§1.3 REVIEW OF QUANTUM MECHANICS

In quantum mechanics the measurements of the physical properties of a system (the observables) are represented by self-adjoint linear operators which will be denoted by A. The eigenvalues $\{a\}$ of A are the possible results of the measurement of A in the various states of the system. Hence, each eigenvalue of A characterizes a state of the system called the eigenstate. We assume that if A is an observable then its eigenvalues completely characterize the states of the system. That is the states of the system,which we denote by Ψ, are represented by vectors in Hilbert space denoted $|\psi>$. The eigenvectors of A, labeled by their associated eigenvalue and denoted by $|a>$, form a complete set in Hilbert space. Thus, $A|a>=a|a>$ and each $|\psi>\in H$ can be expanded as

$$|\psi>=\sum_a <a|\psi>|a>=\sum_a \psi(a)|a> \tag{1.3.1}$$

with $\psi(a)=<a|\psi>$ the wavefunction, which is the inner product of $|\psi>$ with $|a>$. The same comments apply to the case where we have a set of measurements $A_1, A_2, ...A_n$ that can be precisely made simultaneously (in any order). Then the set of eigenvalues $\{a_1, a_2, ...a_n\}$ of $\{A_1, A_2, ...A_n\}$ completely characterizes the possible states of the system. More specifically if several operators commute they can be simultaneously diagonalized and hence, measured precisely; there exists simultaneous eigenvectors $|a_1, a_2, ..., a_n>$ for these $A_1, A_2, ..., A_n$ operators. Since these measurements contain the totality of physical information about the system, the states $|a_1, a_2, ..., a_n>$ form a basis for the Hilbert space of states of the system.

For every system it is assumed that there exists a complete set of commuting observables $A_1, A_2, ...A_n$ whose eigenvectors $|a_1, a_2, ..., a_n>$ form a basis of the Hilbert space, that is, they are complete. Hence, any state may be written as a unique superposition of these states

$$|\psi>=\sum_{a_1,...,a_n} \psi(a_1, ..., a_n)|a_1, ..., a_n> \tag{1.3.2}$$

where the wavefunction $\psi(a_1, a_2, ..., a_n)=<a_1, a_2, ..., a_n|\psi>$. Since the states belonging to different eigenvalues are orthogonal (suppose they are normalized to unity)

$$<a_1, ..., a_n|a'_1, ..., a'_n>=\delta_{a_1 a'_1}\cdots\delta_{a_n a'_n}. \tag{1.3.3}$$

The probability that a system in the state Ψ described by the vector $|\psi>$ is observed in the state Φ desribed by the vector $|\phi>$ is

$$|<\phi|\psi>|^2 \tag{1.3.4}$$

and is called the transition probability. Since the state vector

$$|\hat{\psi}>= e^{i\alpha}|\psi>, \qquad \alpha \in R, \tag{1.3.5}$$

has the same probability to be observed in state Φ as the vector $|\psi>$

$$|<\phi|\hat{\psi}>|^2 = |<\phi|\psi>|^2, \tag{1.3.6}$$

they are equivalent representatives of the state Ψ of the system. Thus, unit rays in Hilbert space are in one-to-one correspondence with the states of the system (the set of vectors differing by a phase is called a unit ray). Any choice of a vector $|\psi>$ from the unit ray is an equivalent representative of the state Ψ, there is no measurable effect for different choices. We can choose this phase for convenience.

Symmetry transformations set up a one-to-one correspondence between the states of a system $\Psi \leftrightarrow \Psi'$ such that all transition probabilities are preserved

$$|<\phi'|\psi'>|^2 = |<\phi|\psi>|^2. \tag{1.3.7}$$

For instance, from the Schrödinger or active view of relativistic transformations, the Poincare' group element $g = (a, \Lambda)$ describes the difference between the position, velocity, and spatial orientation of two identical systems. Then each physical state Ψ corresponds to a transformed state Ψ_g. Ψ_g differs from Ψ by the location of the system and its measuring apparati; that is, depending on g, the state Ψ_g is prepared with the laboratory equipment translated, rotated, and boosted relative to the initial position in which the state Ψ was prepared. Thus, g sets up a one-to-one correspondence between the unit rays in Hilbert space. So the Poincare' transformations can be viewed as leaving the Hilbert space of states invariant. Each vector is mapped into another vector of the space.

Further, the transition probability to go from $\Phi_g \rightarrow \Psi_g$ describes the experiment for measuring the transition probability to go from $\Phi \rightarrow \Psi$ but with the laboratory equipment translated, rotated, and boosted according to $g = (a, \Lambda)$. This correspondence of states describes a Poincare' symmetry transformation if

the transition probability for $\Phi \to \Psi$ is the same as the transition probability for $\Phi_g \to \Psi_g$, that is

$$| < \phi|\psi > |^2 = | < \phi_g|\psi_g > |^2. \tag{1.3.8}$$

The fact that symmetry transformations can be represented by a unique (up to a phase) unitary or antiunitary operator on the state vectors has been proven by Wigner . The theorem is stated here without proof.

Theorem 1. *Wigner's Theorem.*

If the one-to-one symmetry transformation $\Psi \leftrightarrow \Psi'$ mapping unit rays into unit rays in the Hilbert space $\mathcal{H}$ preserves transition probabilities

$$| < \phi'|\psi' > |^2 = | < \phi|\psi > |^2, \tag{1.3.9}$$

then there exists a unitary or antiunitary operator U, unique up to a phase, such that

$$|\psi' >= U|\psi > . \tag{1.3.10}$$

(If U is unitary with $|\psi' >= U|\psi >$ then $< \phi'|\psi' >=< \phi|\psi >$,if V is antiunitary with $|\psi' >= U|\psi >$ then $< \phi'|\psi' >= \overline{< \phi|\psi >} =< \psi|\phi >$. In both cases $| < \phi'|\psi' > |^2 = | < \phi|\psi > |^2$.) The expectation values for an observable A should remain unchanged

$$\begin{aligned} < \psi|A|\psi > &= \sum_{a,b} < \psi|a >< a|A|b >< b|\psi > \\ &= \sum_{a} | < \psi|a > |^2 < a|A|a > \\ &= \sum_{a} \; a \;\; | < \psi|a > |^2 \end{aligned} \tag{1.3.11}$$

by the symmetry operation and that we are summing over all eigenvalues, we have

$$\begin{aligned} < \psi|A|\psi > &= \sum_{a} \; a \;\; | < \psi'|a > |^2 \\ &=< \psi'|A'|\psi' >=< \psi|U^{-1}A'U|\psi > \end{aligned} \tag{1.3.12}$$

so that $A' = UAU^{-1}$.

