§1.2 REVIEW OF SPECIAL RELATIVITY

Einstein's theory of special relativity requires every physical law to be the same in all inertial systems (the form of the law is covariant) and that the speed of light in vacuum is a universal constant independent of the motion of the source. Thus if two frames S and S' are moving at constant velocity relative to each other the relativistic intervals in each frame are equal and given by:

$$ds^2 = c^2 dt^2 - \sum_{i=1}^{3} (dx^i)^2$$

$$ds'^2 = c^2 dt'^2 - \sum_{i=1}^{3} (dx'^i)^2$$

$$ds^2 = ds'^2 \tag{1.2.1}$$

The most general linear transformations which leave this interval invariant are the Lorentz transformations. These can be most simply expressed by introducing four vector notation. Let the space-time coordinates in the frame S be given by $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and in S', $x'^\mu = (ct', x', y', z')$. Then the general linear transformation between S and S' is $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where $\Lambda^\mu{}_\nu$ is a four-by-four matrix. The invariance of the space-time interval, $ds^2 = ds'^2$, implies that $ds'^2 = dx'^\mu g_{\mu\nu} dx'^\nu = dx^\alpha g_{\alpha\beta} dx^\beta = ds^2$ where

$$g_{\mu\nu} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \tag{1.2.2}$$

is the metric tensor. However, $dx'^\mu = \Lambda^\mu{}_\alpha dx^\alpha$ so the equality of the intervals implies a restriction on $\Lambda^\mu{}_\nu$,

$$dx^\alpha \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta dx^\beta = dx^\alpha g_{\alpha\beta} dx^\beta.$$

Thus, $\Lambda^\mu{}_\nu$ must satisfy

$$g_{\alpha\beta} = \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta, \tag{1.2.3}$$

or in matrix notation

$$g = \Lambda^T g \Lambda. \tag{1.2.4}$$

If Λ_1 and Λ_2 satisfy (1.2.3) then so do $\Lambda_1\Lambda_2$ and Λ_1^{-1}. For example, if $\Lambda^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\gamma = \delta^\mu{}_\gamma$ then

$$(\Lambda^{-1})^\alpha{}_\gamma g_{\alpha\beta}(\Lambda^{-1})^\beta{}_\delta = \Lambda^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\gamma g_{\mu\nu}\Lambda^\nu{}_\beta(\Lambda^{-1})^\beta{}_\delta$$

$$= \delta^\mu{}_\gamma g_{\mu\nu}\delta^\nu{}_\delta$$

$$= g_{\gamma\delta}.$$

Thus, relabeling the dummy indices,

$$g_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu g_{\alpha\beta}(\Lambda^{-1})^\beta{}_\nu. \tag{1.2.5}$$

Hence, the Lorentz transformations form a group, the Lorentz group, denoted L (or sometimes SO(1,3)).

In addition to linear transformations, the homogeneity of space-time implies that uniform translations of the frame should not effect experiments. As we see then the transformation of the coordinates

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{1.2.6}$$

with a^μ a constant 4-vector, also leaves the interval invariant, $ds^2 = ds'^2$. Note, above it is understood that the translation a^μ follows the rotation $\Lambda^\mu{}_\nu$. The general inhomogeneous transformation is called a Poincare' transformation. Two such transformations (Λ_1, a_1) and (Λ_2, a_2) obey the composition law

$$x_2^\mu = \Lambda_2^\mu{}_\nu x_1^\nu + a_2^\mu$$

$$= \Lambda_2^\mu{}_\nu \Lambda_1^\nu{}_\rho x^\rho + \Lambda_2^\mu{}_\nu a_1{}_\nu + a_2^\nu$$

$$\equiv \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1.2.7}$$

with

$$\Lambda^\mu{}_\nu \equiv \Lambda_2^\mu{}_\alpha \Lambda_1^\alpha{}_\nu$$

$$a^\mu \equiv \Lambda_2^\mu{}_\nu a_1^\nu + a_2^\mu. \tag{1.2.8}$$

Hence, these transformations also form a group, the Poincare' group denoted by $\mathcal{P}$.

The Lorentz group (and hence, the Poincare' group) contains reflections of the space-time coordinates as well as boosts, rotations, and translations.
1) Space Inversion: Let $\Lambda^\mu_{\ \nu} = P^\mu_{\ \nu}$

$$x'^\mu = P^\mu_{\ \nu} x^\nu \tag{1.2.9}$$

with

$$P^\mu_{\ \nu} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \tag{1.2.10}$$

thus,

$$\begin{aligned} x'^0 &= +x^0 \\ x'^i &= -x^i. \end{aligned} \tag{1.2.11}$$

2) Time Reversal: Let $\Lambda^\mu_{\ \nu} = T^\mu_{\ \nu}$

$$x'^\mu = T^\mu_{\ \nu} x^\nu \tag{1.2.12}$$

with

$$T^\mu_{\ \nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}_{\mu\nu} \tag{1.2.13}$$

thus,

$$\begin{aligned} x'^0 &= -x^0 \\ x'^i &= +x^i. \end{aligned} \tag{1.2.14}$$

3) Space-Time Inversion: Let $\Lambda^\mu_{\ \nu} = I^\mu_{\ \nu}$

$$\begin{aligned} I^\mu_{\ \nu} \equiv P^\mu_{\ \alpha} T^\alpha_{\ \nu} = &\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \\ = &-\delta^\mu_{\ \nu} \end{aligned} \tag{1.2.15}$$

thus,

$$x'^\mu = -x^\mu. \tag{1.2.16}$$

The Lorentz group and, hence, the Poincare' group have four disconnected components. That is not every frame may be related by a sequence of Lorentz transformations differing infinitesimally from each other. More precisely, Lorentz transformations depend on 6 parameters, each transformation can be identified by its particular values for these parameters. One cannot go from one transformation to every other by a continuous change in these parameters. That is the parameter space is disconnected. To see this use the fact that equation (1.2.3) can be written in matrix notation (1.2.4) with Λ a matrix with elements $\Lambda^{\mu}{}_{\nu}$ where μ labels the rows and ν labels the columns. Then with $g = \Lambda^T g \Lambda$, the $\det g = \det \Lambda^T \det g \det \Lambda$, which implies $\det \Lambda = \pm 1$. Furthermore, taking the $\alpha = 0, \beta = 0$ component of equation (1.2.3) we find

