$\mathcal{L} = \mathcal{L}(\Phi_r, \partial_\mu \Phi_r), \tag{2.2.95}$$

with the normal ordered Lagrangian given by

$$\hat{\mathcal{L}} = N[\mathcal{L}]. \tag{2.2.96}$$

The Euler-Lagrange equations are the same in the free field case

$$\frac{\partial \hat{\mathcal{L}}}{\partial \Phi_r} - \partial_\mu \frac{\partial \hat{\mathcal{L}}}{\partial \partial_\mu \Phi_r} = \frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} = 0 \tag{2.2.97}$$

and so is the momentum

$$\Pi_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_r}. \tag{2.2.98}$$

The canonical commutation relations are

$$[\Pi_r(\vec{x}, t), \Phi_s(\vec{y}, t)] = -i\delta_{rs}\delta^3(\vec{x} - \vec{y})$$

$$[\Pi_r(\vec{x}, t), \Pi_s(\vec{y}, t)] = 0 = [\Phi_r(\vec{x}, t), \Phi_s(\vec{y}, t)]. \tag{2.2.99}$$

The Hamiltonian density becomes the normal product of the classical formal Hamiltonian

$$\hat{\mathcal{H}} = N[\mathcal{H}] = N[\Pi_r \dot{\Phi}_r] - \hat{\mathcal{L}}$$

$$\mathcal{H} = \Pi_r \dot{\Phi}_r - \mathcal{L} \tag{2.2.100}$$

and

$$\hat{H} = \int d^3x N[\mathcal{H}]. \tag{2.2.101}$$

The quantum action principle and quantum Noether's theorem have the same form except N-products appear on all operator products (note that operator ordering does not matter in a normal product since the N operator normal orders all products)

$$\hat{T}^{\mu\nu} = N[T^{\mu\nu}]$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \partial^\nu \Phi - g^{\mu\nu} \mathcal{L}. \tag{2.2.102}$$

One can prove that $\partial_\mu \hat{T}^{\mu\nu} = 0$ since $T^{\mu\nu}$ and $\hat{T}^{\mu\nu}$ differ by a constant (the only change is $-E_0$ from H) and since the operators are normal ordered $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$. Further the energy-momentum operator is given by

$$\hat{\mathcal{P}}^\mu = \int d^3x \hat{T}^{0\mu} \tag{2.2.103}$$

with

$$[\hat{\mathcal{P}}^\mu, \Phi(x)] = -i\partial^\mu \Phi(x). \tag{2.2.104}$$

Similarly, the angular momentum tensor is defined by

$$\hat{M}^{\mu\nu\rho} = x^\nu \hat{T}^{\mu\rho} - x^\rho \hat{T}^{\mu\nu} \tag{2.2.105}$$

with

$$\partial_\mu \hat{M}^{\mu\nu\rho} = 0 \tag{2.2.106}$$

and the angular momentum operator is

$$\hat{\mathcal{M}}^{\mu\nu} = \int d^3x \hat{M}^{0\mu\nu} \tag{2.2.107}$$

with

$$[\hat{\mathcal{M}}^{\mu\nu}, \Phi(x)] = -i(x^\mu \partial^\nu - x^\nu \partial^\mu)\Phi(x). \tag{2.2.108}$$

The particle states of the system are given by the vacuum state $|0>$ defined as the no particle state $a(\vec{k})|0>= 0$ with $\hat{H}|0>= 0 = \hat{\vec{\mathcal{P}}}|0>$. The N-particle state is given by $|\vec{k}_1, \cdots, \vec{k}_N>= a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_N)|0>$. The inner product is given in terms of the one particle subspace and $<0|0>= 1$ so that

$$<\vec{k}|\vec{k}'>= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'). \tag{2.2.109}$$

Finally, we can see that the spin of the particle is zero by considering the action of the angular momentum operator (the generator of rotations in coordinate space) on the one particle state $|\vec{k}>$. Recall from the review of relativity and quantum mechanics that

$$\begin{aligned}\left[\hat{\mathcal{M}}^{\mu\nu}, \Phi(x)\right] &= -i(x^\mu \partial^\nu - x^\nu \partial^\mu)\Phi(x) \\ &\equiv -M^{\mu\nu}\Phi(x).\end{aligned} \tag{2.2.110}$$

The angular momentum operator is given by

$$\mathcal{J}_i \equiv \frac{1}{2}\epsilon_{ijk}\hat{\mathcal{M}}_{jk} \tag{2.2.111}$$

and is represented on the fields as space-time differential operators

$$J^i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk} = i(\vec{x}\times\vec{\bigtriangledown})_i. \tag{2.2.112}$$

The operators obey the angular momentum algebra

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k \tag{2.2.113}$$

and $\vec{\mathcal{J}}$ generates spatial rotations on the states and field as represented by the unitary operator

$$U_r(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\mathcal{J}}}. \tag{2.2.114}$$

Fourier transforming the field we find

$$[\hat{\mathcal{M}}^{ij}, \Phi(x)] = -i(x^i\partial^j - x^j\partial^i)\int\frac{d^3k}{(2\pi)^3 2\omega_k}\left[e^{-ikx}a(\vec{k}) + e^{+ikx}a^\dagger(\vec{k})\right]$$

$$\begin{aligned} &= \int\frac{d^3k}{(2\pi)^3 2\omega_k}(-i)[e^{-ikx}(k^j\partial_k^i - k^i\partial_k^j)a(\vec{k}) \\ &\qquad + e^{ikx}(k^j\partial_k^i - k^i\partial_k^j)a^\dagger(\vec{k})]. \end{aligned} \tag{2.2.115}$$

So for the creation operator this yields

$$[\hat{\mathcal{M}}^{ij}, a^\dagger(\vec{k})] = +i(k^i\partial_k^j - k^j\partial_k^i)a^\dagger(\vec{k}). \tag{2.2.116}$$

Hence, for rotations

$$[\mathcal{J}_i, a^\dagger(\vec{k})] = i\epsilon_{ijl}k^j\partial_k^l a^\dagger(\vec{k}) \tag{2.2.117}$$

or in vector notation

$$[\vec{\mathcal{J}}, a^\dagger(\vec{k})] = -i\vec{k}\times\vec{\bigtriangledown}_k a^\dagger(\vec{k}). \tag{2.2.118}$$

For a particle at rest, i.e. in the particle's rest frame, $\vec{k}=0$. So

$$[\vec{\mathcal{J}}, a^\dagger(0)] = 0 \tag{2.2.119}$$

and since $\vec{\mathcal{J}}|0>=0$

$$\vec{\mathcal{J}}|\vec{k}=0>=0. \qquad (2.2.120)$$

The angular momentum as measured in the rest frame is zero and hence the spin is zero, therefore, in general the totat angular momentum

$$\vec{\mathcal{J}}|\vec{k}>=-i\vec{k}\times\vec{\bigtriangledown}_k|\vec{k}> \qquad (2.2.121)$$

is just the orbital angular momentum $\vec{L}=\vec{r}\times\vec{p}$, the spin is equal to zero. Thus, the free hermitian, scalar field $\Phi(x)$ describes a system of non-interacting, mass m, spinless particles.

Let's return to the equal time quantization conditions and show that our quantization procedure, although it singles out time, is Lorentz invariant due to the scalar nature of the field and the invariance of the field equations. (Although we have chosen an equal time surface on which to specify these initial conditions, the quantization conditions can be specified in terms of general space-like surfaces and hence be made to appear explicitly covariant.) The fact that the field is a scalar under restricted Poincare' transformations implies

$$[\mathcal{P}^\mu,\Phi(x)]=-i\partial^\mu\Phi(x)$$

$$[\mathcal{M}^{\mu\nu},\Phi(x)]=-i(x^\mu\partial^\nu-x^\nu\partial^\mu)\Phi(x). \qquad (2.2.122)$$

Thus, finite transformations are given by the action of the unitary operator

$$U(a,\Lambda)=e^{i\mathcal{P}_\mu a^\mu}e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)\mathcal{M}^{\mu\nu}} \qquad (2.2.123)$$

and according to (2.2.122) implies

$$e^{\frac{-i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}}\Phi(x)e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}}=\Phi(\Lambda x)$$

$$e^{i\mathcal{P}_\mu a^\mu}\Phi(y)e^{-i\mathcal{P}_\mu a^\mu}=\Phi(y+a) \qquad (2.2.124)$$

that is

$$U^\dagger(a,\Lambda)\Phi(\Lambda x+a)U(a,\Lambda)=\Phi(x). \qquad (2.2.125)$$

As we have seen the field equation leads to the momentum decomposition

$$\Phi(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(\vec{k})e^{-ikx}+a^\dagger(\vec{k})e^{+ikx}\right]$$

$$= \Phi^+(x) + \Phi^-(x) \tag{2.2.126}$$

which can be used to calculate the all-time commutator of $\Phi(x)$ and $\Phi(y)$

$$[\Phi(x), \Phi(y)] = \left[\Phi^+(x), \Phi^-(y)\right] + \left[\Phi^-(x), \Phi^+(y)\right] \tag{2.2.127}$$

since $[a(\vec{k}), a(\vec{l})] = 0$ or $[\Phi^+(x), \Phi^+(y)] = 0 = [\Phi^-(x), \Phi^-(y)]$. Now

$$[\Phi^+(x), \Phi^-(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} e^{-ikx} e^{+ily} [a(\vec{k}), a^\dagger(\vec{l}\,)], \tag{2.2.128}$$

but the creation and annihilation operator commutation relation

$$[a(\vec{k}), a^\dagger(\vec{k})] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}\,) \tag{2.2.129}$$

may be used to obtain

$$[\Phi^+(x), \Phi^-(y)] = \int \frac{d^3x}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \equiv i\triangle^+(x-y). \tag{2.2.130}$$

This function appears often and so we have given it a special symbol

$$i\triangle^+(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)}. \tag{2.2.131}$$

Using $\delta(k^2 - m^2) = \frac{1}{2\omega_k}[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)]$ we have

$$i\triangle^+(x-y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k^0) e^{-ik(x-y)}. \tag{2.2.132}$$

Note that under restricted Poincare' transformations this is invariant since k^0 does not change sign under such transformations and all other terms are manifestly invariant. We also need

$$[\Phi^-(x), \Phi^+(y)] = -i\triangle^+(y-x) = -\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+ik(x-y)}$$

$$= -\int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(k^2 - m^2)\theta(-k^0) e^{-ik(x-y)}$$

$$\equiv +i\triangle^-(x-y). \tag{2.2.133}$$

As seen from above the function $\triangle^-$ is given by

$$\triangle^-(x-y) = -\triangle^+(y-x) \tag{2.2.134}$$

and is also $\mathcal{P}_+^\uparrow$ invariant. The all time field commutator is given by the sum of $\triangle^+$ and $\triangle^-$ and is just a c-number

$$\begin{aligned}[\Phi(x), \Phi(y)] &= i\triangle^+(x-y) + i\triangle^-(x-y) \\ &= i(\triangle^+(x-y) + \triangle^-(x-y)) \,. \\ &\equiv i\triangle(x-y)\end{aligned} \tag{2.2.135}$$

Thus, $\triangle(x-y)$ is restricted Poincare' ($\mathcal{P}_+^\uparrow$) invariant. This also follows directly from $\Phi(x)$ being a scalar field and $\triangle(x-y)$ being a c-number

$$\begin{aligned}U(a,\Lambda)\,[\Phi(x), \Phi(y)]\,U^\dagger(a,\Lambda) &= [\Phi(\Lambda x + a), \Phi(\Lambda y + a)] = i\triangle(\Lambda(x-y)) \\ &= U(a,\Lambda) i\triangle(x-y) U^\dagger(a,\Lambda) \qquad . \\ &= i\triangle(x-y)\end{aligned} \tag{2.2.136}$$