One of the most important symmetry operations on our system will be that corresponding to relativistic invariance. As seen above that means that transition

probabilities should be unchanged under the action of the Poincare' group. Before discusssing the representations of the Lorentz and Poincare' groups let's return to Wigner's theorem to show that we can always choose a unitary operator on our Hilbert space of states to represent the action of a Poincare' transformation. For the relativistic transformation (a,Λ) each state Ψ of our system is transformed into another state $\Psi_{(a,\Lambda)}$ of our system (active or Schrödinger interpretation). The observables are described the same in each frame they are just translated or rotated from their previous location. Hence, in each frame the measuring apparatus yields the same set of possible states for the system, the totality of eigenvalues is unchanged. That is, the Poincare' transformation (a,Λ) generates a one-to-one correspondence of unit rays in the Hilbert space of states. The transition probability to go from state Ψ to Φ is the same as that measured to go from $\Psi_{(a,\Lambda)}$ to $\Phi_{(a,\Lambda)}$ if this correspondence is to be a symmetry transformation. Consequently, we require that

$$| < \phi_{(a,\Lambda)} | \psi_{(a,\Lambda)} > |^2 = | < \phi | \psi > |^2 \tag{1.3.13}$$

for any state vectors representing the states of our system. By Wigner's theorem the correspondence of states under our symmetry transformation is realized by a unique unitary or antiunitary operator (up to a phase) taking one state vector into another, that is, corresponding to (a,Λ) we have the operator U(a,Λ) such that

$$|\psi > \rightarrow |\psi_{(a,\Lambda)} > = U(a,\Lambda)|\psi > . \tag{1.3.14}$$

Since the transformation of reference frame is a macroscopic process as well, we further assume that the transition probabilities depend continuously on the transformation parameters;

$$| < \phi | \psi_{(a,\Lambda)} > |^2 = | < \psi | U(a,\Lambda) | \psi > |^2 \tag{1.3.15}$$

is a continuous function of (a, Λ). Hence, the matrix elements of U(a, Λ) are continuous functions of the parameters. Since $\mathcal{P}_+^{\uparrow}$ contains the identity we can show that the operator U(a, Λ) representing its transformations is unitary. In general, consider two successive Poincare' transformations $(a_1, \Lambda_1)(a_2, \Lambda_2)$

$$|\psi_{(a_1,\Lambda_1)(a_2,\Lambda_2)} > = U(a_1,\Lambda_1)U(a_2,\Lambda_2)|\psi >, \tag{1.3.16}$$

equivalently we must be able to reach the same frame directly by one Poincare' transformation

$$(a,\Lambda) = (a_1,\Lambda_1)(a_2,\Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2). \tag{1.3.17}$$

Since it is a unit ray that represents a state in the Hilbert space, $|\psi(a,\Lambda) >$ and $\omega|\psi(a,\Lambda) >$, where ω is a phase (i.e. $|\omega| = 1$), are equivalent descriptions of the same state. Thus, we have according to Wigner's theorem

$$U(a,\Lambda)|\psi >= |\psi_{(a,\Lambda)} >= \omega(a_1,\Lambda_1;a_2,\Lambda_2)U(a_1,\Lambda_1)U(a_2,\Lambda_2)|\psi > \tag{1.3.18}$$

where the phase $\omega(1,2)$ depends on our choice of representations $U(a,\Lambda)$. So we find the multiplication law for representations is the same as the group up to a phase, for $(a,\Lambda) = (a_1,\Lambda_1)(a_2,\Lambda_2)$

$$U(a,\Lambda) = \omega(1,2)U(a_1,\Lambda_1)U(a_2,\Lambda_2). \tag{1.3.19}$$

Now every translation or restricted Lorentz transformation can be written as a square; $(a,\Lambda) = (a_1,\Lambda_1)(a_1,\Lambda_1)$ so that

$$U(a,\Lambda) = \omega(1,1)U(a_1,\Lambda_1)^2. \tag{1.3.20}$$

Since the square of a unitary or antiunitary operator is unitary

$$\begin{aligned} U^2(c|\psi_1 > +|\psi_2 >) &= cU^2|\psi > +U^2|\psi_2 > \\ &< U^2\phi|U^2\psi >=< \phi|\psi > \end{aligned} \tag{1.3.21}$$

every restricted Poincare' transformation can be represented by a unitary operator. Note also that since $\mathcal{P}_+^\uparrow$ contains the identity and is connected, similarly, by the continuity assumption, each representation of an element of $\mathcal{P}_+^\uparrow$ can be built up from infinitesimal representations starting from the identity U(0,**1**) = **1**.

Further, the unitary operator may be multiplied by a phase

$$U'(a,\Lambda) = e^{i\alpha}U(a,\Lambda)$$

without changing the physical meaning of the transformation. Hence, the phase ω in the multiplication law is replaced with

$$\omega'(1,2) = e^{i[\alpha(1)+\alpha(2)-\alpha(a,\Lambda)]}\omega(1,2). \tag{1.3.22}$$

Now $\alpha(a,\Lambda)$ can always be chosen so that $\omega' = \pm 1$ for $\mathcal{P}_+^\uparrow$. Further it can be shown that if we represent ISL(2,C) transformations by unitary operators, $(a,S) \to U(a,S)$, then since the parameter space is simply connected α can be chosen so that $\omega' = +1$. To summarize we quote another theorem of Wigner:

Theorem 2. *The Quantum Mechanical Poincare' Group.*

Every continuous unitary representation up to a phase of $\mathcal{P}_+^\uparrow$ can be brought, by an appropriate choice of phase factor, into the form of a continuous representation (a,S)$\rightarrow$ U(a,S) of the inhomogeneous SL(2,C). The multiplication law becoming

$$U(a,S) = U(a_1,S_1)U(a_2,S_2) \tag{1.3.23}$$

for

$$(a,S) = (a_1 + S_1 a_2 S_1^\dagger, S_1 S_2). \tag{1.3.24}$$

Recall that the inhomogeneous SL(2,C) transformations are defined by

$$\not{x}' = S\ \not{x} S^\dagger + \not{a} \tag{1.3.25}$$

where $\not{a}$ is a two-by-two Hermitian matrix corresponding to the space-time translation by the four vector a^μ. It is understood that the SL(2,C) transformations are performed before the translation. So

$$\not{x}_2 = S_2\ \not{x} S_2^\dagger + \not{a}_2 \tag{1.3.26}$$

and

$$\begin{aligned} \not{x}_1 &= S_1\ \not{x}_2 S_1^\dagger + \not{a}_1 \\ &= S_1 S_2\ \not{x} S_2^\dagger S_1^\dagger + \not{a}_1 + S_1\ \not{a}_2 S_1^\dagger \\ &= S_1 S_2\ \not{x} (S_1 S_2)^\dagger + \not{a}_1 + S_1\ \not{a}_2 S_1^\dagger \\ &\equiv S\ \not{x} S^\dagger + \not{a} \end{aligned} . \tag{1.3.27}$$

Hence, $(a,S) = (a_1,S_1)(a_2,S_2) = (a_1 + S_1 a_2 S_2^\dagger, S_1 S_2)$ gives the composition law for ISL(2,C).