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2$$

which implies $(\Lambda^0{}_0)^2 \geq 1$. Hence we have the two possibilities $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. Since $\det \Lambda$ and the sign of $\Lambda^0{}_0$ are continuous functions of the matrix elements, $\Lambda^{\mu}{}_{\nu}$, they must be constant on each component. Hence, we have the four connected components of L denoted by

1) $\mathbf{L}_{+}^{\uparrow}$ in which $\det \Lambda = +1, \Lambda^0{}_0 \geq +1$ and the transformations of which are called the restricted or proper orthochronous Lorentz transformations. This is the only component which contains $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$ the identity and hence is a subgroup of L. The direction of time is unchanged and spatial reflections are absent in this component.

2) $\mathbf{L}_{-}^{\uparrow}$ in which $\det \Lambda = -1, \Lambda^0{}_0 \geq +1$. These transformations can be written as the product of space inversion and $L_{+}^{\uparrow}$, that is $L_{-}^{\uparrow} = P L_{+}^{\uparrow}$.

3) $\mathbf{L}_{+}^{\downarrow}$ in which $\det \Lambda = +1, \Lambda^0{}_0 \leq -1$. These transformations must include both time inversion ($\Lambda^0{}_0 \leq -1$) and space inversion ($\det \Lambda = +1$). They can be written as total coordinate inversions times $L_{+}^{\uparrow}$, that is $L_{+}^{\downarrow} = PT L_{+}^{\uparrow}$.

4) $\mathbf{L}_{-}^{\downarrow}$ in which $\det \Lambda = -1, \Lambda^0{}_0 \leq -1$. The direction of time is changed by this component which can be written as time reversal times $L_{+}^{\uparrow}$, that is $L_{-}^{\downarrow} = T L_{+}^{\uparrow}$.

Symbolically the Lorentz group can be written as the sum of four components:

$$L = L_{+}^{\uparrow} + P L_{+}^{\uparrow} + PT L_{+}^{\uparrow} + T L_{+}^{\uparrow}. \tag{1.2.17}$$

A similar decomposition holds for the Poincare' group $\mathcal{P}$:

$$\mathcal{P} = \mathcal{P}_{+}^{\uparrow} + \mathcal{P}_{-}^{\uparrow} + \mathcal{P}_{+}^{\downarrow} + \mathcal{P}_{-}^{\downarrow}$$

$$= \mathcal{P}_+^\uparrow + P\mathcal{P}_+^\uparrow + PT\mathcal{P}_+^\uparrow + T\mathcal{P}_+^\uparrow, \tag{1.2.18}$$

where the component $\mathcal{P}_+^\uparrow$ is called the restricted or proper orthochronous Poincare' group. It is the only component which is a subgroup of P since it contains the identity $(\Lambda, a) = (\mathbf{1}, 0)$. We will be interested in the restricted Lorentz and Poincare' transformations since we can build up any general transformation from them as indicated above.

Any orthochronous, proper or improper, Lorentz transformation can be written uniquely as a boost followed by a spatial rotation $\Lambda = R\Lambda_B$. Boosts of velocity $\vec{V} = V\vec{v},\ \vec{v}\cdot\vec{v} = 1$ (in what follows we choose units such that c=1) can be written as

$$x'^0 = x^0 \cosh\beta + \vec{v}\cdot\vec{x}\sinh\beta$$

$$\vec{x}' = \vec{x} + \vec{v}(\vec{x}\cdot\vec{v})(\cosh\beta - 1) + x^0\vec{v}\sinh\beta, \tag{1.2.19}$$

with $V = \tanh\beta$. In 4-vector notation this can be written as $x'^\mu = \Lambda_{B\ \nu}^{\mu} x^\nu$ with the pure boost Lorentz transformation matrix given by

$$\Lambda_{B\ \nu}^{\mu} =$$

$$\begin{pmatrix} ch\beta & v^1 sh\beta & v^2 sh\beta & v^3 sh\beta \\ v^1 sh\beta & [1 + v^1v^1(ch\beta - 1)] & v^1v^2(ch\beta - 1) & v^1v^3(ch\beta - 1) \\ v^2 sh\beta & v^2v^1(ch\beta - 1) & [1 + v^2v^2(ch\beta - 1)] & v^2v^3(ch\beta - 1) \\ v^3 sh\beta & v^3v^1(ch\beta - 1) & v^3v^2(ch\beta - 1) & [1 + v^3v^3(ch\beta - 1)] \end{pmatrix}. \tag{1.2.20}$$

Further, recall that any spatial rotation, $x'^i = R_{ij}x^j$, can be represented as a rotation through an angle α about some axis with direction $\vec{u}$ where $\vec{u}\cdot\vec{u} = 1$. For instance, in spherical coordinates $\vec{u}$ is given by $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. For a spatial rotation

$$x'^0 = x^0$$

$$\vec{x}' = \vec{x}\cos\alpha + \vec{u}(\vec{u}\cdot\vec{x})(1 - \cos\alpha) + \vec{x}\times\vec{u}\sin\alpha, \tag{1.2.21}$$

where $0 \le |\alpha| \le \pi$. Hence $x'\ ^\mu = R^\mu{}_\nu x^\nu$ can be written in 4-vector matrix notation as

$$R^\mu{}_\nu =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & [c\alpha + u^1u^1(1-c\alpha)] & [u^1u^2(1-c\alpha) - u^3s\alpha] & [u^1u^3(1-c\alpha) + u^2s\alpha] \\ 0 & [u^2u^1(1-c\alpha) + u^3s\alpha] & [c\alpha + u^2u^2(1-c\alpha)] & [u^2u^3(1-c\alpha) - u^1s\alpha] \\ 0 & [u^3u^1(1-c\alpha) - u^2s\alpha] & [u^3u^2(1-c\alpha) + u^1s\alpha] & [c\alpha + u^3u^3(1-c\alpha)] \end{pmatrix}. \tag{1.2.22}$$

Thus, we see that the Lorentz group is a six parameter group with three components of velocity for boosts and three angles to specify spatial rotations. Since there are four directions for space-time translations, the Poincare' group is a ten parameter group.