So adding and subtracting a^μ, we have

$$\triangle(x-y) = \triangle([\Lambda x + a] - [\Lambda y + a]), \tag{2.2.137}$$

hence $\triangle(x-y)$ is $\mathcal{P}_+^\uparrow$ invariant. We can now represent $\triangle(x-y)$ in several ways which will be useful later on, for instance

$$\begin{aligned}\triangle(x-y) &= -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[e^{-ik(x-y)} - e^{+ik(x-y)}\right] \\ &= -2\int \frac{d^3k}{(2\pi)^3 2\omega_k}\sin k(x-y) \qquad . \\ &= -2\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+i\vec{k}\cdot(\vec{x}-\vec{y})}\sin \omega_k(x^0 - y^0)\end{aligned} \tag{2.2.138}$$

Alternately we can express $\triangle$ as an energy-momentum integral

$$i\triangle(x-y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2) e^{-ik(x-y)}\left[\theta(k^0) - \theta(-k^0)\right]. \tag{2.2.139}$$

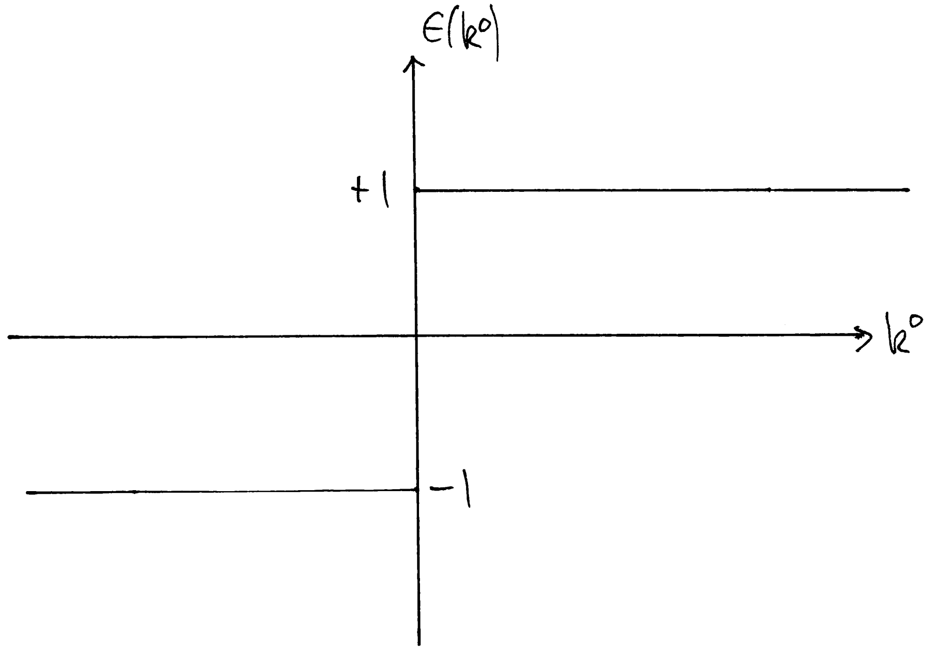

Figure 2.2.1

The difference of the step functions is defined as

$$\epsilon(k^0) \equiv \theta(k^0) - \theta(-k^0) = \begin{cases} +1 & , \text{ for } k^0 > 0 \cdot \\ -1 & , \text{ for } k^0 < 0 \end{cases} \tag{2.2.140}$$

So

$$i\triangle(x-y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\epsilon(k^0)e^{-ik(x-y)}, \tag{2.2.141}$$

again revealing the $\mathcal{P}_+^\uparrow$ invariance of $\triangle$. $\triangle(x-y)$ is a genralized function, performing the integrals we find

$$\triangle(x) = \frac{-1}{2\pi}\epsilon(x^0)\left\{\delta(x^2) - \frac{m^2}{2}\theta(x^2)\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right\} \tag{2.2.142}$$

where J_1 is a Bessel function (see Bogoliubov and Shirkov).

Finally we see that the equal-time commutation relations are just the initial conditions that $\triangle$ must satisfy. That is, $\triangle(x)$ is a solution to the Klein-Gordon equation

$$\left(\partial^2 + m^2\right)\triangle(x) = 0 \tag{2.2.143}$$

since

$$\left(\partial^2 + m^2\right)\Phi(x) = 0. \tag{2.2.144}$$

It satisfies the initial conditions

$$\triangle(\vec{x},0) = \triangle(\vec{0},0) = 0 \tag{2.2.145}$$

since

$$[\Phi(\vec{x},t),\Phi(0,t)] = 0. \tag{2.2.146}$$

Further

$$\frac{\partial}{\partial x^0}\triangle(\vec{x},t) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{\omega_k}\omega_k e^{+i\vec{k}\cdot\vec{x}}\cos\omega_k t, \tag{2.2.147}$$

so at time t = 0, multiplying by i, we find

$$\frac{\partial}{\partial t} i\triangle(\vec{x},t)|_{t=0} = \frac{-i}{(2\pi)^3}\int d^3k e^{+i\vec{k}\cdot\vec{x}} = -i\delta^3(\vec{x}). \tag{2.2.148}$$

This is just the initial condition

$$i\dot{\triangle}(\vec{x},0) = \left[\dot{\Phi}(\vec{x},t),\Phi(\vec{0},t)\right] = -i\delta^3(\vec{x}), \tag{2.2.149}$$

and of course $\ddot{\triangle}(\vec{x},t)|_{t=0} = 0$ since $\left[\dot{\Phi}(\vec{x},t),\dot{\Phi}(0,t)\right] = 0$. Furthermore, we note that since

$$\triangle(x-y) = \triangle(\Lambda(x-y)), \tag{2.2.150}$$

and, for any space-like separation $(x-y)^2 < 0$, we can always find a Lorentz transformation such that $x'^0 = y'^0$, we have

$$\triangle(x-y) = \triangle(\vec{x}\,'-\vec{y}\,',0). \tag{2.2.151}$$

¿From equation (2.2.146) $\triangle(\vec{x},0) = 0$, thus

$$\triangle(x-y) = 0 \qquad \text{for } (x-y)^2 < 0 \tag{2.2.152}$$

and consequently

$$[\Phi(x),\Phi(y)] = 0 \qquad \text{for } (x-y)^2 < 0. \tag{2.2.153}$$

This space-like commutivity property is known as the principle of microcausality. For observables depending on $\Phi(x)$, two observations performed at space-like separations cannot interfere with each other. This is a statement of relativistic invariance since the theory of special relativity forbids signals with $v > c$ from being sent between points and this would have been required for the disturbance due to one measurement at point x to propagate to space-like separated point y to effect the measurement there.

Besides various integral representations for $\triangle$ we can also write $\triangle^{\pm}$ in various ways. Recall we defined

$$i\triangle^{\pm}(x) = \pm\int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(\pm k^0)e^{-ikx}. \tag{2.2.154}$$

In particular for

$$i\triangle^{+}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ikx}, \tag{2.2.155}$$

we can express$\triangle^{+}$ as the contour integral

$$i\triangle^{+}(x) = \int_{C_+} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{(k^2 - m^2)} \tag{2.2.156}$$

where the C_+ contour is chosen in the imaginary k^0-plane as shown below.

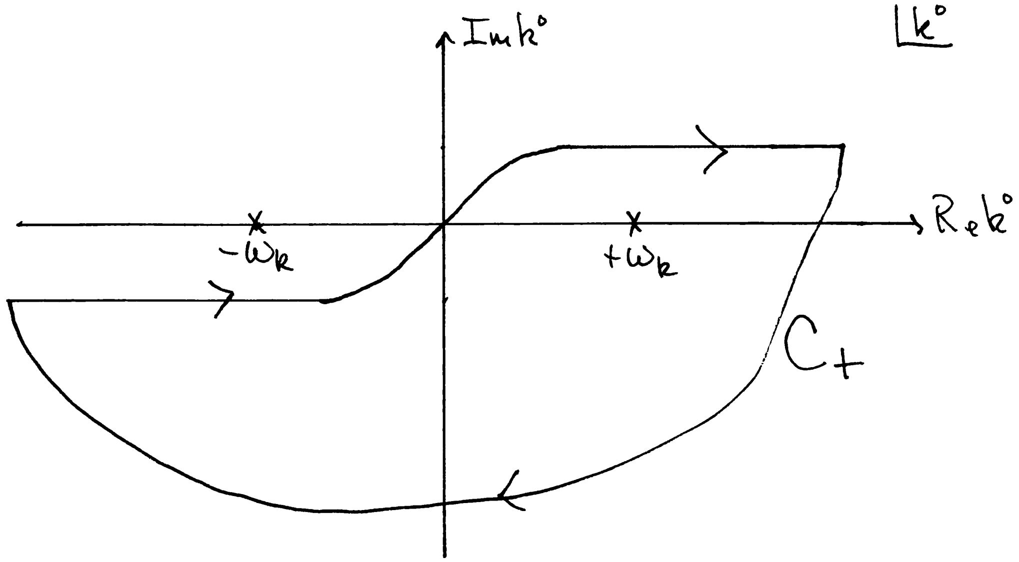

Figure 2.2.2

Performing the k^0 integral first we have

$$i\triangle^{+}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{dk_0}{(2\pi)} e^{-ik^0x^0 + i\vec{k}\cdot\vec{x}} \frac{i}{[k^0 + \omega_k][k^0 - \omega_k]} \tag{2.2.157}$$

since $k^2 - m^2 = k^{0^2} - \vec{k}^2 - m^2 = k^{02} - \omega_k^2 = (k^0 + \omega_k)(k^0 - \omega_k)$ and noting that the path is taken clockwise and hence the integral is minus the residue at the $k^0 = \omega_k$ pole

$$i\triangle^+(x) = \int \frac{d^3k}{(2\pi)^3} \frac{2\pi i}{2\pi} \left(\frac{-ie^{-ik^0x^0 + i\vec{k}\cdot\vec{x}}}{(k^0 + \omega_k)} \right) |_{k^0=\omega_k}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ikx}, \tag{2.2.158}$$

as claimed. Note that it is understood, as usual, that $k^0 = \omega_k$ in the final integral. Similarly $\triangle^-(x)$ can be written as

$$\begin{aligned} i\triangle^-(x) &= \int_{C_-} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{(k^2 - m^2)} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{dk^0}{(2\pi)} e^{-ik^0x^0 + i\vec{k}\cdot\vec{x}} \frac{i}{[k^0 + \omega_k][k^0 - \omega_k]} \end{aligned} \tag{2.2.159}$$

with the k^0 integration performed first over the contour C_- below.