Finally, let's just point out that if we have an operator, perhaps depending on space-time, $A(x)$, an observer in another frame describes the operator in the same way, it is only translated or rotated as compared to the original frame. That is, an observer S' uses $A(x')$ to study states Ψ' while an observer S uses $A(x)$ to study states Ψ. Since the theory is to be relativistically invariant the corresponding matrix elements, the experimental observations, should transform covariantly like a tensor or a spinor, hence,

$$< \phi_{(a,S)} | A^{(\alpha)}(x') | \psi_{(a,s)} > = D^{(\alpha)}_{\ (\beta)}(S) < \phi | A^{(\beta)}(x) | \psi > \tag{1.3.28}$$

where $x'^{\mu} = \Lambda^{\mu}_{\ \nu}(S)x^{\nu} + a^{\mu}$ and $|\psi_{(a,S)}> = U(a,S)|\psi>$ so that

$$U^{-1}(a,S)A^{(\alpha)}(x')U(a,S) = D^{(\alpha)}_{\ \ (\beta)}(S)A^{(\beta)}(x) \tag{1.3.29}$$

is the corresponding transformation law for operators; in particular our field operators will transform thusly. Note that $< \phi_{(a,S)}|A^{(\alpha)}(x')|\psi_{(a,S)} >$ is like the classical field $\varphi'^{(\alpha)}(x')$ transforming as

$$\varphi'^{(\alpha)}(x') = D^{(\alpha)}_{\ \ (\beta)}(S)\varphi^{(\beta)}(x). \tag{1.3.30}$$

Thus quantum mechanical operators transform as

$$U(a,S)A^{(\alpha)}(x)U^{-1}(a,S) = D^{-1(\alpha)}_{\ \ \ \ (\beta)}(S)A^{(\beta)}(x'). \tag{1.3.31}$$

We are now ready to study the quantum mechanical representations of the inhomogeneous SL(2,C). Since U(a,S) is unitary we can always write it as the exponential map. In addition, we have that

$$U(a,S) = U(a,\mathbf{1})U(0,S)$$

where

$$\begin{aligned} U(a,\mathbf{1}) &\equiv e^{ia_{\mu}\mathcal{P}^{\mu}} \\ U(0,S) &\equiv e^{\frac{-i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} \end{aligned} \tag{1.3.32}$$

where the Hermitian (since U is unitary) operators are $\mathcal{P}^{\mu}$ the space-time translation generators identified with the energy and momentum operators and $\mathcal{M}^{\mu\nu}$ are the Lorentz transformation (rotation) generators identified with the angular momentum operators. a_{μ} is just the translation vector $a^{\mu} = \frac{1}{2}Tr[\not{a}\bar{\sigma}^{\mu}]$ while $\omega_{\mu\nu}(S)$ are just the angles of rotation in the $x_{\mu} - x_{\nu}$ plane parameterizing the finite SL(2,C) transformation S, that is

$$S = e^{-\frac{i}{4}\omega_{\mu\nu}(S)\sigma^{\mu\nu}}, \tag{1.3.33}$$

this is related to $\Lambda^{\mu\nu}$ by equation (1.2.122) with

$$\Lambda^{\mu\nu} = \Lambda^{\mu\nu}(S) = \frac{1}{2}Tr\left[S\sigma^{\nu}S^{\dagger}\bar{\sigma}^{\mu}\right]. \tag{1.3.34}$$

For infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}, \tag{1.3.35}$$

with ϵ^{μ} and $\omega^{\mu\nu}$ infinitesimal parameters, we can expand the unitary operators to first order

$$U(a, S) = \mathbf{1} + i\epsilon_{\mu}\mathcal{P}^{\mu} - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} \tag{1.3.36}$$

where now $\omega_{\mu\nu}(S) = \omega_{\mu\nu}$, the infinitesimal rotation angles. Recall that since

$$\begin{aligned} g_{\mu\nu} &= (g_{\alpha\mu} + \omega_{\alpha\mu})g^{\alpha\beta}(g_{\beta\nu} + \omega_{\beta\nu}) \\ &= g_{\nu\mu} + \omega_{\nu\mu} + \omega_{\mu\nu} \end{aligned}, \tag{1.3.37}$$

we find $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric and hence so is the generator $\mathcal{M}_{\mu\nu} = -\mathcal{M}_{\mu\nu}$.

To make more explicit the identification of $\mathcal{M}_{\mu\nu}$ with rotations consider the transformation $U = \mathbf{1} - i\omega_{12}\mathcal{M}^{12}$ describing the change in the state vector. The corresponding infinitesimal coordinate change is

$$\begin{aligned} x'^{0} &= x^{0} \\ x'^{1} &= x^{1} - \omega^{12}x^{2} \\ x'^{2} &= x^{2} + w^{12}x^{1} \\ x'^{2} &= x^{3}. \end{aligned} \tag{1.3.38}$$

This is a rotation in the $x_1 - x_2$ plane. Hence, $-i\mathcal{M}_{ij}$ is the generator of rotations in the $x_i - x_j$ plane for the state vectors and corresponds to the total angular momentum operator

$$\mathcal{J}^{i} \equiv \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk} = (\mathcal{M}_{23}, \mathcal{M}_{31}, \mathcal{M}_{12}). \tag{1.3.39}$$

For an infinitesimal Lorentz boost along the x^1 direction

$$\begin{aligned} x'^{0} &= x^{0} - x^{1}\omega^{01} \\ x'^{1} &= x^{1} - x^{0}\omega^{01} \\ x'^{2} &= x^{2} \end{aligned}$$

$$x'^3 = x^3, \tag{1.3.40}$$

the state vector is transformed by

$$U = 1 - i\omega_{01}\mathcal{M}^{01}. \tag{1.3.41}$$

Hence, $-i\mathcal{M}_{0i}$ generates Lorentz boosts along the i^{th} axis for the state vectors. We write $\mathcal{K}^i$ for the three-vector $\mathcal{K}^i \equiv \mathcal{M}^{0i}$.

Since $\mathcal{P}^\mu, \mathcal{M}^{\mu\nu}$ are the genrators of the Poincare' or SL(2,C) group they obey commutation relations which characterize their group multiplication law (the commutation relations for $\mathcal{P}_+^\uparrow$ and SL(2,C) generators are the same)

$$U(a_1, S_1)U(a_2, S_2) = U(a_1 + S_1 a_2 S_1^\dagger, S_1 S_2)$$

$$U(a, S)^{-1} = U(-S^{-1}aS^{-1\dagger}, S^{-1}). \tag{1.3.42}$$

Using the above laws we find

$$U(a_1, 1)U(a_2, 1) = U(a_1 + a_2, 1) \tag{1.3.43}$$

which implies $[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$. Further, we have

$$U(0, S^{-1})U(a, 1)U(0, S) = U(S^{-1}aS^{-1\dagger}, 1) \tag{1.3.44}$$

that is

$$U(0, S^{-1})e^{ia_\mu \mathcal{P}^\mu}U(0, S) = e^{(\Lambda^{-1}(S)a)_\mu \mathcal{P}^\mu}. \tag{1.3.45}$$

For infinitesimal a^μ this yields

$$U^{-1}(0, S)a_\mu \mathcal{P}^\mu U(0, S) = \Lambda_\mu^{-1\nu}(S)a_\nu \mathcal{P}^\mu \tag{1.3.46}$$

or

$$e^{\frac{i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} a_\lambda \mathcal{P}^\lambda e^{\frac{-i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} = \Lambda_\mu^{-1\nu}(S)a_\nu \mathcal{P}^\mu. \tag{1.3.47}$$