Since the laws of physics are covariant in form, physical quantities and the equations they obey are most easily described and managed by tensors and tensorial equations. Tensors are defined to have specific Lorentz transformation properties which, as with tensorial equations, are maintained in every inertial frame. We can define contravariant and covariant tensor fields according to their specific Lorentz transformation properties

$$T'^{\ \mu_1\cdots\mu_n}(x') = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}\ T^{\nu_1\cdots\nu_n}(x) \tag{1.2.23}$$

for a contravariant rank n tensor field and

$$T'_{\mu_1\cdots\mu_n}(x') = (\Lambda^{-1})^{\nu_1}{}_{\mu_1}\cdots(\Lambda^{-1})^{\nu_n}{}_{\mu_n}\ T_{\nu_1\cdots\nu_n}(x)$$
$$(= T_{\nu_1\cdots\nu_n}(x)(\Lambda^{-1})^{\nu_1}{}_{\mu_1}\cdots(\Lambda^{-1})^{\nu_n}{}_{\mu_n}). \tag{1.2.24}$$

for a covariant rank n tensor field and

$$T'^{\ \mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}(x') = \Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda^{\mu_m}{}_{\alpha_m}T^{\alpha_1\cdots\alpha_m}{}_{\beta_1\cdots\beta_n}(x)(\Lambda^{-1})^{\beta_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\beta_n}{}_{\nu_n} \tag{1.2.25}$$

for a mixed (m,n) tensor of rank (m+n), where as before

$$\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\rho} = \delta^{\mu}{}_{\rho}. \tag{1.2.26}$$

Note that

$$V^{'\mu}W^{'}{}_{\mu} = \Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\rho}{}_{\mu}V^{\nu}W_{\rho}$$
$$= \delta^{\rho}{}_{\nu}V^{\nu}W_{\rho} = V^{\mu}W_{\mu}$$

is a Lorentz invariant. We can define the contravariant metric tensor $g^{\mu\nu}$ as the inverse of the covariant metric tensor $g_{\mu\nu}$

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}. \tag{1.2.27}$$

Note that $g = \Lambda^T g \Lambda$ implies that (equation (1.2.5))

$$g'_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} g_{\alpha\beta} = g_{\mu\nu} \tag{1.2.28}$$

and similarly $g'^{\mu\nu} = g^{\mu\nu}$; the contravariant metric tensor is invariant.

Then for every contravariant vector there is an associated covariant vector, and vice versa, obtained by lowering or raising an index with $g_{\mu\nu}$ or $g^{\mu\nu}$. That is if V^μ is a contravariant vector, $V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$, then $V_\mu \equiv g_{\mu\nu} V^\nu$ is a covariant vector since

$$V'_{\mu} = g_{\mu\nu} V'^{\nu} = g_{\mu\nu} \Lambda^{\nu}{}_{\rho} V^{\rho}$$

but, equation (1.2.28), $g_{\mu\nu}\Lambda^{\nu}{}_{\rho} = (\Lambda^{-1})^{\beta}{}_{\mu} g_{\beta\rho}$ so that $V'_{\mu} = (\Lambda^{-1})^{\beta}{}_{\mu} g_{\beta\rho} V^{\rho}$. Hence, $V'_{\mu} = (\Lambda^{-1})^{\beta}{}_{\mu} V_{\beta}$, is the transformation law for covariant vectors. Using this notation to define $\Lambda_{\mu\nu} = g_{\mu\alpha}\Lambda^{\alpha}{}_{\nu}$ etc. we find $\delta^{\alpha}{}_{\beta} = \Lambda^{\mu\alpha}\Lambda_{\mu\beta}$, that is

$$(\Lambda^{-1})^{\mu\nu} = (\Lambda^T)^{\mu\nu} = \Lambda^{\nu\mu}. \tag{1.2.29}$$

We can also relate the contravariant, covariant and mixed tensors by raising and lowering indices with the metric tensors.

Since $\Lambda \in L_{+}^{\uparrow}$ is continuously connected to the identity we can build up any finite Lorentz transformation by making many consecutive small Lorentz transformations. Each coordinate system differing from the previous, as well as the following, by an infinitesimal amount. The properties of Λ can be inferred from the properties of infinitesimal Lorentz transformations. From our experience with spatial rotations we know that every Λ can be written as the exponential of the 6 transformation parameters (the "angles" of rotation), denoted by $\omega^{\alpha\beta}(\Lambda)$, times the six (4x4) matrices that generate the infintesimal Lorentz transformations, denoted by $(D^{\alpha\beta})_{\mu\nu}$. That Lorentz transformations leave the metric invariant implies that the parameters for infinitesimal Lorentz transformations, $\omega^{\alpha\beta}$ with the infinitesimal Lorentz transformation written as $\Lambda^{\alpha\beta} = g^{\alpha\beta} + \omega^{\alpha\beta}$ and the $\omega^{\alpha\beta}$ are infinitesimal, are anti-symmetric

$$g_{\alpha\beta} = \Lambda_{\mu\alpha} g^{\mu\nu} \Lambda_{\nu\beta} = (g_{\mu\alpha} + \omega_{\mu\alpha}) g^{\mu\nu} (g_{\nu\beta} + \omega_{\nu\beta})$$

$$= g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta}$$

which implies that $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$, that is $\omega^{\alpha\beta}$ (and of course $\omega_{\alpha\beta}$) is antisymmetric. Further we know that the Lorentz group multiplication law implies that the $D^{\alpha\beta}$ matrices must obey specific commutation relations. Alternatively, the Lorentz algebra completely characterizes the Lorentz group multiplication law. It is sufficient to find the matrices that obey the desired commutation relations in order to construct the finite Lorentz transformations by exponentiation with appropriate parameters. In fact instead of characterizing tensors by their finite Lorentz transformation properties we can equivalently specify their transformation properties under infinitesimal transformations. That is we will define the tensor by specifying its general $D^{\alpha\beta}$ matrix. This matrix obeys the Lorentz algebra so that when appropriately exponentiated we will recover the finite Lorentz transformation definition of the tensor.