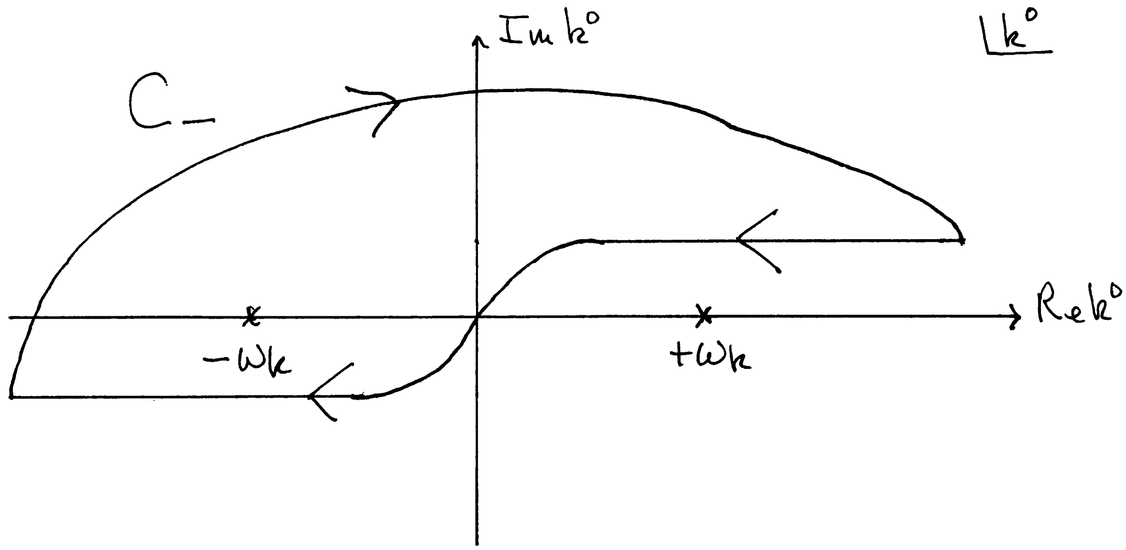

Figure 2.2.3

This yields the desired result

$$\begin{aligned} i\triangle^-(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+i\omega_k x^0 + i\vec{k}\cdot\vec{x}}(-i) \\ &= -\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+ikx} \end{aligned} \tag{2.2.160}$$

where $k^0 = \omega_k$ is understood in the final integral.

In addition to these covariant commutators we can also evaluate the vacuum expectation values of the fields and time ordered products of the fields. These play a fundamental role in quantum field theory since they are related to matrix elements of observables as we will see. First the vacuum expectation value of $\Phi(x)\Phi(y)$,

$$< 0|\Phi(x)\Phi(y)|0 >, \tag{2.2.161}$$

is simply related to $\triangle^+$

$$\begin{aligned} < 0|\Phi(x)\Phi(y)|0 > &=< 0|(\Phi^+(x) + \Phi^-(x))(\Phi^+(y) + \Phi^-(y))|0 > \\ &=< 0|\Phi^+(x)\Phi^-(y)|0 > \end{aligned} \tag{2.2.162}$$

since $\Phi^+|0 >= 0$ and $< 0|\Phi^- = 0$. Then since $\Phi^+(x)|0 >= 0$ we can replace $\Phi^+\Phi^-$ with the commutator

$$< 0|\Phi(x)\Phi(y)|0 >=< 0|[\Phi^+(x), \Phi^-(y)]|0 > . \tag{2.2.163}$$

Thus

$$\begin{aligned} < 0|\Phi(x)\Phi(y)|0 > &= [\Phi^+(x), \Phi^-(y)] \\ &= i\triangle^+(x - y) \end{aligned} \tag{2.2.164}$$

since $< 0|0 >= 1$ and the commutator is a c-number. This VEV of two fields is called the two point Wightman function, and is denoted $W^{(2)}(x, y)$. In general the n-point Wightman function is

$$W^{(n)}(x_1, \cdots, x_n) \equiv< 0|\Phi(x_1)\cdots\Phi(x_n)|0 > . \tag{2.2.165}$$

We can evaluate this product by using the creation and annihiltion properties of Φ. Hence, using $< 0|\Phi^-(x_1) = 0$ and $\Phi^+(x_1)|0 >= 0$ and the fact that the commutator is a c-number, we have

$$< 0|\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)|0 >$$

$$=< 0|\Phi^+(x_1)\Phi(x_2)\cdots\Phi(x_n)|0 >=< 0| \left[\Phi^+(x_1), \Phi(x_2)\cdots\Phi(x_n)\right] |0 >$$

$$=< 0| \left[\Phi^+(x_1), \Phi(x_2)\right] \Phi(x_3)\cdots\Phi(x_n)|0 >$$

$$+ < 0|\Phi(x_2)\left[\Phi^+(x_1), \Phi(x_3)\right]\Phi(x_4)\cdots\Phi(x_n)|0> + \cdots$$

$$+ < 0|\Phi(x_2)\cdots\Phi(x_{i-1})\left[\Phi^+(x_1), \Phi(x_i)\right]\Phi(x_{i+1})\cdots\Phi(x_n)|0> + \cdots$$

$$= \sum_{i=2}^{n}\left[\Phi^+(x_1), \Phi^-(x_i)\right] < 0|\Phi(x_2)\cdots\Phi(x_{i-1})\Phi(x_{i+1})\cdots\Phi(x_n)|0> . \quad (2.2.166)$$

Thus, we have reduced $W^{(n)}(x_1, \cdots, x_n)$ to $\triangle^+$ times $W^{(n-2)}(x_{i_1}, \cdots, x_{i_{n-2}})$. Proceeding until no fields are left we find

$$W^{(n)}(x_1, \cdots, x_n) = \begin{cases} \sum_P W^{(2)}(x_{i_1}, x_{j_1})\cdots W^{(2)}(x_{i_{\frac{n}{2}}}, x_{j_{\frac{n}{2}}}) & \text{, n even} \\ 0 & \text{, n odd} \end{cases} \quad (2.2.167)$$

where $\sum_P$ is a sum over all permutations P of $(1, \cdots, n)$ into $\frac{n}{2}$ pairs $(i_1, j_1)\cdots (i_{\frac{n}{2}}, j_{\frac{n}{2}})$ with $i_1 < j_1, \cdots, i_{\frac{n}{2}} < j_{\frac{n}{2}}$ and $i_1 < i_2 < \cdots < i_{\frac{n}{2}}$. For example, the 4-point Wightman function is

$$\begin{aligned} W^{(4)}(x_1, x_2, x_3, x_4) &= \left[\Phi^+(x_1), \Phi^-(x_2)\right] < 0|\Phi(x_3)\Phi(x_4)|0> \\ &+ \left[\Phi^+(x_1), \Phi^-(x_3)\right] < 0|\Phi(x_3)\Phi(x_4)|0> \\ &+ \left[\Phi^+(x_1), \Phi^-(x_4)\right] < 0|\Phi(x_2)\Phi(x_3)|0> \\ &= W^{(2)}(x_1, x_2)W^{(2)}(x_3, x_4) \\ &\quad + W^{(2)}(x_1, x_3)W^{(2)}(x_2, x_4) \\ &\quad + W^{(2)}(x_1, x_4)W^{(2)}(x_2, x_3). \end{aligned} \quad (2.2.168)$$

These product formulae for the Wightman functions are just special cases of the general reduction of a product of free fields in terms of Wick or Normal products of the free fields. The reduction formula is known as Wick's Theorem and is given by (here we have generalized to possibly different free fields denoted by subscript i, Φ_i)

Wick's Theorem:

$$\Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n) = N\left[\Phi_1(x_1)\cdots\Phi_n(x_n)\right]$$

$$+ \sum_{\substack{1\ pairing \\ 1\le i<j\le n}} < 0|\Phi_i(x_i)\Phi_j(x_j)|0> N\left[\frac{\Phi_1(x_1)\cdots\Phi_n(x_n)}{\Phi_i(x_i)\Phi_j(x_j)}\right]$$

$$+ \sum_{\substack{2\ pairings \\ i_1<i_2, i_1<j_1, i_2<j_2}} < 0|\Phi_{i_1}(x_{i_1})\Phi_{j_1}(x_{j_1})|0>< 0|\Phi_{i_2}(x_{i_2})\Phi_{j_2}(x_{j_2})|0>$$

$$\times N\left[\frac{\Phi_1(x_1)\cdots\Phi_n(x_n)}{\Phi_{i_1}\Phi_{i_2}\Phi_{j_1}\Phi_{j_2}}\right]$$

$$+\sum_{3\ pairings}\cdots+\sum_{\substack{P\\ n\ even}} <0|\Phi_{i_1}(x_{i_1})\Phi_{j_1}(x_{j_1})|0>\cdots<0|\Phi_{i_{\frac{n}{2}}}(x_{i_{\frac{n}{2}}})\Phi_{j_{\frac{n}{2}}}(x_{j_{\frac{n}{2}}})|0>. \qquad (2.2.169)$$

The ratio of products of fields in the normal products is just shorthand notation to indicate that the fields in the denominator are absent from their location in the product in the numerator. Equation (2.2.167) is obtained from Wick's Theorem by simply taking the vacuum expectation value of (2.2.169) and noting that only the last term on the right hand side is non-zero for n even and for n odd the VEV of the right hand side is zero. A more compact notation is one which represents the Wightman function between two fields by a curved line connecting them

$$\underbrace{\Phi(x)\Phi(y)} \equiv <0|\Phi(x)\Phi(y)|0> \qquad (2.2.170)$$

with fields in the order left to right $\Phi(x)\Phi(y)$ and in a normal product

$$N\left[\Phi(x)\cdots\underbrace{\Phi(y)\cdots\Phi(z)}\cdots\Phi(w)\right] = <0|\Phi(y)\Phi(z)|0> N\left[\frac{\Phi(x)\cdots\Phi(w)}{\Phi(y)\Phi(z)}\right]. \qquad (2.2.171)$$

So Wick's Theorem can be written as

$$\Phi_1(x_1)\cdots\Phi_n(x_n) = N\left[\Phi_1(x_1)\cdots\Phi_n(x_n)\right] + \sum_{1\ pair} N\left[\Phi_1\cdots\underbrace{\Phi_i\cdots\Phi_j}\cdots\Phi_n\right]$$

$$+\sum_{2\ pairs} N\left[\Phi_1\cdots\Phi_{i_1}\cdots\Phi_{i_2}\cdots\Phi_{j_1}\cdots\Phi_{j_2}\cdots\Phi_n\right]+\cdots \quad . \qquad (2.2.172)$$

For the proof of Wick's Theorem we must first prove a Lemma that allows us to bring a field inside a Normal product for outside the product. We will then be able to prove our main result by induction.

Lemma:

$$\left(N\left[\Phi_1\cdots\Phi_n\right]\right)\Phi = N\left[\Phi_1\cdots\Phi_n\Phi\right] + \sum_{1\leq i\leq n} N\left[\Phi_1\cdots\underbrace{\Phi_i\cdots\Phi_n\Phi}\right]. \qquad (2.2.173)$$

Since the normal product is linear the lemma follows as soon as we prove it for each Φ_i and Φ being either creation or annihilation operators. If Φ is an annihilation operator then the identity is true since Φ^+ is in its normal order already

$$N\left[\Phi_1\cdots\Phi_n\right]\Phi^+ = N\left[\Phi_1\cdots\Phi_n\Phi^+\right]. \qquad (2.2.174)$$

Further $\underbrace{\Phi_i\Phi^+} =< 0|\Phi_i\Phi^+|0 > \equiv 0$ since $\Phi^+|0 >= 0$. Thus we only need consider the case where $\Phi = \Phi^-$, a creation operator. In addition, if any of the Φ_i are creation operators they can be brought outside of the normal product to the left. Also the pairings of any of these creation operators with Φ^- is zero since $\underbrace{\Phi^-\Phi^-} =< 0|\Phi^-\Phi^-|0 > \equiv 0$ because $< 0|\Phi^- = 0$. Hence we have

$$N\left[\Phi_{i_1}^- \cdots \Phi_{i_n}^+\right]\Phi^- = \Phi_{i_1}^- \cdots \Phi_{i_l}^- N\left[\Phi_{i_{l+1}}^+ \cdots \Phi_{i_n}^+\right]\Phi^-$$

$$= \Phi_{i_1}^- \cdots \Phi_{i_l}^- N\left[\Phi_{i_{l+1}}^+ \cdots \Phi_{i_n}^+\right]\Phi^- + \sum_{1\le j\le l} N\left[\Phi_{i_1}^- \cdots \underbrace{\Phi_{i_j}^- \cdots \Phi_{i_n}^+ \Phi^-}\right]. \qquad (2.2.175)$$

Hence to obtain the lemma we only need to prove it for all annihilation operators inside the normal product and a creation operator outside the normal product.