For infinitesimal S we have

$$\begin{aligned} a_\lambda \mathcal{P}^\lambda + a_\lambda \frac{i}{2}\omega_{\mu\nu}[\mathcal{M}^{\mu\nu}, \mathcal{P}^\lambda] &= (\delta_\mu{}^\nu - \omega_\mu{}^\nu)a_\nu \mathcal{P}^\mu \\ &= a_\lambda \mathcal{P}^\lambda - \frac{a^\lambda}{2}(\omega_{\mu\lambda} - \omega_{\lambda\mu})\mathcal{P}^\mu \end{aligned}. \tag{1.3.48}$$

Thus we obtain the commutator

$$[\mathcal{M}^{\mu\nu}, \mathcal{P}^{\lambda}] = i[g^{\lambda\nu}\mathcal{P}^{\mu} - g^{\lambda\mu}\mathcal{P}^{\nu}]. \tag{1.3.49}$$

Finally we obtain the angular momentum commutation relations by considering the infinitesimal S' transformations

$$U(0, S^{-1})U(0, S')U(0, S) = U(0, S^{-1}S'S) \tag{1.3.50}$$

or

$$\begin{aligned} & e^{\frac{i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} e^{\frac{-i}{2}\omega'_{\rho\lambda}\mathcal{M}^{\rho\lambda}} e^{\frac{-i}{2}\omega_{\alpha\beta}(S)\mathcal{M}^{\alpha\beta}} \\ & \qquad = e^{\frac{-i}{2}\omega_{\mu\nu}(S^{-1}S'S)\mathcal{M}^{\mu\nu}}. \end{aligned} \tag{1.3.51}$$

Now the parameter describing the product of Lorentz transformations $S^{-1}S'S$ is found by considering the action of the three successive transformations on x^{μ} . First we transform to

$$x_1^{\mu} = \Lambda^{\mu\nu}(S)x_{\nu}, \tag{1.3.52}$$

then to

$$x_2^{\alpha} = \Lambda^{\alpha\beta}(S')\Lambda_{\beta\nu}(S)x^{\nu}, \tag{1.3.53}$$

and finally back by

$$x_3^{\mu} = \Lambda^{-1\mu\alpha}(S)\Lambda_{\alpha\beta}(S')\Lambda^{\beta\nu}(S)x_{\nu}. \tag{1.3.54}$$

For S' infinitesimal we have

$$\Lambda_{\alpha\beta}(S') = g_{\alpha\beta} + \omega'_{\alpha\beta} \tag{1.3.55}$$

so

$$\begin{aligned} x_3^{\mu} &= \Lambda^{-1\mu\alpha}(S)g_{\alpha\beta}\Lambda^{\beta\nu}(S)x_{\nu} + \Lambda^{-1\mu\alpha}(S)\omega'_{\alpha\beta}\Lambda^{\beta\nu}(S)x_{\nu} \\ &= (g^{\mu\nu} + \Lambda^{-1\mu\alpha}(S)\omega'_{\alpha\beta}\Lambda^{\beta\nu}(S))x_{\nu} \end{aligned}. \tag{1.3.56}$$

Hence, we have that

$$\omega_{\mu\nu}(S^{-1}S'S) = \Lambda^{-1}_{\mu\alpha}(S)\omega'^{\alpha\beta}\Lambda_{\beta\nu}(S) \tag{1.3.57}$$

and thus,

$$\begin{aligned} U(0, S^{-1})\mathcal{M}^{\mu\nu}U(0, S) &= \Lambda^{-1\alpha\mu}(S)\Lambda^{\nu\beta}(S)\mathcal{M}_{\alpha\beta} \\ &= \Lambda^{\mu\alpha}(S)\Lambda^{\nu\beta}(S)\mathcal{M}_{\alpha\beta} \end{aligned}. \tag{1.3.58}$$

Taking S to be infinitesimal also, we find

$$\begin{aligned}\omega_{\rho\lambda}\frac{i}{2}[\mathcal{M}^{\rho\lambda},\mathcal{M}^{\mu\nu}] &= (\omega^{\mu\alpha}g^{\nu\beta}+g^{\mu\alpha}\omega^{\nu\beta})\mathcal{M}_{\alpha\beta}\\ &= \frac{1}{2}\omega_{\rho\lambda}[g^{\rho\mu}g^{\lambda\alpha}g^{\nu\beta}-g^{\lambda\mu}g^{\rho\alpha}g^{\nu\beta}+g^{\mu\alpha}g^{\rho\nu}g^{\lambda\beta}\\ &\qquad - g^{\mu\alpha}g^{\lambda\nu}g^{\rho\beta}]\mathcal{M}_{\alpha\beta}\\ &= \frac{1}{2}\omega_{\rho\lambda}[g^{\rho\mu}\mathcal{M}^{\lambda\nu}-g^{\lambda\mu}\mathcal{M}^{\rho\nu}+g^{\rho\nu}\mathcal{M}^{\mu\lambda}-g^{\lambda\nu}\mathcal{M}^{\mu\rho}].\end{aligned} \tag{1.3.59}$$

We finally secure the angular momentum commutation relations

$$[\mathcal{M}^{\mu\nu},\mathcal{M}^{\rho\lambda}] = i(g^{\mu\lambda}\mathcal{M}^{\nu\rho}-g^{\mu\rho}\mathcal{M}^{\nu\lambda}+g^{\nu\rho}\mathcal{M}^{\mu\lambda}-g^{\nu\lambda}\mathcal{M}^{\mu\rho}). \tag{1.3.60}$$

As before with the space-time differential operators we define

$$\begin{aligned}\mathcal{J}^i &\equiv \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}\\ \mathcal{K}^i &\equiv \mathcal{M}^{0i}\end{aligned} \tag{1.3.61}$$

and see that they obey the algebra

$$\begin{aligned}[\mathcal{J}_i,\mathcal{J}_j] &= +i\epsilon_{ijk}\mathcal{J}_k\\ [\mathcal{K}_i,\mathcal{K}_j] &= -i\epsilon_{ijk}\mathcal{J}_k\\ [\mathcal{J}_i,\mathcal{K}_j] &= +i\epsilon_{ijk}\mathcal{K}_k.\end{aligned} \tag{1.3.62}$$

Hence $\vec{\mathcal{J}}$ are the angular momentum operators, $\vec{\mathcal{K}}$ the boost operators and $\mathcal{P}_\mu$ the translation operators.