To begin let's go back to just 4x4 matrices and study what is called the fundamental or vector representation of the Lorentz group. It is given by the infinitesimal coordinate transformations, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}$$

$$\equiv x^{\mu} + \frac{\omega^{\beta}{}_{\alpha}}{2} \left(D^{\alpha}{}_{\beta}\right)^{\mu}{}_{\nu} x^{\nu}. \qquad (1.2.30)$$

Thus

$$\left(D^{\alpha}{}_{\beta}\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\beta}\delta^{\alpha}{}_{\nu} - g^{\mu\alpha} g_{\nu\beta}$$

or raising β and lowering μ

$$\left(D^{\alpha\beta}\right)_{\mu\nu} = \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\nu} - \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}. \qquad (1.2.31)$$

We see that since $\omega^{\alpha\beta}$ is anti-symmetric so is $D^{\alpha\beta} = -D^{\beta\alpha}$. From this vector representation we can obtain the commutation relations for the $D^{\mu\nu}$ which then all further representations of the Lorentz group must obey

$$[D^{\mu\nu}, D^{\rho\sigma}]_{\alpha\beta} = (D^{\mu\nu})_{\alpha}{}^{\gamma}(D^{\rho\sigma})_{\gamma\beta} - (D^{\rho\sigma})_{\alpha}{}^{\gamma}(D^{\mu\nu})_{\gamma\beta}$$

$$= (\delta^{\nu}{}_{\alpha} g^{\mu\gamma} - \delta^{\mu}{}_{\alpha} g^{\nu\gamma}) \left(\delta^{\sigma}{}_{\gamma}\delta^{\rho}{}_{\beta} - \delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\beta}\right) - (D^{\rho\sigma} D^{\mu\nu})_{\alpha\beta}$$

$$= g^{\mu\sigma} \left(\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\beta} - \delta^{\rho}{}_{\alpha}\delta^{\nu}{}_{\beta}\right) - g^{\mu\rho} \left(\delta^{\nu}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\sigma}{}_{\alpha}\delta^{\nu}{}_{\beta}\right)$$

$$-g^{\nu\sigma}\left(\delta^{\mu}{}_{\alpha}\delta^{\rho}{}_{\beta}-\delta^{\rho}{}_{\alpha}\delta^{\mu}{}_{\beta}\right)+g^{\nu\rho}\left(\delta^{\mu}{}_{\alpha}\delta^{\sigma}{}_{\beta}-\delta^{\sigma}{}_{\alpha}\delta^{\mu}{}_{\beta}\right)$$
$$=\left(g^{\mu\sigma}D^{\rho\nu}-g^{\mu\rho}D^{\sigma\nu}-g^{\nu\sigma}D^{\rho\mu}+g^{\nu\rho}D^{\sigma\mu}\right)_{\alpha\beta}. \tag{1.2.32}$$

Thus, surpressing the matrix element indices, the defining commutation relations for the Lorentz group are

$$[D^{\mu\nu},D^{\rho\sigma}]=g^{\mu\rho}D^{\nu\sigma}+g^{\nu\sigma}D^{\mu\rho}-g^{\mu\sigma}D^{\nu\rho}-g^{\nu\rho}D^{\mu\sigma}. \tag{1.2.33}$$

For a general tensor field we can find its Lorentz representation matrix in a similar way. Consider the rank n tensor transformation law, equation (1.2.23), for infinitesimal $\Lambda^{\mu\nu}$

$$T'^{\mu_1\cdots\mu_n}(x')=(g^{\mu_1\nu_1}+\omega^{\mu_1\nu_1})\cdots(g^{\mu_n\nu_n}+\omega^{\mu_n\nu_n})T_{\nu_1\cdots\nu_n}(x)$$
$$=T^{\mu_1\cdots\mu_n}(x)+\sum_{i=1}^{n}g^{\mu_1\nu_1}\cdots\omega^{\mu_i\nu_i}\cdots g^{\mu_n\nu_n}T_{\nu_1\cdots\nu_i\cdots\nu_n}(x)$$
$$=T^{\mu_1\cdots\mu_n}(x)+\frac{\omega_{\alpha\beta}}{2}\sum_{i=1}^{n}g^{\mu_1\nu_1}\cdots[g^{\alpha\mu_i}g^{\beta\nu_i}-g^{\alpha\nu_i}g^{\beta\mu_i}]\cdots g^{\mu_n\nu_n}T_{\nu_1\cdots\nu_n}(x). \tag{1.2.34}$$

Thus using the notation $(\mu)=\mu_1\cdots\mu_n$ we have

$$T'^{(\mu)}(x')-T^{(\mu)}(x)\equiv\frac{\omega_{\beta\alpha}}{2}(D^{\alpha\beta})^{(\mu)(\nu)}T_{(\nu)}(x) \tag{1.2.35}$$

where

$$(D^{\alpha\beta})^{(\mu)(\nu)}\equiv\sum_{i=1}^{n}g^{\mu_1\nu_1}\cdots[g^{\alpha\nu_i}g^{\beta\mu_i}-g^{\alpha\mu_i}g^{\beta\nu_i}]\cdots g^{\mu_n\nu_n}. \tag{1.2.36}$$

As in the vector case, the $(D^{\alpha\beta})^{(\mu)(\nu)}$ obey the Lorentz group commutation relations equation (1.2.33).