Thus we consider $N\left[\Phi_1^+ \cdots \Phi_m^+\right]\Phi^-$. The lemma applied to it will be proven by induction. First for $m = 1$

$$\begin{aligned} \Phi_1^+\Phi^- &= \Phi^-\Phi_1^+ + \left[\Phi_1^+, \Phi^-\right] \\ &= N\left[\Phi^-\Phi_1^+\right] + \left[\Phi_1^+, \Phi^-\right]. \end{aligned} \qquad (2.2.176)$$

But we have that

$$\left[\Phi_1^+, \Phi^-\right] = W^{(2)}(x_1, x) =< 0|\Phi_1(x_1)\Phi(x)|0 > \qquad (2.2.177)$$

and $N\left[\Phi^-\Phi_1^+\right] = N\left[\Phi_1^+\Phi^-\right]$ and $N\left[\Phi_1^+\right] = \Phi_1^+$. So we have the first step in the lemma

$$\begin{aligned} N\left[\Phi_1^+\right]\Phi^- &= N\left[\Phi_1^+\Phi^-\right] + < 0|\Phi_1(x_1)\Phi(x)|0 > \\ &= N\left[\Phi_1^+\Phi^-\right] + N\left[\underbrace{\Phi_1\Phi}\right]. \end{aligned} \qquad (2.2.178)$$

Next we assume the lemma true for $m \geq 2$ annihilation operators Φ_i^+

$$N\left[\Phi_1^+ \cdots \Phi_m^+\right]\Phi^- = N\left[\Phi_1^+ \cdots \Phi_m^+\Phi^-\right] + \sum_{1\le i\le m} N\left[\Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+\Phi^-}\right]. \qquad (2.2.179)$$

In order to prove the lemma for $m + 1$ annihilation operators we multiply this by Φ_0^+ on the left

$$\begin{aligned} \Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\right]\Phi^- &= \Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\Phi^-\right] \\ &+ \sum_{1\le i\le m} \Phi_0^+ N\left[\Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+\Phi^-}\right]. \end{aligned} \qquad (2.2.180)$$

First Φ^- is the only creation operator in the second term on the right hand side and it is paired with Φ_i^+, it is now a c-number factor. Hence, only annihilation operators appear in this last term. But they all commute amongst themselves, so their Wick product is their ordinary product

$$\sum_{1\le i\le m} \Phi_0^+ N\left[\Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+ \Phi^-}\right] = \sum_{1\le i\le m} N\left[\Phi_0^+ \Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+ \Phi^-}\right]. \tag{2.2.181}$$

Next we analyze $\Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+ \Phi^-\right]$, clearly since

$$N\left[\Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] = N\left[\Phi^- \Phi_1^+ \cdots \Phi_m^+\right] = \Phi^- N\left[\Phi_1^+ \cdots \Phi_m^+\right] \tag{2.2.182}$$

we find

$$\begin{aligned}\Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] &= \Phi_0^+ \Phi^- N\left[\Phi_1^+ \cdots \Phi_m^+\right] \\ &= \Phi^- \Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\right] + \left[\Phi_0^+, \Phi^-\right] N\left[\Phi_1^+ \cdots \Phi_m^+\right].\end{aligned} \tag{2.2.183}$$

Now the first term on the right hand side has the operators in their normal order, hence

$$\begin{aligned}\Phi^- \Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\right] &= N\left[\Phi^- \Phi_0^+ \Phi_1^+ \cdots \Phi_m^+\right] \\ &= N\left[\Phi_0^+ \cdots \Phi_m^+ \Phi^-\right].\end{aligned} \tag{2.2.184}$$

As before $\left[\Phi_0^+, \Phi^-\right]$ is just the pairing of Φ^- with Φ_0^+, so

$$\Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] = N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] + N\left[\underbrace{\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-}\right]. \tag{2.2.185}$$

Putting this together we have

$$\begin{aligned}&\Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\right] \Phi^- \\ &= \Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] + \sum_{1\le i\le m} N\left[\Phi_0^+ \Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+ \Phi^-}\right] \\ &= N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] + N\left[\underbrace{\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-}\right] \\ &+ \sum_{1\le i\le m} N\left[\Phi_0^+ \Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+ \Phi^-}\right] \\ &= N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] + \sum_{0\le i\le m} N\left[\Phi_0^+ \Phi_1^+ \cdots \underbrace{\Phi_i^+ \cdots \Phi_m^+ \Phi^-}\right].\end{aligned} \tag{2.2.186}$$

Now Φ_0^+ is an annihilation operator as are the Φ_i^+, so the left hand side is simply

$$\Phi_0^+ N\left[\Phi_1^+ \cdots \Phi_m^+\right] = N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+\right]. \tag{2.2.187}$$

Thus we have proven the lemma for $m+1$ annihilation operators Φ_i^+

$$\begin{aligned} N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+\right] \Phi^- &= N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_m^+ \Phi^-\right] \\ &+ \sum_{0 \le i \le m} N\left[\Phi_0^+ \Phi_1^+ \cdots \Phi_i^+ \cdots \Phi_m^+ \Phi^-\right]. \end{aligned} \tag{2.2.188}$$

Now then we have the lemma we desire

$$N\left[\Phi_1 \cdots \Phi_n\right] \Phi = N\left[\Phi_1 \cdots \Phi_n \Phi\right] + \sum_{1 \le i \le n} N\left[\Phi_1 \cdots \underbrace{\Phi_i \cdots \Phi_n \Phi}_{}\right]. \tag{2.2.189}$$

We now turn to the proof of Wick's Theorem, again for $n = 2$ we verify the theorem directly

$$\Phi_1(x_1)\Phi_2(x_2) = N\left[\Phi_1(x_1)\Phi_2(x_2)\right] + < 0|\Phi_1(x_1)\Phi_2(x_2)|0 > . \tag{2.2.190}$$

The proof of the theorem proceeds by induction. Assuming the theorem valid for n operators

$$\begin{aligned} \Phi_1 \cdots \Phi_n &= N\left[\Phi_1 \cdots \Phi_n\right] + \sum_{1\ pair} N\left[\Phi_1 \cdots \underbrace{\Phi_i \cdots \Phi_j}_{} \cdots \Phi_n\right] \\ &+ \sum_{2\ pairs} N\left[\Phi_1 \cdots \underbrace{\Phi_i \cdots \Phi_j}_{} \cdots \underbrace{\Phi_k \cdots \Phi_l}_{} \cdots \Phi_n\right] + \cdots, \end{aligned} \tag{2.2.191}$$

we multiply by Φ_{n+1} on the right and apply our lemma in order to show that the theorem is valid for $n+1$ operators. Since Wick's Theorem for the $\Phi_1 \cdots \Phi_n$ already has all possible pairings amongst these fields, multiplying by Φ_{n+1} and applying our lemma yields the sum over all additional pairings with $\Phi_1 \cdots \Phi_n$ and Φ_{n+1} as well as the non-paired normal product of them all. Hence we have immediately the proof of Wick's Theorem.

In addition to pure products of fields we see that Wick's Theorem is immediately applicable to the case of products of normal products of fields also, such as

$$N\left[\Phi_1 \cdots \Phi_{i_1}\right] N\left[\Phi_{i_1+1} \cdots \Phi_{i_2}\right] \cdots N\left[\Phi_{i_m+1} \cdots \Phi_n\right]. \tag{2.2.192}$$

The proof is as in the pure field product case, the only difference being we now no longer have pairings between fields in the same normal product since their "internal order" is already normal (thus we have "no self-pairings"). We only have pairings between fields in different normal products.

Besides Wightman functions we can also evaluate the time ordered functions for free fields, that is, the vacuum expectation value of the time ordered products of the fields, which, as we recall, will be relevant when we consider the perturbative expansion of the S-matrix. Recall the definition of the time ordering operator for one and two operators

$$T\Phi(x) = \Phi(x)$$

$$T\Phi(x_1)\Phi(x_2) = \theta(x_1^0 - x_2^0)\Phi(x_1)\Phi(x_2) + \theta(x_2^0 - x_1^0)\Phi(x_2)\Phi(x_1), \tag{2.2.193}$$

and in general

$$T\Phi(x_1)\cdots\Phi(x_n)$$

$$= \sum_{(1,\cdots,n)\to(i_1,\cdots,i_n)} \theta(x_{i_1}^0 - x_{i_2}^0)\theta(x_{i_2}^0 - x_{i_3}^0)\cdots\theta(x_{i_{(n-1)}}^0 - x_{i_n}^0)\Phi(x_{i_1})\cdots\Phi(x_{i_n}). \tag{2.2.194}$$

The vacuum expectation values of the time ordered product of operators are called the time ordered functions, the Green functions, the τ-functions, or the n-point functions and are denoted

$$G^{(n)}(x_1,\cdots,x_n) \equiv < 0|T\Phi(x_1)\cdots\Phi(x_n)|0 > . \tag{2.2.195}$$

Since $G^{(n)}$ is expressible in terms of Wightman functions we see that $G^{(2n+1)} = 0$. Also as with the Wightman functions, we will express the n-point function for free fields in terms of products of the free field 2-point functions. Hence we start by evaluating the 2-point function, also known as the Feynman propagator

$$\begin{aligned} < 0|T\Phi(x_1)\Phi(x_2)|0 > &= \theta(x_1^0 - x_2^0) < 0|\Phi(x_1)\Phi(x_2)|0 > \\ &\quad + \theta(x_2^0 - x_1^0) < 0|\Phi(x_2)\Phi(x_1)|0 > \\ &= \theta(x_1^0 - x_2^0)i\triangle^+(x_1 - x_2) + \theta(x_2^0 - x_1^0)i\triangle^+(x_2 - x_1) \\ &= i\left[\theta(x_1^0 - x_2^0)\triangle^+(x_1 - x_2) - \theta(x_2^0 - x_1^0)\triangle^-(x_1 - x_2)\right] \\ &\equiv \Delta_F(x_1 - x_2). \end{aligned} \tag{2.2.196}$$

Since this combination of $\triangle^+$ and $\triangle^-$ occurs frequently we have denoted it by a special symbol, $\triangle_F(x_1 - x_2)$, called the Feynman propagator,

$$\triangle_F(x) = \theta(x^0) i\triangle^+(x) - i\theta(-x^0)\triangle^-(x). \tag{2.2.197}$$

Hence, we find that

$$\triangle_F(x) = \begin{cases} i\triangle^+(x) & \text{, if } x^0 > 0 \\ -i\triangle^-(x) & \text{, if } x^0 < 0. \end{cases} \tag{2.2.198}$$

Using our integral representation for $\triangle^{\pm}$, equations (2.2.156) and (2.2.159), we have

$$\triangle_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{(k^2 - m^2)} \tag{2.2.199}$$

where we use C_+ when $x^0 > 0$ and $-C_-$ when $x^0 < 0$ for the contour of integration in the complex k^0-plane shown in Figure 2.2.4.