In particular let's consider the action of the space-time translations further. In the Heisenberg representation the states are independent of time while the operators depend on time. Thus the time evolution of our operators is determined by the action of P^0. Recall Poincare' invariance implies equation (1.3.28)

$$<\phi_{(a,S)}|A^{(\alpha)}(x')|\psi_{(a,S)}>= D^{(\alpha)}{}_{(\beta)}(S)<\phi|A^{(\beta)}(x)|\psi> \tag{1.3.63}$$

or equation (1.3.31)

$$U(a,S)A^{(\alpha)}(x)U^{-1}(a,S) = D^{-1(\alpha)}{}_{(\beta)}(S)A^{(\beta)}(x'). \tag{1.3.64}$$

For $\not{x}' = \not{x} + \not{a}$ we find

$$e^{ia_\mu \mathcal{P}^\mu} A^{(\alpha)}(x) e^{-ia_\mu \mathcal{P}^\mu} = A^{(\alpha)}(x+a). \tag{1.3.65}$$

For a_μ infinitesimal we expand the exponentials and Taylor expand the operator

$$(1 + ia_\mu \mathcal{P}^\mu) A^{(\alpha)}(x)(1 - ia_\mu \mathcal{P}^\mu) = A^{(\alpha)}(x) + a^\mu \partial_\mu A^{(\alpha)}(x), \tag{1.3.66}$$

which implies

$$ia_\mu [\mathcal{P}^\mu, A^{(\alpha)}(x)] = a^\mu \partial_\mu A^{(\alpha)}(x). \tag{1.3.67}$$

Thus for the translation operator we find

$$[\mathcal{P}^\mu, A^{(\alpha)}(x)] = -i\partial^\mu A^{(\alpha)}(x) = -P^\mu A^{(\alpha)}(x) \tag{1.3.68}$$

with P^μ given by equation (1.2.52). In particular

$$[\mathcal{P}^0, A^{(\alpha)}(x)] = -i\partial^0 A^{(\alpha)}(x). \tag{1.3.69}$$

Calling $P^0 = H$,the Hamiltonian operator, we find the Heisenberg equations of motion for operators in the Heisenberg representation

$$[H, A^{(\alpha)}(x)] = -i\frac{\partial}{\partial t} A^{(\alpha)}(x). \tag{1.3.70}$$

Thus, the quantum mechanical law of dynamics is contained in the representation of the Poincare' group.

We have been considering the Heisenberg representation in which the states are independent of time while the field operators depend on time. Since we are working with fields that depend on space as well, this approach can be cast into a relativistically covariant form more readily than if we worked in the Schrödinger representation in which the states depend on time and the field operators depend on the spatial coordinates only. Even so, it is sometimes useful to work in the Schrödinger as well as other representations. When the states depend on time, $|\psi(t)>_S$ we have that (the subscipt S denotes the Schrödinger picture and the subscript H denotes the Heisenberg picture)

$$|\psi(t-\epsilon)>_S = U(\epsilon,1)|\psi(t)>_S = e^{i\epsilon\mathcal{P}^0}|\psi(t)>_S = |\psi(t)>_S + i\epsilon H|\psi(t)>_S$$

$$= |\psi(t) >_S -\epsilon\frac{\partial}{\partial t}|\psi(t) >_S \ . \tag{1.3.71}$$

This implies

$$+i\frac{\partial}{\partial t}|\psi(t) >_S = H|\psi(t) >_S, \tag{1.3.72}$$

the Schrödinger equation. Now the operators are independent of time and we denote that they are in the Schrödinger representation by a superscript S, so

$$A^S = A^S(\vec{x}) \tag{1.3.73}$$

only depending on $\vec{x}$ with $\frac{dA^S}{dt} = 0$. The time translation operator $U(a^\mu = (t_0 - t, 0, 0, 0), \mathbf{1})$ relates the states at time t to the states at time t_0, that is the time evolution operator, denoted $U(t, t_0)$, solves the Schrödinger equation

$$U(t, t_0) \equiv U(t_0 - t, 1) \tag{1.3.74}$$

and

$$|\psi(t) >_S = U(t, t_0)|\psi(t_0) >_S$$

$$= e^{-iH(t-t_0)}|\psi(t_0) >_S \ . \tag{1.3.75}$$

The time translation operator $U(t, t_0)$ also allows us to relate the Heisenberg and Schrödinger picture operators and states

$$|\psi(t_0) >_H = U^\dagger(t, t_0)|\psi(t) >_S \ . \tag{1.3.76}$$

Conventionally $t_0 = 0$ so $|\psi(0) >_S = |\psi >_H$ and $U(t, 0) \equiv U(t)$ so that

$$|\psi(t) >_S = U(t)|\psi >_H, \tag{1.3.77}$$

and the Heisenberg picture state vectors are independent of time $\frac{d}{dt}|\psi >_H = 0$. Again matrix elements are invariant so we find

$$_S < \phi(t)|A^S(\vec{x})|\psi(t) >_S$$

$$= _H < \phi|U^\dagger(t)A^S(\vec{x})U(t)|\psi >_H$$

$$\equiv _H < \phi|A^H(\vec{x}, t)|\psi >_H \ . \tag{1.3.78}$$

Hence, the operators in each picture are related by

$$A^H(\vec{x},t) = U^\dagger(t)A^S(\vec{x},0)U(t) \tag{1.3.79}$$

and thus, the Schrödinger representation operators are equal to the Heisenberg representation operators at t = 0. ¿From this we see that the Hamiltonian is the same in both representations

$$H^H = U^\dagger(t)H^S U(t) \tag{1.3.80}$$

but

$$[H, U(t)] = 0 \tag{1.3.81}$$

so

$$H^H = H^S = H. \tag{1.3.82}$$

Further the time evolution of the Heisenberg picture operators is governed by the Heisenberg equations of motion

$$-i\frac{\partial}{\partial t}A^H(t) = [H, A^H(t)], \tag{1.3.83}$$

obtained by differentiating equation (1.3.79). Also recall that the canonical commutation relations (CCR) between generalized momentum, P_i, and coordinate, Q_i, operators in the Schrödinger representation are

$$[P_i^S, Q_j^S] = -i\hbar\delta_{ij}. \tag{1.3.84}$$

Operating on the left by $U^\dagger(t)$ and on the right by $U(t)$ we obtain

$$\begin{aligned} e^{+iHt}[P_i^S, Q_j^S]e^{-iHt} &= [P_i^H(t), Q_j^H(t)] \\ &= -i\hbar\delta_{ij}. \end{aligned} \tag{1.3.85}$$

The CCR become equal time commutation relations (ETCR) in the Heisenberg representation

$$\delta(t-t')[P_i^H(t), Q_j^H(t')] = -i\hbar\delta_{ij}\delta(t-t'). \tag{1.3.86}$$