In general, the difference $\delta T(x)$,

$$\delta T(x)\equiv T'(x')-T(x), \tag{1.2.37}$$

is called the total variation of T. The rank n tensor is then defined so that

$$\delta T^{(\mu)}(x)\equiv\frac{\omega_{\beta\alpha}}{2}(D^{\alpha\beta})^{(\mu)(\nu)}T_{(\nu)}(x) \tag{1.2.38}$$

with $(D^{\alpha\beta})^{(\mu)(\nu)}$ given in equation (1.2.36). When the $D^{\alpha\beta}$ matrices are exponentiated with the finite parameter matrices $\omega_{\beta\alpha}(\Lambda)$ for finite Lorentz transformations Λ the commutation relations (1.2.33) guarantee the correct tensor transformation law (1.2.23).

Notice the total variation of T evaluates the change in the tensor field at the same space-time point, x^μ in S and x'^μ in S'. It is physically more meaningful to compare tensors not at the same space-time point but at the same numerical value, say x^μ, of their argument. This intrinsic variation of T, denoted by $\bar{\delta}T$, is related to the total variation of T, δT, by a Taylor expansion. Thus the intrinsic variation of T is defined by

$$\begin{aligned}\bar{\delta}T(x) &\equiv T'(x) - T(x) \\ &= [T'(x') - T(x)] - [T'(x') - T'(x)] \\ &= \delta T(x) - [T'(x') - T'(x)]. \end{aligned} \tag{1.2.39}$$

Again, $\bar{\delta}T(x)$ is the change in T(x) for the same value of the argument not the same point in space-time. Since these variations are infinitesimal, we need only keep first order in changes. Writing the space-time point transformation as $x'^\mu = x^\mu + \delta x^\mu$ so that the Taylor expansion of T'(x') about x is $T'(x') = T'(x) + \delta x^\mu \partial_\mu T(x)$ we conclude that

$$\bar{\delta}T(x) = \delta T(x) - \delta x^\mu \partial_\mu T(x). \tag{1.2.40}$$

For a Lorentz transformation we can write δx^μ as the defining 4x4 vector representation matrix $D^{\alpha\beta}$

$$x'^\mu = x^\mu + \frac{\omega^{\beta\alpha}}{2}[g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}]x_\nu .$$

However this variation is just the first Taylor expansion term of the function x^μ, hence it is equivalent, and later more useful when considering variations of more general tensor fields, to write the x-variation as a differential Taylor operator so that

$$x'^\mu = x^\mu - \frac{\omega_{\alpha\beta}}{2}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)x^\mu . \tag{1.2.41}$$

Thus the intrinsic variation of a rank n tensor $T^{(\mu)}$, equation (1.2.40), can be written as

$$\bar{\delta}T^{(\mu)} = \frac{\omega_{\beta\alpha}}{2}\left[(D^{\alpha\beta})^{(\mu)(\nu)} - g^{(\mu)(\nu)}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\right]T_{(\nu)} . \tag{1.2.43}$$

We find the differential operator $(x^\mu \partial^\nu - x^\nu \partial^\mu)$ obeys the algebra, cf. equation (1.2.33), associated with the Lorentz group also. Thus we define the differential matrix (angular momentum) operator $M^{\mu\nu}$ for rank n tensors as

$$(M^{\mu\nu})^{(\alpha)(\beta)} T_{(\beta)} \equiv -i \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) g^{(\alpha)(\beta)} - (D^{\mu\nu})^{(\alpha)(\beta)} \right] T_{(\beta)}. \tag{1.2.43}$$

We check explicitly that the angular momentum commutation relations are satisfied

$$[M^{\mu\nu}, M^{\rho\sigma}] = +i[g^{\mu\rho} M^{\nu\sigma} + g^{\nu\sigma} M^{\mu\rho} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\rho}]. \tag{1.2.44}$$

The intrinsic variation can then be used to define rank n tensors

$$\bar{\delta} T^{(\mu)}(x) \equiv \frac{-i\omega_{\beta\alpha}}{2} (M^{\alpha\beta})^{(\mu)(\nu)} T_{(\nu)}(x) \tag{1.2.45}$$

where $M^{\alpha\beta}$ is the total angular momentum carried by the field while $D^{\alpha\beta}$ is related to its intrinsic angular momentum or spin. Thus we can characterize the Lorentz group tensors by their spin, the $D^{\alpha\beta}$ matrices, and their orbital angular momentum as represented by the action of the space-time differential operator $(x^\mu \partial^\nu - x^\nu \partial^\mu)$ on their argument. Hence we can represent the Lorentz group by finite matrices $D^{\alpha\beta}$ obeying the algebra (1.2.33) and by space-time differential operators acting on tensor fields $T^{(\alpha)}(x)$.

Similarly we can consider infinitesimal space-time translations

$$x'^\mu = x^\mu + \delta x^\mu = x^\mu + \epsilon^\mu. \tag{1.2.46}$$

Then for translationally invariant fields $T'(x') = T(x)$, which is all that is defined, we have

$$\begin{aligned} \bar{\delta} T &= -\delta x^\mu \partial_\mu T = -\epsilon^\mu \partial_\mu T \\ &\equiv +i\epsilon^\mu P_\mu T \end{aligned} \tag{1.2.47}$$

that is the momentum operator P^μ given by

$$P_\mu \equiv +i\partial_\mu \tag{1.2.48}$$

represents the generator of space-time translations. Thus for any representation of the Lorentz group $M^{\mu\nu}$ we can calculate the commutator of P^μ with it to find

$$[M^{\mu\nu}, P^\rho] = -i[P^\mu g^{\nu\rho} - P^\nu g^{\mu\rho}]. \tag{1.2.49}$$

Along with $[P^\mu, P^\nu] = 0$, these three sets of commutation relations define the action of the Poincare' group on tensor fields $T^{(\mu)}(x)$. As usual the field after a finite Poincare' transformation is obtained by exponentiation of the translation generator. For finite translations $x'^\mu = x^\mu + a^\mu$, we have

$$\begin{aligned} T'(x) &= \lim_{n\to\infty}\left(1 + (\frac{+ia^\mu}{n})P_\mu\right)^n T(x) \\ &= e^{+ia^\mu P_\mu} T(x) \\ &= e^{-a^\mu\partial_\mu} T(x) = T(x-a) \end{aligned} \tag{1.2.50}$$

(which is just the definition of a translationally invariant field).