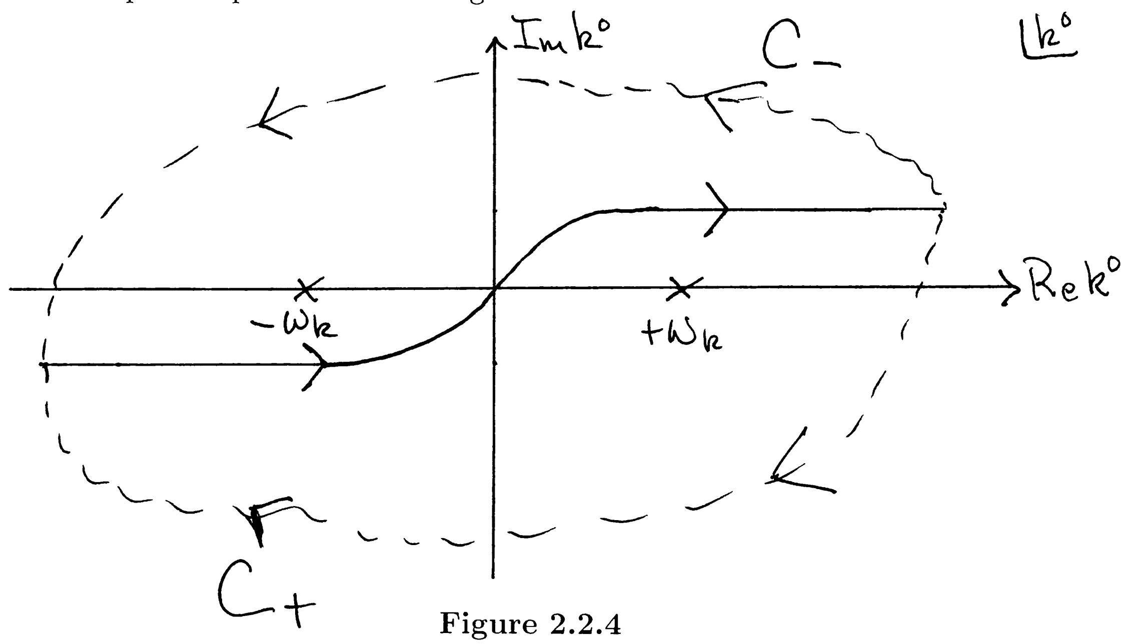

Figure 2.2.4

Hence, we can write

$$\triangle_F(x) = \int_{C_F} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{(k^2 - m^2)} \tag{2.2.200}$$

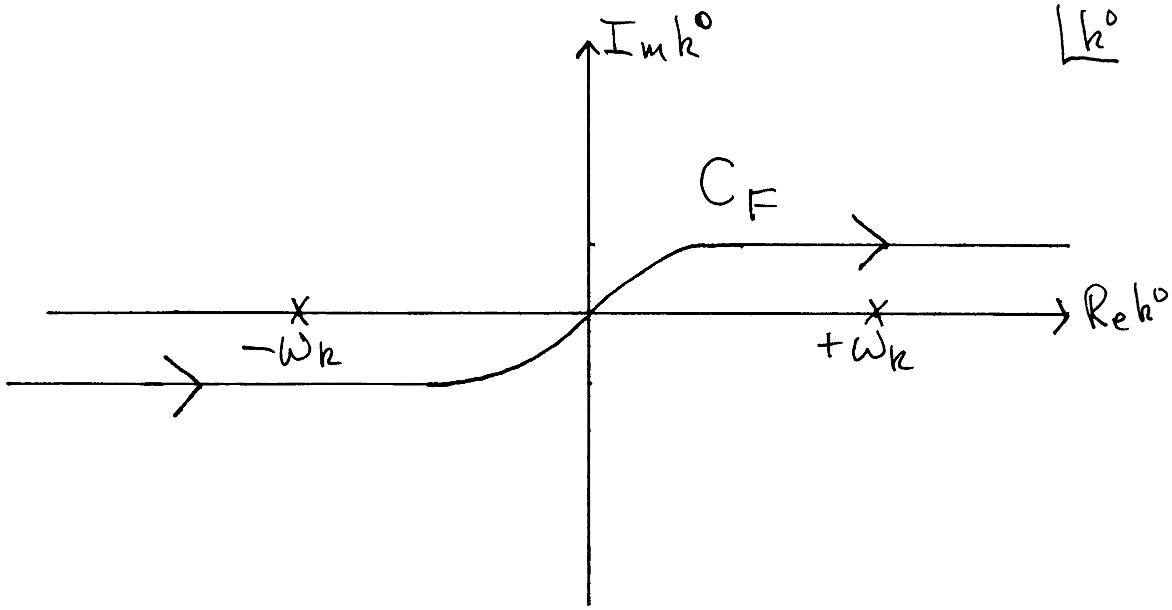

Figure 2.2.5

with the contour C_F in the complex k^0-plane given in Figure 2.2.5.

Thus when evaluating the integral for $x^0 > 0$ the contour is closed in the lower half plane to give $i\triangle^+$. This is the case since the contribution from the semi-circle vanishes for an infinite radius ($e^{-ik^0x^0} \to e^{-|k^0|x^0} \to 0$ for $x^0 > 0$ and $k^0 \to -i\infty$). For $x^0 < 0$ the contour is closed in the upper half plane to give $-i\triangle^-$. Similarly this is the case since the contribution from this semi-circle vanishes for infinite radius ($e^{-ik^0x^0} \to e^{-|k^0||x^0|} \to 0$ for $x^0 < 0$ and $k^0 \to +i\infty$). Note this contour integration is equivalent to the integration over the contour given by Figure 2.2.6.

Equivalently, we can add a small imaginary part to the denominator in order to miss the poles at $k^0 = \pm\omega_k$ and integrate along the real axis

$$\triangle_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{(k^2 - m^2 + i\epsilon)} \tag{2.2.201}$$

where it is understood that $\epsilon \to 0^+$ at the end of all calculations. Since the poles are now at

$$k^0 = \begin{cases} \omega_k - i\epsilon' \\ -\omega_k + i\epsilon' \end{cases}, \tag{2.2.202}$$

since we have

$$\begin{aligned}(k^0 - \omega_k + i\epsilon')(k^0 + \omega_k - i\epsilon') &= k^{02} - \omega_k^2 + 2i\epsilon'\omega_k + \epsilon'^2 \\ &\equiv k^2 - m^2 + i\epsilon\end{aligned} \tag{2.2.203}$$

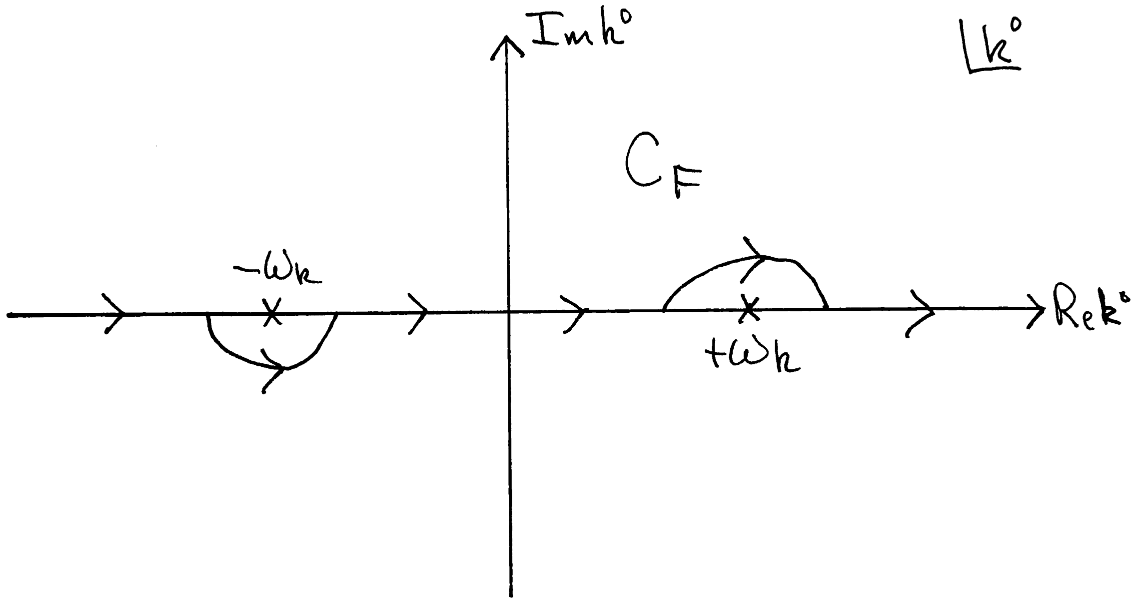

Figure 2.2.6

we integrate along the real axis $(-\infty, +\infty)$ as shown in Figure 2.2.7 and take $\epsilon \to 0^+$ at the end. As done previously one can perform the k^0-integration first by closing the contour in the lower half plane for $x^0 > 0$ and in the upper half plane for $x^0 < 0$, then subsequently taking $\epsilon \to 0^+$ and we find again

$$\triangle_F(x) = \begin{cases} +i\triangle^+(x) & , \text{ for } x^0 > 0 \\ -i\triangle^-(x) & , \text{ for } x^0 < 0 \end{cases} . \tag{2.2.204}$$

Returning to the n-point functions, we would like to show that we can reduce them to products of Feynman propagators. That is we will have for the 2-point and 4-point functions

$$\begin{aligned} < 0|T\Phi(x_1)\Phi(x_2)|0 > &= \triangle_F(x_1 - x_2) \\ < 0|T\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0 > &= < 0|T\Phi(x_1)\Phi(x_2)|0 >< 0|T\Phi(x_3)\Phi(x_4)|0 > \\ &+ < 0|T\Phi(x_1)\Phi(x_3)|0 >< 0|T\Phi(x_2)\Phi(x_4)|0 > \\ &+ < 0|T\Phi(x_1)\Phi(x_4)|0 >< 0|T\Phi(x_2)\Phi(x_3)|0 > . \end{aligned} \tag{2.2.205}$$

In general we will derive the result for the n-point function

$$< 0|T\Phi(x_1).....\Phi(x_n)|0 >$$

$$= \begin{cases} \sum_P < 0|T\Phi(x_{i_1})\Phi(x_{j_1})|0 > \cdots < 0|T\Phi(x_{i_{\frac{n}{2}}})\Phi(x_{j_{\frac{n}{2}}})|0 > & , \text{n = even} \\ 0 & , \text{n = odd} \end{cases} \tag{2.2.206}$$

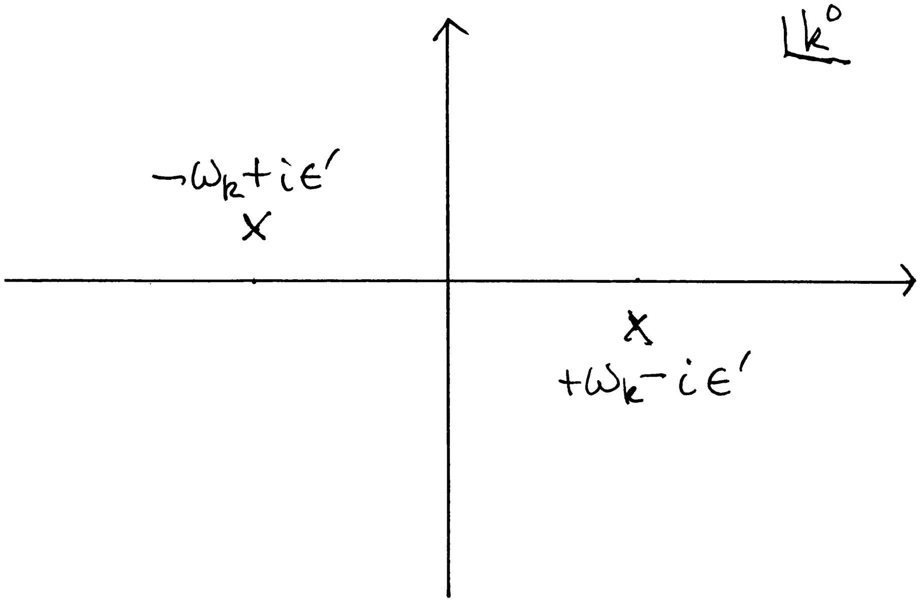

Figure 2.2.7

where again $\sum_P$ is a sum over all permutations P of $(1, \cdots, n)$ into $\frac{n}{2}$ pairs $(i_1, j_1) \cdots (i_{\frac{n}{2}}, j_{\frac{n}{2}})$ with $i_1 < j_1, \cdots, i_{\frac{n}{2}} < j_{\frac{n}{2}}$ and $i_1 < i_2 < \cdots < i_{\frac{n}{2}}$. As in the Wightman function case we can prove this result directly by using the fact that $\Phi^+|0>= 0$ and $< 0|\Phi^- = 0$. That is without loss of generality we may consider a particular chronological ordering, say $x_2^0 > x_3^0 > \cdots > x_m^0 > x_1^0 > x_{m+1}^0 > \cdots x_n^0$, then the time ordered function reduces to the ordinary Wightman function with the fields in that chronological order

$$< 0|T\Phi(x_1) \cdots \Phi(x_n)|0 >=< 0|\Phi(x_2) \cdots \Phi(x_m)\Phi(x_1)\Phi(x_{m+1}) \cdots \Phi(x_n)|0 > . \tag{2.2.207}$$