Besides the Schrödinger and the Heisenberg representations the interaction representation introduced by Dirac will be useful in developing a perturbation

theory for the time evolution operator. In order to introduce this representation the Hamiltonian is first split into a sum of a free or noninteracting term, H_0, and an interacting term, H_I, $H = H_0 + H_I$, true in any representation. In particular in the Schrödinger representation we have

$$H = H_0^S + H_I^S. \tag{1.3.87}$$

In the interaction picture the state vectors are to evolve in time as Schrödinger picture states with H_I^{iP} as the Hamiltonian while the operators are to evolve in time as in the Heisenberg representation with H_0^{iP} as the Hamiltonian as if they were free fields. Note that if the interaction were turned off there is no scattering and the interaction representation becomes the Heisenberg representation with the states and operators evolving as free quantities. In the interaction representation then the state vectors have only the time dependence due to scattering. The interaction representation state vectors will be obtained by removing the H_0^{iP} time dependence from the Schrödinger representation state vectors. Consider

$$|\psi(t)>_{iP} \equiv U_0^\dagger(t,t_0)|\psi(t)>_S \tag{1.3.88}$$

where

$$U_0(t,t_0) = e^{-iH_0^S(t-t_0)} \tag{1.3.89}$$

and again conventionally $t_0 = 0$ and

$$U_0(t) \equiv U_0(t,0). \tag{1.3.90}$$

Furthermore, we define the interaction picture operators so as to leave matrix elements invariant

$$_S<\phi(t)|A^S|\psi(t)>_S = {}_{iP}<\phi(t)|A^{iP}(t)|\psi(t)>_{iP} \tag{1.3.91}$$

which implies

$$A^{iP}(t) = U_0^\dagger(t) A^S U_0(t). \tag{1.3.92}$$

Applying this to H_0^S we see that

$$H_0^{iP} = H_0^S \equiv H_0. \tag{1.3.93}$$

While applying it to H_I^S, we have

$$H_I^{iP}(t) = e^{+iH_0 t} H_I^S e^{-iH_0 t}, \tag{1.3.94}$$

generally $[H_0, H_I^S] \neq 0$ so $H_I^{iP}(t) \neq H_I^S$. Differentiating equation (1.3.92) we find

$$-i\frac{\partial}{\partial t} A^{iP}(t) = [H_0, A^{iP}(t)], \tag{1.3.95}$$

the Heisenberg equation of motion with H_0 as the Hamiltonian. Furthermore,

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi(t)>_{iP} &= -H_0|\psi(t)>_{iP} + U_0^\dagger(t) H |\psi(t)>_S \\ &= -H_0|\psi(t)>_{iP} + U_0^\dagger(t) H U_0(t) U_0^\dagger(t)|\psi(t)>_S \\ &= -H_0|\psi(t)>_{iP} + U_0^\dagger(t)(H_0 + H_I^S) U_0(t)|\psi(t)>_{iP} . \end{aligned} \tag{1.3.96}$$

Now $[H_0, U_0(t, t_0)] = 0$, so the free Hamiltonian terms cancel while we define the interaction Hamiltonian in the interaction picture as in equation (1.3.94)

$$H_I^{iP}(t) \equiv U_0^\dagger(t) H_I^s U_0(t) \tag{1.3.97}$$

so we secure

$$i\frac{\partial}{\partial t}|\psi(t)>_{iP} = H_I^{iP}(t)|\psi(t)>_{iP} . \tag{1.3.98}$$

The states in the interaction picture evolve in time according to the Schrödinger equation using the interaction Hamiltonian. Finally, note that the CCR of the Schrödinger picture again become ETCR in the interaction picture

$$\delta(t - t')[P_i^{iP}(t), Q_j^{iP}(t')] = -i\hbar\delta_{ij}\delta(t - t'). \tag{1.3.99}$$

When we try to solve the above Schrödinger equation for the interaction picture time evolution operator $U(t, t_0)$ (here and in what follows we have dropped the "iP" superscript from the time evolution operator, that is we define $U(t, t_0) \equiv U^{iP}(t, t_0) = U_0^\dagger(t) U^S(t, t_0) U_0(t_0)$) so that

$$|\psi(t)>_{iP} = U(t, t_0)|\psi(t_0)>_{iP} \quad) \tag{1.3.100}$$

the time dependence of $H_I^{iP}(t)$ prevents us from simply exponentiating this Hamiltonian to find a solution for $U(t, t_0)$. More directly we can obtain the equation

of evolution for $U(t,t_0)$ by substituting the above into the Schrödinger equation (1.3.98) as follows

$$i\frac{\partial}{\partial t}U(t,t_0)|\psi(t_0)>_{iP}= H_I^{iP}(t)U(t,t_0)|\psi(t_0)>_{iP} \tag{1.3.101}$$

implying

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I^{iP}(t)U(t,t_0). \tag{1.3.102}$$

Since at $t=t_0$ the states are the same, we have the initial condition

$$U(t_0,t_0) = \mathbf{1}. \tag{1.3.103}$$

This differential equation and boundary condition can be written as an integral equation

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I^{iP}(t_1)U(t_1,t_0) \tag{1.3.104}$$

from which we can check directly that

$$U(t_0,t_0) = \mathbf{1}$$

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I^{iP}(t)U(t,t_0). \tag{1.3.105}$$

Note, from the definition of the evolution of the Schrödinger representation states, we have that

$$|\psi(t)>_S= e^{-iH(t-t_0)}|\psi(t_0)>_S\ . \tag{1.3.106}$$

Converting the states to the interaction representation yields

$$U_0(t)|\psi(t)>_{iP}= e^{-iH(t-t_0)}U_0(t_0)|\psi(t_0)>_{iP} \tag{1.3.107}$$

implying

$$U(t,t_0) = U_0^{\dagger}(t)e^{-iH(t-t_0)}U_0(t_0) \tag{1.3.108}$$

that is

$$U(t,t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}. \tag{1.3.109}$$

Since H_0 and H_I do not commute this is an awkward expression which can only be simplified iteratively as will now be done.

This integral equation can be solved iteratively to yield

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I^{iP}(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I^{iP}(t_1) H_I^{iP}(t_2) + \cdots$$

$$+(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I^{iP}(t_1) H_I^{iP}(t_2) \cdots H_I^{iP}(t_n) + \cdots . \tag{1.3.110}$$

To see this consider $H_I^{iP}(t)$ to be replaced by a parameter λ times itself, $\lambda H_I^{iP}(t)$, and U to be a power series in λ

$$U(t) = \sum_{n=0}^{\infty} U_n(t)\lambda^n. \tag{1.3.111}$$

The integral equation for U becomes

$$\sum_{n=0}^{\infty} \lambda^n U_n(t) = 1 - i\sum_{n=1}^{\infty} \lambda^n \int_{t_0}^{t} dt_1 H_I^{iP}(t_1) U_{n-1}(t_1). \tag{1.3.112}$$

Equating like powers of λ, we find for n=0

$$U_0(t) = 1 \tag{1.3.113}$$

and for $n > 0$

$$U_n(t) = -i\int_{t_0}^{t} dt_1 H_I^{iP}(t_1) U_{n-1}(t_1) \tag{1.3.114}$$

yielding

$$U_0(t) = 1$$

$$U_1(t) = -i\int_{t_0}^{t} dt_1 H_I^{iP}(t_1)$$

$$U_2(t) = (-i)^2 \int_{t_0}^{t} dt_1 H_I^{iP}(t_1) \int_{t_0}^{t_1} dt_2 H_I^{iP}(t_2)$$

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$$U_n(t) = (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I^{iP}(t_1) \cdots H_I^{iP}(t_n) \tag{1.3.115}$$

and so on, securing the above iterative solution. Notice that in each term U_n the integration interval is such that the Hamiltonians are ordered by decreasing time; the time of each Hamiltonian is later than the one to its right, $t_1 > t_2 > t_3 > ... > t_n$. They are said to be time ordered.