To summarize then, the Poincare' group is defined by the algebra its generators P^μ, the energy-momentum operator which is the generator of space-time translations, and $M^{\mu\nu}$, the angular momentum operator, which is the generator of Lorentz transformations and space rotations, obey

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\rho] &= -i(P^\mu g^{\nu\rho} - P^\nu g^{\mu\rho}) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= +i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma}). \end{aligned} \tag{1.2.51}$$

For the tensor representations of the Lorentz group this algebra is realized by the intrinsic variation of the fields as

$$\begin{aligned} P_\mu T^{(\alpha)}(x) &= +i\partial_\mu T^{(\alpha)}(x) \\ (M^{\mu\nu})^{(\alpha)(\beta)} T_{(\beta)}(x) &= -i\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)g^{(\alpha)(\beta)} - (D^{\mu\nu})^{(\alpha)(\beta)}\right] T_{(\beta)}(x) \end{aligned} \tag{1.2.52}$$

where $(\alpha) = \alpha_1 \cdots \alpha_n$ for a rank n tensor,

$$g^{(\alpha)(\beta)} = g^{\alpha_1\beta_1}\cdots g^{\alpha_n\beta_n} \tag{1.2.53}$$

and

$$(D^{\mu\nu})^{(\alpha)(\beta)} = \sum_{i=1}^n g^{\alpha_1\beta_1}\cdots[g^{\mu\beta_i}g^{\nu\alpha_i} - g^{\nu\beta_i}g^{\mu\alpha_i}]\cdots g^{\alpha_n\beta_n} \tag{1.2.54}$$

is the matrix for the finite dimensional rank n tensor representation of the Lorentz group where the $D^{\mu\nu}$ obey the algebra

$$[D^{\mu\nu}, D^{\rho\sigma}] = g^{\mu\rho}D^{\nu\sigma} - g^{\mu\sigma}D^{\nu\rho} + g^{\nu\sigma}D^{\mu\rho} - g^{\nu\rho}D^{\mu\sigma}. \tag{1.2.55}$$

The transformations induced in $T^{(\alpha)}$ when finite Poincare' transformations are made $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ are obtained by exponentiating the operators

$$\begin{aligned} T'^{\alpha_1\cdots\alpha_n}(x) &= \Lambda^{\alpha_1}{}_{\beta_1}\cdots\Lambda^{\alpha_n}{}_{\beta_n}T^{\beta_1\cdots\beta_n}(\Lambda^{-1}(x-a)) \\ &= \left(e^{+ia^\mu P_\mu}e^{+\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}\right)^{(\alpha)(\beta)}T_{(\beta)}(x) \end{aligned} \tag{1.2.56}$$

where $\omega_{\mu\nu}(\Lambda)$ is defined so that

$$e^{+\frac{1}{2}\omega_{\mu\nu}(\Lambda)(x^\mu\partial^\nu - x^\nu\partial^\mu)}x^\rho = (\Lambda^{-1})^\rho{}_\sigma x^\sigma. \tag{1.2.57}$$

The tensor representations are not the only realizations of the algebra possible; there are also the spinor representations. That is there are other objects besides tensors that have the same transformation law in all inertial frames; the Lorentz group multiplication law is valid for these quantities. We will find that spinorial equations will be covariant and hence will be useful for describing physical laws. Spin one half particles such as electrons will be described by spinor quantum fields. To introduce spinor transformation laws consider the operators $M_{\mu\nu}$ more closely. The M_{ij} are associated with spatial rotations in the $x^i - x^j$ plane. For instance, for $\omega_{12} \neq 0$ only, we have

$$\begin{aligned} x'^0 &= x^0 \\ x'^1 &= x^1 + \omega^{12}x_2 = x^1 - \omega^{12}x^2 \\ x'^2 &= x^2 + \omega^{21}x_1 = x^2 + \omega^{12}x^1 \\ x'^3 &= x^3, \end{aligned} \tag{1.2.58}$$

a rotation of the coordinate system about the z-axis through an infinitesimal angle ω^{12}. Hence the operator

$$J^i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk} = (M_{23}, M_{31}, M_{12}) \tag{1.2.59}$$

corresponds to the total angular momentum operator, that is, the generator of spatial rotations for our functions of space-time. Similarly then the M^{0i} are associated with boosts that is pure Lorentz transformations in the x^i direction. For example, for infinitesimal $\omega^{01} \neq 0$ only we have

$$x'^0 = x^0 - \omega^{01}x^1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(ct - \frac{v}{c}x^1) \approx ct - \frac{v}{c}x^1$$

$$x'^1 = x^1 - \omega^{01}x^0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x^1 - vt) \approx x^1 - \frac{v}{c}ct$$

$$x'^2 = x^2$$

$$x'^3 = x^3. \tag{1.2.60}$$

Hence, $K^i \equiv M^{0i} = (M^{01}, M^{02}, M^{03})$ are the generators of Lorentz transformations for our functions of space-time.