We can write $\Phi(x)$ in terms of its positive and negative frequency components, $\Phi(x) = \Phi^+(x) + \Phi^-(x)$, and exploit the definition of the vacuum state to obtain

$$\begin{aligned}
&< 0|T\Phi(x_1) \cdots \Phi(x_n)|0 > \\
&=< 0|\Phi(x_2) \cdots \Phi(x_m) \left[\Phi^+(x_1), \Phi(x_{m+1}) \cdots \Phi(x_n)\right] |0 > \\
&+ < 0| \left[\Phi(x_2) \cdots \Phi(x_m), \Phi^-(x_1)\right] \Phi(x_{m+1}) \cdots \Phi(x_n)|0 > \\
&= \left(\sum_{i=m+1}^{n} \left[\Phi^+(x_1), \Phi(x_i)\right] + \sum_{i=2}^{m} \left[\Phi(x_i), \Phi^-(x_1)\right] \right) \times \\
&\times < 0|\frac{\Phi(x_1) \cdots \Phi(x_n)}{\Phi(x_1)\Phi(x_i)}|0 > .
\end{aligned} \tag{2.2.208}$$

Now the remaining fields in the Wightman function on the right hand side of equation (2.2.208) are still in chronological order, thus the functions are time ordered functions. Further since $x_1^0 > x_i^0$ for $i > m$, the commutator is just the expression for the Feynman propagator

$$\left[\Phi^+(x_1), \Phi(x_i)\right] = < 0|T\Phi(x_1)\Phi(x_i)|0 > \qquad \text{for } x_1^0 > x_i^0. \tag{2.2.209}$$

Similarly since $x_1^0 < x_i^0$ for $i \leq m$, the commutator is again just the Feynman propagator for that ordering

$$\left[\Phi(x_i), \Phi^-(x_1)\right] = < 0|T\Phi(x_1)\Phi(x_i)|0 > \qquad \text{for } x_1^0 < x_i^0. \tag{2.2.210}$$

Thus equation (2.2.208) becomes

$$< 0|T\Phi(x_1)\cdots\Phi(x_n)|0 >=$$

$$\sum_{i=2}^{n} < 0|T\Phi(x_1)\Phi(x_i)|0 >< 0|T\Phi(x_2)\cdots \cancel{\Phi(x_i)} \cdots\Phi(x_n)|0 > . \tag{2.2.211}$$

Similar results are obtained for any chronological order, hence we obtain the desired lemma, writing this out again

$$< 0|T\Phi(x)\Phi(x_1)\cdots\Phi(x_n)|0 >$$

$$= \sum_{j=1}^{n} < 0|T\Phi(x)\Phi(x_j)|0 >< 0|T\Phi(x_1)\cdots \cancel{\Phi(x_j)} \cdots\Phi(x_n)|0 > . \tag{2.2.212}$$

So applying the lemma again to the $< 0|T\Phi(x_1)...\cancel{\Phi(x_j)}...\Phi(x_n)|0 >$ function we can "contract" out two more fields. Continuing until all fields are chronologically paired we obtain the reduction of the free field n-point function into product of free field 2-point functions, that is equation (2.2.206) (again we see that the n-point function vanishes for n odd). Writing this in a cryptic fashion we have for n even

$$< 0|T\Phi(x_1)\cdots\Phi(x_n)|0 >$$

$$= \sum_{pairs} < 0|T\Phi(x_{i_1})\Phi(x_{j_1})|0 > \cdots < 0|T\Phi(x_{i_{n/2}})\Phi(x_{j_{n/2}})|0 >$$

$$= \sum_{pairs} \Delta_F(x_{i_1} - x_{j_1})\cdots\Delta_F(x_{i_{\frac{n}{2}}} - x_{j_{\frac{n}{2}}}). \tag{2.2.213}$$

Recall that the time ordered functions are symmetric $< 0|T\Phi(x)\Phi(y)|0 >=< 0|T\Phi(y)\Phi(x)|0 >$, so the summation includes each pair only once. Hence, the time ordered function of free fields is just the sum over all possible pairings of the coordinates of the product of the associated two point functions. Often this process is written with straight lines joining the chronologically paired fields which in turn are said to be contracted. For example, one possible term in the sum for $< 0|T\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0 >$ is given by the contraction

$$< 0|T\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0 >=< 0|T\Phi(x_1)\Phi(x_3)|0 >< 0|T\Phi(x_2)\Phi(x_4)|0 > . \tag{2.2.214}$$

These product formulae for the time ordered functions are just special cases of the general reduction of the time ordered product of free fields in terms of Wick or Normal products of free fields. As with the ordinary product of fields, Wick's Theorem applies to the time ordered product also (here again we have generalized to possibly different free fields denoted by the subscript i, Φ_i)

$$\begin{aligned} &T\Phi_1(x_1)\cdots\Phi_n(x_n) \\ &= N\left[\Phi_1(x_1)\cdots\Phi_n(x_n)\right] + \sum_{1\ contraction} N\left[\Phi_1\cdots\Phi_i\cdots\Phi_j\cdots\Phi_n\right] \\ &+ \sum_{2\ contractions} N\left[\Phi_1\cdots\Phi_{i_1}\cdots\Phi_{i_2}\cdots\Phi_{j_1}\cdots\Phi_{j_2}\cdots\Phi_n\right] + \cdots \end{aligned} \tag{2.2.215}$$

where the contracted fields in the normal products stand for the time ordered function between the two fields. In order to prove this all we need to recall again is that for a specific chronological ordering the T-product reduces to an ordinary product of fields

$$T\Phi_1(x_1)\cdots\Phi_n(x_n) = \Phi_{i_1}(x_{i_1})\cdots\Phi_{i_n}(x_{i_n}) \qquad \text{for } x^0_{i_1} > x^0_{i_2} > \cdots x^0_{i_n} . \tag{2.2.216}$$

We then apply Wick's Theorem to this ordinary product with all ordinary pairings of the fields. However, since the fields are chronologically ordered already when you pair (recall pairing is in a definite order $< 0|\Phi_{i_a}\Phi_{j_a}|0 >$ with $i_a < j_a$), hence the pairing is a chronological pairing, a contraction, that is a Feynman propagator for the fields. Since inside the T-product the order of the field operators does not matter as is the case inside the normal product, the fields can be brought into the same order as initially $\Phi_1(x_1)\cdots\Phi_n(x_n)$. That is a similar reduction occurs for each chronological ordering. Thus we obtain the Wick Theorem expansion for the time ordered product of free fields.

Similar to the product of normal products, we can derive a Wick expansion for the chronological product of normal products

$$T\left(N\left[\Phi_1(x)\cdots\Phi_a(x)\right]N\left[\Phi_{a+1}(y)\cdots\Phi_{a+b}(y)\right]\cdots\right.$$
$$\left.\cdots N\left[\Phi_{a+b+\cdots+1}(z)\cdots\Phi_n(z)\right]\right). \tag{2.2.217}$$

Wick's Theorem has just the same form as for the time ordered products of the fields themselves except now there are no contractions between fields from the same normal product allowed on the right hand side of the Wick expansion. Our theorem for Green functions follows directly from Wick's Theorem by taking the VEV of equation (2.2.215). The VEV of all normal products on the right hand side vanish and hence we are left with

$$< 0|T\Phi(x_1)\cdots\Phi(x_n)|0 >$$
$$= \begin{cases} 0 & \text{, n odd} \\ \sum_P < 0|T\Phi(x_{i_1})\Phi(x_{j_1})|0 > \cdots < 0|T\Phi(x_{i_{\frac{n}{2}}})\Phi(x_{j_{\frac{n}{2}}})|0 > & \text{, n even} \end{cases} \tag{2.2.218}$$

where $\sum_P$ is a sum over all permutations P of $(1,\cdots,n)$ into $\frac{n}{2}$ pairs $(i_1,j_1)\cdots(i_{\frac{n}{2}},j_{\frac{n}{2}})$ with $i_1 < j_1,\cdots,i_{\frac{n}{2}} < j_{\frac{n}{2}}$ and $i_1 < i_2 < \cdots < i_{\frac{n}{2}}$.

Finally let's see why these time ordered functions are called Green functions. They are the Green functions of the Klein-Gordon equation, that is,

$$\begin{aligned}\left(\partial_x^2 + m^2\right) < 0|T\Phi(x)\Phi(y)|0 > &= (\partial_x^2 + m_2)\Delta_F(x-y) \\ &= \int \frac{d^4k}{(2\pi)^4}(-k^2+m^2)e^{-ik(x-y)}\frac{i}{k^2-m^2+i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}(-i)e^{-ik(x-y)} = -i\delta^4(x-y).\end{aligned} \tag{2.2.219}$$

This also follows directly from the definition of the time ordered operators and the equal time commutation relations. Consider the time derivative of

$$T\Phi(x)\Phi(y) = \theta(x^0-y^0)\Phi(x)\Phi(y) + \theta(y^0-x^0)\Phi(y)\Phi(x),$$

$$\begin{aligned}\partial_0^x T\Phi(x)\Phi(y) = {}& \theta(x^0-y^0)\dot{\Phi}(x)\Phi(y) + \theta(y^0-x^0)\Phi(y)\dot{\Phi}(x) \\ &+ \left[\partial_0^x\theta(x^0-y^0)\right]\Phi(x)\Phi(y) + \left[\partial_0^x\theta(y^0-x^0)\right]\Phi(y)\Phi(x).\end{aligned} \tag{2.2.220}$$

Recall that $\partial_0^x \theta(x^0 - y^0) = \delta(x^0 - y^0)$ and hence,

$$\partial_0^x T\Phi(x)\Phi(y) = \theta(x^0 - y^0)\dot{\Phi}(x)\Phi(y) + \theta(y^0 - x^0)\Phi(y)\dot{\Phi}(x)$$

$$+\delta(x^0 - y^0)\left[\Phi(x), \Phi(y)\right]. \tag{2.2.221}$$

The last term on the right-hand side of equation (2.2.221) vanishes by the ETCR. Taking another time derivative we have

$$\partial_0^x \partial_x^0 T\Phi(x)\Phi(y) = \theta(x^0 - y^0)\ddot{\Phi}(x)\Phi(y) + \theta(y^0 - x^0)\Phi(y)\ddot{\Phi}(x)$$

$$+\delta(x^0 - y^0)\left[\dot{\Phi}(x), \Phi(y)\right]. \tag{2.2.222}$$

By the ETCR

$$\delta(x^0 - y^0)\left[\dot{\Phi}(x), \Phi(y)\right] = -i\delta^4(x - y) \tag{2.2.223}$$

this yields

$$\partial_x^{0^2} T\Phi(x)\Phi(y) = T\ddot{\Phi}(x)\Phi(y) - i\delta^4(x - y) \tag{2.2.224}$$

and thus,

$$(\partial_x^2 + m^2)T\Phi(x)\Phi(y) = T(\partial_x^2 + m^2)\Phi(x)\Phi(y) - i\delta^4(x - y). \tag{2.2.225}$$

The first term on the right-hand side vanishes by the Euler-Lagrange field equations. Taking the vacuum expectation value with and using $< 0|0 >= 1$, we obtain

$$(\partial_x^2 + m^2) < 0|T\Phi(x)\Phi(y)|0 >= -i\delta^4(x - y). \tag{2.2.226}$$

Further, applying the Klein-Gordon equation to our lemma equation (2.2.212), we find

$$(\partial_x^2 + m^2) < 0|T\Phi(x)\Phi(x_1)\cdots\Phi(x_n)|0 >$$

$$= \sum_{j=1}^{n} -i\delta^4(x - x_j) < 0|T\Phi(x_1)\cdots\cancel{\Phi(x_j)}\cdots\Phi(x_n)|0 > . \tag{2.2.227}$$

Alternatively one can derive equation (2.2.227) from the general definition of the time ordered product of operators, the Euler-Lagrange equations and the ETCR. This could then be viewed as the definition of the Green functions, the solution to these equations will simply be equation (2.2.213), that is the time ordered function Wick's Theorem with the two point function given by the Feynman propagator.