We can introduce a time ordering operator T such that for arbitrary operators $A_1(t_1) \cdots A_n(t_n)$ it orders the operators chronologically with later times to the left, and earlier times to the right. Specifically we have first the trivial case

$$TA(t) = A(t). \tag{1.3.116}$$

Then for products of two operators we have

$$TA_1(t_1)A_2(t_2) = \begin{cases} A_1(t_1)A_2(t_2), & \text{if } t_1 > t_2; \\ A_2(t_2)A_1(t_1), & \text{if } t_2 > t_1. \end{cases} \tag{1.3.117}$$

These are the 2! ways to order (t_1, t_2). With the help of the step function $\theta(t_1 - t_2)$

$$\theta(t_1 - t_2) = \begin{cases} 1, & \text{if } t_1 > t_2; \\ 0, & \text{if } t_2 > t_1, \end{cases}$$

we write these cases as

$$TA_1(t_1)A_2(t_2) = \theta(t_1 - t_2)A_1(t_1)A_2(t_2) + \theta(t_2 - t_1)A_2(t_2)A_1(t_1). \tag{1.3.118}$$

We can more compactly write the time ordered product as a sum over these two permutations of the times

$$TA_1(t_1)A_2(t_2) = \sum_{(1,2) \overset{P}{\to} (i_1, i_2)} \theta(t_{i_1} - t_{i_2})A_{i_1}(t_{i_1})A_{i_2}(t_{i_2}). \tag{1.3.119}$$

Next for products of three operators we define

$$TA_1(t_1)A_2(t_2)A_3(t_3) = \begin{cases} A_1(t_1)A_2(t_2)A_3(t_3), & \text{if } t_1 > t_2 > t_3; \\ A_1(t_1)A_3(t_3)A_2(t_2), & \text{if } t_1 > t_3 > t_2; \\ A_2(t_2)A_1(t_1)A_3(t_3), & \text{if } t_2 > t_1 > t_3; \\ A_2(t_2)A_3(t_3)A_1(t_1), & \text{if } t_2 > t_3 > t_1; \\ A_3(t_3)A_1(t_1)A_2(t_2), & \text{if } t_3 > t_1 > t_2; \\ A_3(t_3)A_2(t_2)A_1(t_1), & \text{if } t_3 > t_2 > t_1. \end{cases} \tag{1.3.120}$$

These are the 3! ways to order (t_1, t_2, t_3), again we can use the step function to write this as

$$TA_1(t_1)A_2(t_2)A_3(t_3)$$

$$= \theta(t_1 - t_2)\theta(t_2 - t_3)A_1(t_1)A_2(t_2)A_3(t_3)$$

$$+\theta(t_1 - t_3)\theta(t_3 - t_2)A_1(t_1)A_3(t_3)A_2(t_2) + \cdots. \tag{1.3.121}$$

We can more compactly write this by summing over the six permutations of the times

$$TA_1(t_1)A_2(t_2)A_3(t_3)$$

$$= \sum_{(1,2,3)\overset{P}{\rightarrow}(i_1,i_2,i_3)} \theta(t_{i_1} - t_{i_2})\theta(t_{i_2} - t_{i_3})A_{i_1}(t_{i_1})A_{i_2}(t_{i_2})A_{i_3}(t_{i_3}). \tag{1.3.122}$$

And in general we define the time ordering operator to yield

$$TA_1(t_1)....A_n(t_n)$$

$$= \sum_{(1,\ldots,n)\overset{P}{\rightarrow}(i_1,\ldots i_n)} \theta(t_{i_1} - t_{i_2})\theta(t_{i_2} - t_{i_3})\cdots\theta(t_{i_n-1} - t_{i_n})\ A_{i_1}(t_{i_1})......A_{i_n}(t_{i_n}), \tag{1.3.123}$$

where the sum is over all n! ways to order $t_1, ..., t_n$. That is $\sum_P$ is the sum over all permutations P of the n integers $(1, \cdots, n)$ into the order $(i_1, \cdots, i_n)$, each i_j being one of the n integers from 1 through n.

Hence, we can apply the time ordering operator to our product of Hamiltonians

$$TH_I^{iP}(t_1)\cdots H_I^{iP}(t_n) = \sum_P \theta(t_{i_1} - t_{i_2})\cdots\theta(t_{i_n-1} - t_{i_n})H_I^{iP}(t_{i_1})\cdots H_I^{iP}(t_{i_n})$$

$$= H_I^{iP}(t_{a_1})\cdots H_I^{iP}(t_{a_n}), \qquad \text{for } t_{a_1} > t_{a_2} > t_{a_3} > \cdots > t_{a_n}. \tag{1.3.124}$$

Thus, we can use the T operator to extend our region of integration on each integral in U_n, equation (1.3.115), from $t_0 < t_i < t$ since it chronologically orders the Hamiltonian factors

$$U_n(t) = (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I^{iP}(t_1)\cdots H_I^{iP}(t_n)$$

$$= (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_n - 1} dt_n T H_I^{iP}(t_1) \cdots H_I^{iP}(t_n)$$

$$= \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \; T H_I^{iP}(t_1) \cdots H_I^{iP}(t_n). \tag{1.3.125}$$

The $\frac{1}{n!}$ arises from the fact that by integrating from $t_0 < t_i < t$ we are just doing the original integral n! times, just relabeling the dummy integration variables each time we have one of the n! permutations of the times from the definition of T. To make this perfectly clear let's do the U_2 case in detail

$$U_2(t) = \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T H_I^{iP}(t_1) H_I^{iP}(t_2)$$

$$= \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 [\theta(t_1 - t_2) H_I^{iP}(t_1) H_I^{iP}(t_2) + \theta(t_2 - t_1) H_I^{iP}(t_2) H_I^{iP}(t_1)]. \tag{1.3.126}$$

Now in the second term let $t_1 = u_2$ and $t_2 = u_1$ and interchange the order of integration so that

$$U_2(t) = \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \theta(t_1 - t_2) H_I^{iP}(t_1) H_I^{iP}(t_2)$$

$$+ \frac{(-i)^2}{2!} \int_{t_0}^{t} du_1 \int_{t_0}^{t} du_2 \theta(u_1 - u_2) H_I^{iP}(u_1) H_I^{iP}(u_2). \tag{1.3.127}$$

Relabel $u_1 \to t_1$ and $u_2 \to t_2$ to obtain

$$U_2(t) = \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \theta(t_1 - t_2) H_I^{iP}(t_1) H_I^{iP}(t_2)$$

$$+ \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \theta(t_1 - t_2) H_I^{iP}(t_1) H_I^{iP}(t_2)$$

$$= (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \theta(t_1 - t_2) H_I^{iP}(t_1) H_I^{iP}(t_2). \tag{1.3.128}$$

Since

$$\theta(t_1 - t_2) = \begin{cases} 1, & \text{if } t_1 > t_2; \\ 0, & \text{if } t_2 > t_1, \end{cases}$$

we obtain

$$U_2(t) = (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I^{iP}(t_1) H_I^{iP}(t_2) \tag{1.3.129}$$

as desired. Similar arguments apply to $U_n(t)$.