The algebra for $M^{\mu\nu}$, equation (1.2.51), implies that $\vec{J}$ and $\vec{K}$ obey the algebra

$$[J_i, J_j] = +i\epsilon_{ijk}J_k$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

$$[J_i, K_j] = +i\epsilon_{ijk}K_k. \tag{1.2.61}$$

Complex (non-hermitian) generators can be defined from these real (hermitian) generators

$$\vec{N} \equiv \frac{1}{2}(\vec{J} + i\vec{K})$$

$$\vec{N}^\dagger \equiv \frac{1}{2}(\vec{J} - i\vec{K}). \tag{1.2.62}$$

Then the commutation relations become disentangled

$$[N_i, N_j] = +i\epsilon_{ijk}N_k$$

$$[N_i^\dagger, N_j^\dagger] = +i\epsilon_{ijk}N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0. \tag{1.2.63}$$

Hence, each set of $\vec{N}$ and $\vec{N}^\dagger$ obey the SU(2) spatial rotation algebra of ordinary angular momentum separately. Eigenvalues of $\vec{N}\cdot\vec{N}$ and $\vec{N}^\dagger\cdot\vec{N}^\dagger$ label the representation. These eigenvalues have values m(m+1) for $\vec{N}\cdot\vec{N}$ and n(n+1) for $\vec{N}^\dagger\cdot\vec{N}^\dagger$, where $m,n=0,\frac{1}{2},1,\frac{3}{2},2,\cdots$. So the pair (m,n) will label the representation of the Lorentz group about which we speak. The functions within each representation are distinguished by the eigenvalue of one of the components of each of the operators $\vec{N}$ and $\vec{N}^\dagger$. For instance, for the third component the eigenvalues of N_3 have the range $-m,-m+1,\cdots,+m-1,+m$ and those of $\vec{N}_3^\dagger$ have the range $-n,-n+1,\cdots,+n-1,+n$, the (m, n) representation has (2m+1)(2n+1) components. For example the (0,0) representation consists of only one function, T(x), and is totally invariant, $\delta T=0$. The vector representation consists of four functions, T^μ, where $\delta T^\mu=\omega^{\mu\nu}T_\nu$. We can show that this corresponds to the $(\frac{1}{2},\frac{1}{2})$ representation. The (1,0) representation can be shown to be that of the self-dual, antisymmetric tensor, $F^{\mu\nu}=\tilde{F}^{\mu\nu}$ with $\tilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. (By the conventions used in these notes we define the totally antisymmetric permutation or Levi-Civita tensor, denoted $\epsilon^{\mu\nu\rho\sigma}$, so that $\epsilon^{0123}\equiv+1$.) Since $\vec{J}=\vec{N}+\vec{N}^\dagger$, we see that (m+n) will denote the total spin of the representation. We can show that all integer values of (m+n) i.e. $(m+n)=0,1,2,3,\cdots$ can be described by our tensor representations. However, we can also have representations for which $(m+n)=\frac{(2k+1)}{2}$ for any integer $k=0,1,2,\cdots$. These all can be built up from products of the basic spinor representations (as we built up the tensor representations from the fundamental vector representation); the $(\frac{1}{2},0)$ representation called left-handed spinors and the $(0,\frac{1}{2})$ representation called right-handed spinors. In this way we will have obtained all possible (finite dimensional) representations of the Lorentz group.

In order to obtain the transformation law for spinors we will consider the set of 2x2 complex matrices with determinant =1. These form the group called SL(2,C). We will represent the Lorentz group by the action of these matrices on two component complex spinors. Equivalently we could build the spinor transformation law from the spin $\frac{1}{2}$ angular momentum representation matrices familiar from quantum mechanics, the Pauli matrices. Once we know the action of $\vec{N}$ and $\vec{N}^\dagger$ on the spinors we can reconstruct that of $M^{\mu\nu}$. However, it is more useful and to the point to proceed by considering directly SL(2,C). To obtain the relation of the Lorentz group to SL(2,C) we must first recall that there exists a one to one correspondence between 2x2 Hermitian matrices and space-time points. The

Pauli matrices

$$
\begin{aligned}
(\sigma^0)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^1)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^2)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^3)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha\dot{\alpha}} ,
\end{aligned}
\tag{1.2.64}
$$

where $\alpha = 1, 2$ labels the two rows and $\dot{\alpha} = 1, 2$ labels the two columns, form a basis for 2x2 Hermitian matrices. Let $X_{\alpha\dot{\alpha}}$ be a Hermitian matrix, that is,

$$
\begin{aligned}
X^\dagger &= X \\
(X^*)_{\dot{\alpha}\alpha} &= (X)_{\alpha\dot{\alpha}} .
\end{aligned}
\tag{1.2.65}
$$

It has the general form

$$
\begin{aligned}
X_{\alpha\dot{\alpha}} &= \begin{pmatrix} (x_0 + x_3) & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3) \end{pmatrix}_{\alpha\dot{\alpha}} \\
&= x_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \equiv \not{x}_{\alpha\dot{\alpha}}
\end{aligned}
\tag{1.2.66}
$$

for x_μ real with $\not{x}$ called “x slash”. Thus corresponding to any 4 vector x^μ we associate a 2x2 Hermitian matrix $X_{\alpha\dot{\alpha}}$ by equation (1.2.66). Using the trace relation for the product of two Pauli matrices

$$
(\sigma^\mu)_{\alpha\dot{\alpha}} (i\sigma^2)_{\dot{\alpha}\dot{\beta}} (\sigma^{\nu\ T})_{\dot{\beta}\beta} (i\sigma^2)_{\beta\alpha} = -2g^{\mu\nu} ,
\tag{1.2.67}
$$

or more succinctly written

$$
Tr[\sigma^\mu (i\sigma^2) \sigma^{\nu\ T} (i\sigma^2)] = -2g^{\mu\nu} ,
\tag{1.2.68}
$$

we have for every Hermitian matrix $X_{\alpha\dot{\alpha}}$ an associated four vector x^μ

$$
x^\mu = -\frac{1}{2} Tr[X (i\sigma^2) \sigma^{\mu\ T} (i\sigma^2)] .
\tag{1.2.69}
$$

This correspondence is one to one (we will use $X = \not{x}$ in what follows to underscore this correspondence).