Before leaving the scalar field let's consider the case where we have two hermitian scalar fields Φ_1 and Φ_2. Suppose the Lagrangian is

$$\hat{\mathcal{L}} = N[\mathcal{L}]$$
$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1 + \frac{1}{2}\partial_\mu\Phi_2\partial^\mu\Phi_2 - \frac{1}{2}m_1^2\Phi_1^2 - \frac{1}{2}m_2^2\Phi_2^2. \qquad (2.2.228)$$

In addition to our usual space-time symmetries generated by $\hat{\mathcal{P}}^\mu$ and $\hat{\mathcal{M}}^{\mu\nu}$ suppose we require this system to have an internal symmetry which considers Φ_1 and Φ_2 as components of a two dimensional vector, and the symmetry to be rotational invariance of these components into each other. More specifically suppose $U(\alpha) = e^{i\alpha Q}$ is the unitary operator generating these internal rotations. Only one charge Q is needed since the vector is two dimensional. We now define

$$U^\dagger(\alpha)\Phi(x)U(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \Phi(x) \qquad (2.2.229)$$

where

$$\Phi(x) \equiv \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}. \qquad (2.2.230)$$

Pictorially we can imagine rotations in "field space" as depicted in Figure 2.2.8.

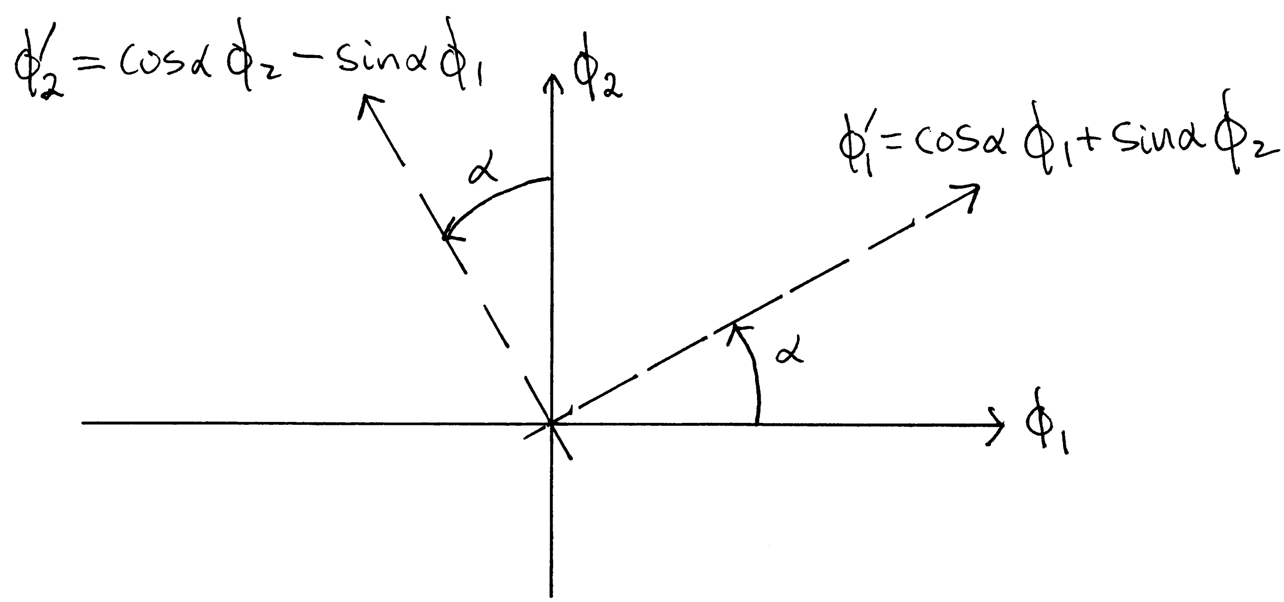

Figure 2.2.8

For infinitesimal rotations, that is, infinitesimal α, $U = 1 + i\alpha Q$ and thus,

$$(1 - i\alpha Q)\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}(1 + i\alpha Q) = \begin{pmatrix}1 & \alpha\\ -\alpha & 1\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}\Phi_1 + \alpha\Phi_2\\ \Phi_2 - \alpha\Phi_1\end{pmatrix}. \tag{2.2.231}$$

This implies

$$\begin{aligned}-i\alpha\,[Q,\Phi_1] &= \alpha\Phi_2\\ -i\alpha\,[Q,\Phi_2] &= -\alpha\Phi_1\end{aligned} \tag{2.2.232}$$

or alternatively,

$$\begin{aligned}[Q,\Phi_1] &= i\Phi_2 \equiv -i\overline{\delta}\Phi_1\\ [Q,\Phi_2] &= -i\Phi_1 \equiv -i\overline{\delta}\Phi_2.\end{aligned} \tag{2.2.233}$$

We may express these relations in the notation discussed in section **2.1** on Noether's Theorem

$$[Q,\Phi_r] \equiv -iD_{rs}\Phi_s \tag{2.2.234}$$

where $r = s = 1, 2$ and

$$D_{rs} = \begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}. \tag{2.2.235}$$

Now we check if the Lagrangian is invariant under such rotations. It is sufficient to consider infinitesimal rotations. So

$$\delta\mathcal{L} = \overline{\delta}\mathcal{L} = -i\,[Q,\mathcal{L}] \tag{2.2.236}$$

and performing the variations of the Lagrangian we find (note, we have no operator ordering problems since $[\delta\Phi_1,\Phi_1] = 0 = [\delta\Phi_2,\Phi_2]$)

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\Phi_r}\delta\Phi_r + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}\partial_\mu\delta\Phi_r\\ &= \partial_\mu\Phi_1\left(\partial^\mu\delta\Phi_1\right) + \partial_\mu\Phi_2\left(\partial^\mu\delta\Phi_2\right) - m_1^2\Phi_1\delta\Phi_1 - m_2^2\Phi_2\delta\Phi_2\\ &= \partial_\mu\Phi_1\left(-\partial^\mu\Phi_2\right) + \partial_\mu\Phi_2\left(\partial^\mu\Phi_1\right) + m_1^2\Phi_1\Phi_2 - m_2^2\Phi_2\Phi_1\\ &= \left(m_1^2 - m_2^2\right)\Phi_1\Phi_2.\end{aligned} \tag{2.2.237}$$

If the masses of the particles are equal, $m_1 = m_2 \equiv m$, then $\delta\mathcal{L} = 0$ and internal rotational symmetry is a good symmetry of the system. By Noether's theorem we have a conserved current

$$\begin{aligned}J^\mu &\equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_1}\delta\Phi_1 + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_2}\delta\Phi_2\\ &= \partial^\mu\Phi_1\left(-\Phi_2\right) + \partial^\mu\Phi_2\left(\Phi_1\right)\\ &= \Phi_1 \overleftrightarrow{\partial^\mu}\,\Phi_2.\end{aligned} \tag{2.2.238}$$

(Again note that $\hat{J}^\mu = N[J^\mu] = J^\mu$ since Φ_1 and Φ_2 commute.) Let's check explicitly that this current is conserved

$$\partial_\mu J^\mu = -\left(\partial^2 \Phi_1\right) \Phi_2 + \Phi_1 \partial^2 \Phi_2 \tag{2.2.239}$$

where the cross terms cancel. However, each field obeys the Klein-Gordon equation

$$\frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} = 0 = -\left(\partial^2 + m_r^2\right) \Phi_r \tag{2.2.240}$$

where r=1,2. Thus,

$$\begin{aligned} \partial_\mu J^\mu &= m_1^2 \Phi_1 \Phi_2 - m_2^2 \Phi_1 \Phi_2 \\ &= \left(m_1^2 - m_2^2\right) \Phi_1 \Phi_2 \end{aligned} \tag{2.2.241}$$

and

$$\delta \mathcal{L} = \partial_\mu J^\mu \tag{2.2.242}$$

as Noether's Theorem required. Hence for equal masses, $m_1 = m_2 = m$, we have an invariant Lagrangian and a conserved current

$$\begin{aligned} \delta \mathcal{L} &= 0 \\ \partial_\mu J^\mu &= 0. \end{aligned} \tag{2.2.243}$$

The charge Q is given by Noether's Theorem accordingly

$$\begin{aligned} Q &= \int d^3x J^0 \\ &= \int d^3x \left[\Phi_1 \Pi_2 - \Phi_2 \Pi_1\right] \end{aligned} \tag{2.2.244}$$

which is independent of time, that is, $\dot{Q} = 0$. So we check using the ETCR that this is indeed Q

$$\begin{aligned} [Q, \Phi_1] &= \int d^3y \left[-\Phi_2 \Pi_1, \Phi_1\right] \\ &= i \int d^3y \delta^3(x-y) \Phi_2 \\ &= i \Phi_2 \\ [Q, \Phi_2] &= \int d^3y \left[\Phi_1 \Pi_2, \Phi_2\right] \\ &= -i \Phi_1. \end{aligned} \tag{2.2.245}$$

The group multiplication law that $U(\alpha)$ obeys is

$$\begin{aligned} U(\alpha_1)U(\alpha_2) &= e^{i\alpha_1 Q}e^{i\alpha_2 Q} \\ &= e^{i(\alpha_1+\alpha_2)Q} \\ &= U(\alpha_1+\alpha_2). \end{aligned} \tag{2.2.246}$$

This is just the product law for the Abelian group O(2) or more generally called U(1), the Abelian group of phase transformations. To make this explicit consider defining the non-hermitian (that is, complex) fields Φ and $\Phi^\dagger$ in terms of the two real fields in the following manner

$$\begin{aligned} \Phi &\equiv \frac{1}{\sqrt{2}}\left(\Phi_1 - i\Phi_2\right) \\ \Phi^\dagger &\equiv \frac{1}{\sqrt{2}}\left(\Phi_1 + i\Phi_2\right). \end{aligned} \tag{2.2.247}$$