Hence, the time evolution operator in the interaction representation can be written as

$$U(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n T H_I^{iP}(t_1) \cdots H_I^{iP}(t_n). \tag{1.3.130}$$

Formally, we write the sum as an exponential

$$U(t,t_0) = T e^{-i \int_{t_0}^{t} dt' H_I^{iP}(t')} \tag{1.3.131}$$

where again the exponential is understood to stand for the above series expansion equation (1.3.130).

The utility of the interaction representation is realized when we consider scattering experiments. Given any state $|\psi(t_0) >_{iP}$, we can now calculate to any order in the interaction its time evolution

$$|\psi(t) >_{iP} = U(t,t_0)|\psi(t_0) >_{iP} . \tag{1.3.132}$$

In scattering experiments the initial states are prepared in the remote past (as $t \to -\infty$). For instance, two beams of particles of specified momenta and spins can be prepared spatially separate (opposite ends of the lab) so that initially they are not interacting. We imagine, according to the formal theory of scattering, that the interaction Hamiltonian $H_I^{iP}(t)$ is adiabatically, that is very slowly compared to the characteristic interaction times to avoid any energy absorption or emission during the process, switched off in the remote past. A complete set of initial states are then the eigenstates of the free Hamiltonian H_0 since $|\psi >_{iP}$ is time independent if $H_I^{iP} = 0$ and the interaction picture states coincide with the Heisenberg picture states for a system described by H_0 . As time proceeds the interaction is slowly turned on and eventually the particles collide. As time runs foward the collision products start to separate and after a sufficiently long time the interaction adiabatically turns off so that in the remote future ($t \to +\infty$) the final states are again eigenstates of the free Hamiltonian H_0 . This slow switching on and off of

the interaction without disturbing the results of the particle collisions is called the adiabatic hypothesis. Initially the particles are well separated and noninteracting (eigenstates of H_0). They approach each other and interact, producing new particles , scattering and annihilating. The reaction products then separate again into noninteracting final states (also eigenstates of H_0). We shall denote these special non-interacting initial and final states in the interaction representation, which are described by the eigenstates of H_0, by rounded brackets with no subscripts, $|i\supset$ and $|f\supset$, respectively. The transition probabilty amplitude for the system to go from the initial state $|i\supset$ at time $t=-\infty$ to the final state $|f\supset$ at time $t=+\infty$ is given by the scalar product

$$S_{fi}=\subset f|U(+\infty,-\infty)|i\supset . \tag{1.3.133}$$

That is, $U(t,-\infty)$ takes the initial state $|i\supset=|i(-\infty)>_{iP}$ from time $t=-\infty$ to the time t

$$|i(t)\supset=U(t,-\infty)|i(-\infty)>_{iP} . \tag{1.3.134}$$

Hence the probability amplitude for the initial state $|i\supset$ to evolve from $t=-\infty$ into the final state $|f\supset$ at $t=+\infty$ is just

$$\begin{aligned} S_{fi}&=\lim_{t\to+\infty}\subset f|i(t)\supset=\subset f|i(+\infty)\supset \\ &=\subset f|U(+\infty,-\infty)|i\supset . \end{aligned} \tag{1.3.135}$$

S_{fi} is called the scattering matrix element or simply the S-matrix element.

$$S\equiv U(+\infty,-\infty) \tag{1.3.136}$$

is known as the scattering or S-operator. From the perturbation expansion for the time evolution operator $U(t,t_0,)$ we have that

$$\begin{aligned} S&=Te^{-i\int_{-\infty}^{+\infty}dtH_I^{iP}(t)} \\ &=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1\cdots\int_{-\infty}^{+\infty}dt_nTH_I^{iP}(t_1)\cdots H_I^{iP}(t_n). \end{aligned} \tag{1.3.137}$$

Hence the S-matrix element is given by

$$S_{fi}=\subset f|Te^{-i\int_{-\infty}^{+\infty}dtH_I^{iP}(t)}|i\supset . \tag{1.3.138}$$

Recall that the operators in the interaction picture have time dependence determined by the free Hamiltonian H_0 . Hence, we can calculate the transition probability amplitude for the process $i \to f$ by using states and operators determined in terms of the free dynamical variables appearing in H_0 . The price we pay for using noninteracting or free quantities is the insertion of the term $Te^{-i\int dtH_I^{iP}}$ in the transition amplitude. Further, $|S_{fi}|^2$ is a measurable quantity for any reaction. However, it depends upon the details of the initial state preparation, for example it depends on the flux of incoming particles, and on the target particle density. A more intrinsic quantity derived from $|S_{fi}|^2$ is the cross section for a process $\sigma_{fi} \propto |S_{fi}|^2$. We will discuss and calculate cross sections in excruciating detail later.

Next we must apply these perturbative schemes more directly to our quantum field theoretic systems. As seen above, the Hamiltonian plays a central role in the discussion of transition probabilities. For field theories, as we have seen in the introduction, these Hamiltonians are for systems with infinte degrees of freedom. Furthermore, they are relativistic systems and the Hamiltonian approach while singling out the time coordinate often hides the Lorentz invariance of the amplitudes. To maintain manifest Lorentz invariance at every step we will often work in terms of the Lagrangian for the system. The formula for S_{fi} is invariant in either formulation. For interaction Hamiltonians that are independent of momenta we have that $H_I^{iP} = -L_I^{iP}$ and

$$\int_{-\infty}^{+\infty} dt L_I^{iP}$$

is just the interaction action of the system which is manifestly Lorentz invariant as seen by writing it in terms of the Lagrangian density $\mathcal{L}_I^{iP}$

$$L_I^{iP} = \int_{-\infty}^{+\infty} d^3x \mathcal{L}_I^{iP}$$

so

$$\int_{-\infty}^{+\infty} dt L_I^{iP} = \int_{-\infty}^{+\infty} d^4x \mathcal{L}_I^{iP}.$$

The time ordering operator however singles out the time also and it is not clear that it preserves the manifest Lorentz invariance. However, restricted Lorentz transformations can only alter the chronological ordering if we consider operators at space-like separated points. At space-like separations operators must commute,

that is, they must be simultaneously measurable since no signal with $v \leq c$ can be sent between them. That is, the measurement of one operator cannot interfere with the measurement of the other since if the second measurement observed the effect of the first measurement the fact that it was even performed must have been transmitted at $v \geq c$ in order to arrive at the second space-like separated point. This microcausality principle then guarantees that the time ordering for space-like separations is immaterial, that is

$$[\mathcal{H}_I^{iP}(\vec{x}_1, t_1), \mathcal{H}_I^{iP}(\vec{x}_2, t_2)] = 0$$

for $(x_1 - x_2)^2 < 0$. Hence, S_{fi} is in fact Lorentz invariant. (The argument goes through for H_I^{iP} momentum dependent also, the definition of T must change by terms that involve Dirac delta functions in the time.) We are now ready to systematically develop Lagrangian quantum field theory.