Under these $O(2)$ or $U(1)$ rotations we find

$$\begin{aligned} [Q,\Phi] &= \frac{1}{\sqrt{2}}\left([Q,\Phi_1] - i\,[Q,\Phi_2]\right) \\ &= \frac{1}{\sqrt{2}}\left(i\Phi_2 - i\,(-i)\,\Phi_1\right) \\ &= \frac{-1}{\sqrt{2}}\left(\Phi_1 - i\Phi_2\right) \\ [Q,\Phi] &= -\Phi, \end{aligned} \tag{2.2.248}$$

and similarly

$$\left[Q,\Phi^\dagger\right] = +\Phi^\dagger. \tag{2.2.249}$$

So calculating the multiple commutators trivially, we have for finite phase transformations

$$U^\dagger(\alpha)\Phi U(\alpha) = e^{-i\alpha Q}\Phi e^{i\alpha Q} = e^{i\alpha}\Phi \tag{2.2.250}$$

and

$$U^\dagger(\alpha)\Phi^\dagger U(\alpha) = e^{-i\alpha}\Phi^\dagger. \tag{2.2.251}$$

The O(2) rotation transformation of the real fields is just the U(1) phase symmetry transformation for the complex fields. Futhermore, all the quantities we have can be rewritten in terms of Φ and $\Phi^\dagger$ since

$$\Phi_1 = \frac{1}{\sqrt{2}}\left(\Phi + \Phi^\dagger\right) \tag{2.2.252}$$

and

$$\Phi_2 = \frac{i}{\sqrt{2}} \left(\Phi - \Phi^\dagger \right). \tag{2.2.253}$$

Hence,

$$\hat{\mathcal{L}} = N[\mathcal{L}]$$

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi \tag{2.2.254}$$

where the factors of $\frac{1}{2}$ disappear and as expected $\Phi \to e^{+i\alpha}\Phi$; $\Phi^\dagger \to e^{-i\alpha}\Phi^\dagger$ is a symmetry of this Langrangian.

The Euler-Lagrange equations and commutation relations can now be formulated in terms of Φ and $\Phi^\dagger$. We obtain the results directly by treating Φ and $\Phi^\dagger$ as independent fields

$$\frac{\partial \mathcal{L}}{\partial \Phi^\dagger} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^\dagger} = 0 = -\left(\partial^2 + m^2 \right) \Phi$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} = 0 = -\left(\partial^2 + m^2 \right) \Phi^\dagger. \tag{2.2.255}$$

The canonical momenta for the complex fields are Π and $\Pi^\dagger$. So

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^\dagger$$

$$\Pi^\dagger \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\dagger} = \dot{\Phi}. \tag{2.2.256}$$

and the equal time commutation relations become

$$\left[\dot{\Phi}^\dagger(\vec{x}, t), \Phi(\vec{y}, t) \right] = -i\delta^3(\vec{x} - \vec{y}) \tag{2.2.257}$$

or conjugating

$$\left[\dot{\Phi}(\vec{x}, t), \Phi^\dagger(\vec{y}, t) \right] = -i\delta^3(\vec{x} - \vec{y}) \tag{2.2.258}$$

while

$$[\Phi(\vec{x}, t), \Phi(\vec{y}, t)] = 0$$

$$\left[\Phi(\vec{x}, t), \Phi^\dagger(\vec{y}, t) \right] = 0$$

$$\left[\dot{\Phi}(\vec{x}, t), \dot{\Phi}(\vec{y}, t) \right] = 0$$

$$\left[\dot{\Phi}(\vec{x},t),\dot{\Phi}^\dagger(\vec{y},t)\right]=0. \tag{2.2.259}$$

In addition, we have the current and associated charge in terms of the complex fields

$$J^\mu = iN\left[\Phi^\dagger \overleftrightarrow{\partial^\mu} \Phi\right]$$

$$Q = i\int d^3xN\left[\Phi^\dagger\Pi^\dagger - \Pi\Phi\right]. \tag{2.2.260}$$

To further physically interpret this complex scalar field let's Fourier transform to momentum space. As previously derived we have the momentum decomposition for the two Hermitian fields

$$\Phi_1(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a_1(\vec{k})e^{-ikx} + a_1^\dagger(\vec{k})e^{+ikx}\right] \tag{2.2.261}$$

and

$$\Phi_2(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a_2(\vec{k})e^{-ikx} + a_2^\dagger(\vec{k})e^{+ikx}\right]. \tag{2.2.262}$$

The operator $a_r^\dagger(\vec{k})$ creates particles of type r with momentum $\vec{k}$, mass m, spin zero and energy ω_k. Since $[H,Q]=0$, the Q eigenvalues will also label the states. As a consequence of the fact that Φ and $\Phi^\dagger$ diagonalize the "rotation matrix", the states created by them will have a definite charge, that is

$$\begin{aligned}\Phi &= \frac{1}{\sqrt{2}}\left(\Phi_1 - i\Phi_2\right)\\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\frac{1}{\sqrt{2}}\left(a_1(\vec{k}) - ia_2(\vec{k})\right)e^{-ikx} + \frac{1}{\sqrt{2}}\left(a_1^\dagger(\vec{k}) - ia_2^\dagger(\vec{k})\right)e^{+ikx}\right]\end{aligned} \tag{2.2.263}$$

where the complex creation and annihilation operators are given by

$$a(\vec{k}) \equiv \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - ia_2(\vec{k})\right]$$

$$b^\dagger(\vec{k}) \equiv \frac{1}{\sqrt{2}}\left[a_1^\dagger(\vec{k}) - ia_2^\dagger(\vec{k})\right]. \tag{2.2.264}$$

Note that $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$ are independent, that is,

$$a^\dagger(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1^\dagger(\vec{k}) + ia_2^\dagger(\vec{k})\right] \neq b^\dagger(\vec{k}). \tag{2.2.265}$$

So we have the momentum decomposition of the complex fields

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{+ikx}\right] \tag{2.2.266}$$

and

$$\Phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[b(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{+ikx}\right]. \tag{2.2.267}$$

Hence, by inverting our Fourier transform we have

$$\begin{aligned}
a(\vec{k}) &= i\int d^3x[e^{ikx} \overleftrightarrow{\partial_0} \Phi(x)] \\
a^\dagger(\vec{k}) &= i\int d^3x[\Phi^\dagger(x) \overleftrightarrow{\partial_0} e^{-ikx}] \\
b(\vec{k}) &= i\int d^3x[e^{ikx} \overleftrightarrow{\partial_0} \Phi^\dagger(x)] \\
b^\dagger(\vec{k}) &= i\int d^3x[\Phi(x) \overleftrightarrow{\partial_0} e^{-ikx}].
\end{aligned} \tag{2.2.268}$$

The CCR for a and b follow as usual from the Φ and Π ETCR

$$\begin{aligned}
\left[a(\vec{k}), a^\dagger(\vec{l}\,)\right] &= (2\pi)^3 2\omega_k \delta^3(\vec{x} - \vec{l}\,) \\
\left[b(\vec{k}), b^\dagger(\vec{l}\,)\right] &= (2\pi)^3 2\omega_k \delta^3(\vec{x} - \vec{l}\,)
\end{aligned} \tag{2.2.269}$$

all other commutators vanish. As before

$$\begin{aligned}
\hat{H} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] \\
\vec{\mathcal{P}} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\vec{k} \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right]
\end{aligned} \tag{2.2.270}$$

and

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right]. \tag{2.2.271}$$

Thus, we again have two types of particles "a" and "b" type. In addition note that

$$\left[Q, a(\vec{k})\right] = -a(\vec{k})$$

$$\left[Q, a^\dagger(\vec{k})\right] = +a^\dagger(\vec{k})$$
$$\left[Q, b(\vec{k})\right] = +b(\vec{k})$$
$$\left[Q, b^\dagger(\vec{k})\right] = -b^\dagger(\vec{k}). \tag{2.2.272}$$

Since $[\mathcal{P}^\mu, Q] = 0$ we label the one particle states with $\vec{k}$, m^2, and the eigenvalues of Q which are ± 1 on this subspace. So the Hilbert space of states is built up from the vacuum state, $|0>$, which is defined as the no particle state, $a(\vec{k})|0>= 0$ and $b(\vec{k})|0>= 0$. As a result, $\hat{H}|0>= \vec{\mathcal{P}}|0>= Q|0>= 0$. The one particle states are constructed by

$$|\vec{k}, +> \equiv a^\dagger(\vec{k})|0>$$
$$|\vec{k}, -> \equiv b^\dagger(\vec{k})|0> . \tag{2.2.273}$$

So using the charge commutation relations $\left[Q, a(\vec{k})\right] = -a(\vec{k})$, etc., we find

$$\begin{aligned} Q|\vec{k}, +> &= Qa^\dagger(\vec{k})|0>= \left[Q, a^\dagger(\vec{k})\right]|0> \\ &= +a^\dagger(\vec{k})|0>= +|k, +> \end{aligned} \tag{2.2.274}$$

and similarly $Q|\vec{k}, ->= -|\vec{k}, ->$. Thus, "a" particles have a U(1) charge of +1 unit and "b" particles have a unit of negative charge, −1. This could be electric charge or hypercharge.

As we have done previously we can introduce a positively charged particle number operator

$$N_+ \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger(\vec{k})a(\vec{k}) \tag{2.2.275}$$

and a negatively charged particle number operator

$$N_- \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} b^\dagger(\vec{k})b(\vec{k}). \tag{2.2.276}$$

The total number operator is given by $N_\infty = N_+ + N_-$ while the charge operator is $Q = N_+ - N_-$. Thus, the many particle state is

$$\begin{aligned} |n_+, n_- > &\equiv |(\vec{k}_1, +); (\vec{k}_2, +); \cdots; (\vec{k}_{n_+}, +); (\vec{l}_1, -); (\vec{l}_2, -); \cdots; (\vec{l}_{n_-}, -)> \\ &= a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_{n_+}) b^\dagger(\vec{l}_1)\cdots b^\dagger(\vec{l}_{n_-})|0> \end{aligned} \tag{2.2.277}$$

and as usual

$$\hat{H}|n_+, n_- >= \left(\sum_{i=1}^{n_+} \omega_{k_i} + \sum_{j=1}^{n_-} \omega_{l_j}\right) |n_+, n_- >$$

$$\vec{\mathcal{P}}|n_+, n_- >= \left(\sum_{i=1}^{n_+} \vec{k}_i + \sum_{j=1}^{n-} \vec{l}_j\right) |n_+, n_- >$$

$$Q|n_+, n_- >= (n_+ - n_-)\,|n_+, n_- >$$

$$N_\infty|n_+, n_- >= (n_+ + n_-)\,|n_+, n_- > . \tag{2.2.278}$$

The particles are identical except for their charge. This pairing of oppositely charged particles has a deep reason associated with CPT invariance and the “a” and “b” particles are called particle and antiparticle (which is which is convention).

Finally, the CCR imply the (all time) covariant commutation relations

$$[\Phi(x), \Phi(y)] = 0 \tag{2.2.279}$$

and

$$\left[\Phi(x), \Phi^\dagger(y)\right] = i\Delta(x - y). \tag{2.2.280}$$

Furthermore,

$$< 0|\Phi(x)\Phi^\dagger(y)|0 >= i\Delta^+(x - y) \tag{2.2.281}$$

and

$$< 0|T\Phi(x)\Phi^\dagger(y)|0 >= i\Delta_F(x - y) \tag{2.2.282}$$

while

$$< 0|T\Phi(x)\Phi(y)|0 >=< 0|T\Phi^\dagger(x)\Phi^\dagger(y)|0 >= 0. \tag{2.2.283}$$

Wick’s theorem has an analogous form to that previously obtained. We are now ready to consider particles with spin $\frac{1}{2}$.