Simplifying the notation, since we would like to keep our dotted and undotted indices separate that is when we sum over indices we would like them to be of the same type in order to avoid extra confusion, we introduce an antisymmetric tensor $\epsilon^{\alpha\beta}$, that is, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ with $\epsilon^{12} = +1$ and with lowered indices

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = -\epsilon_{\beta\alpha}, \tag{1.2.70}$$

that is $\epsilon_{12} = -\epsilon^{12} = -1$. Note that the matrix is the same when we use dotted indices, that is,

$$\epsilon^{\alpha\beta} = (i\sigma^2)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta}$$
$$\epsilon^{\dot{\alpha}\dot{\beta}} = (i\sigma^2)_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\dot{\alpha}\dot{\beta}}. \tag{1.2.71}$$

Also note that

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\ \gamma}$$
$$\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\ \dot{\gamma}}. \tag{1.2.72}$$

Then we can define the Pauli matrices with upper indices

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^{\mu})_{\beta\dot{\beta}}$$
$$= -(i\sigma^2)_{\dot{\alpha}\dot{\beta}}(\sigma^{\mu\ T})_{\dot{\beta}\beta}(i\sigma^2)_{\beta\alpha}. \tag{1.2.73}$$

We can write the trace condition as

$$(\sigma^{\mu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\nu})^{\dot{\alpha}\alpha} = +2g^{\mu\nu} \tag{1.2.74}$$

and equation (1.2.69) has the simple form

$$x^{\mu} = +\frac{1}{2}(\not{x})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = +\frac{1}{2}Tr[\not{x}\bar{\sigma}^{\mu}]. \tag{1.2.75}$$

The $\bar{\sigma}^{\mu}$ matrices are given by

$$(\bar{\sigma}^0)^{\dot{\alpha}\alpha} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\dot{\alpha}\alpha} = +(\sigma^0)_{\dot{\alpha}\alpha}$$

$$(\bar{\sigma}^1)^{\dot{\alpha}\alpha} \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{\dot{\alpha}\alpha} = -(\sigma^1)_{\dot{\alpha}\alpha}$$

$$(\bar{\sigma}^2)^{\dot{\alpha}\alpha} \equiv \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}_{\dot{\alpha}\alpha} = -(\sigma^2)_{\dot{\alpha}\alpha}$$

$$(\bar{\sigma}^3)^{\dot{\alpha}\alpha} \equiv \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}_{\dot{\alpha}\alpha} = -(\sigma^3)_{\dot{\alpha}\alpha}. \tag{1.2.76}$$

We can readily derive the completeness properties of the Pauli matrices

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\alpha} = +2g^{\mu\nu} \tag{1.2.77}$$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = +2\delta_\alpha^{\ \beta}\delta^{\dot{\beta}}_{\ \dot{\alpha}}. \tag{1.2.78}$$

Further products of two yield

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta} + (\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = 2g^{\mu\nu}\delta_\alpha^{\ \beta}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\sigma^\nu)_{\alpha\dot{\beta}} + (\bar{\sigma}^\nu)^{\dot{\alpha}\alpha}(\sigma^\mu)_{\alpha\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}}. \tag{1.2.79}$$

If S is an element of SL(2,C) (that is 2x2 complex matrices with determinant equal to one) with matrix elements $S_\alpha^{\ \beta}$, where α labels the rows and β labels the columns, and $\not{x}$ is a Hermitian matrix, then we can define the transformed matrix $\not{x}'$ as

$$(\not{x}')_{\alpha\dot{\alpha}} = S_\alpha^{\ \beta}(\not{x})_{\beta\dot{\beta}}S^{*\dot{\beta}}_{\dot{\alpha}} \tag{1.2.80}$$

with S^* the complex conjugate of S, again with $\dot{\alpha}$ labelling the rows and $\dot{\beta}$ labelling the columns, or taking the transpose we have $(S^\dagger)^{\dot{\beta}}_{\ \dot{\alpha}} = (S^*)_{\dot{\alpha}}^{\ \dot{\beta}}$, with $\dot{\beta}$ labelling the rows and $\dot{\alpha}$ labeling the columns of $S^\dagger$. Since $\det S = S_1^{\ 1}S_2^{\ 2} - S_1^{\ 2}S_2^{\ 1} = 1$ we have

$$\det \not{x}' = \det \not{x}. \tag{1.2.81}$$

Calculating the determinant we find

$$\det \not{x} = (x_0 + x_3)(x_0 - x_3) - (x_1 - ix_2)(x_1 + ix_2) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$$= x_\mu x^\mu$$

$$= \det \not{x}' = x'_\mu x'^\mu; \tag{1.2.82}$$

the determinant is the Minkowski interval and is invariant. Thus, the transformation

$$\not{x}' = S\ \not{x} S^\dagger \tag{1.2.83}$$

corresponds to a Lorentz transformation, $\Lambda^{\mu\nu}$, of the coordinates. In order to determine it in terms of the SL(2,C) matrix S consider

$$\begin{aligned} x'^\mu &= \frac{1}{2}(\not{x}')_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \\ = \frac{1}{2}S_\alpha^{\ \beta}(\not{x})_{\beta\dot{\beta}}S^{*\dot{\beta}}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} &= \frac{1}{2}S_\alpha^{\ \beta}S^{*\dot{\beta}}_{\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}x_\nu \\ &\equiv \Lambda^{\mu\nu}x_\nu \end{aligned} \tag{1.2.84}$$

where we identify

$$\Lambda^{\mu\nu} \equiv \frac{1}{2}Tr[S\sigma^\nu S^\dagger \bar{\sigma}^\mu] \tag{1.2.85}$$

that is,

$$\begin{aligned} \Lambda^{\mu\nu}(\sigma_\mu)_{\alpha\dot{\alpha}} = \frac{1}{2}S_\beta^{\ \gamma}S^{*\dot{\gamma}}_{\dot{\beta}}(\sigma^\nu)_{\gamma\dot{\gamma}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta}(\sigma_\mu)_{\alpha\dot{\alpha}} &= S_\beta^{\ \gamma}S^{*\dot{\gamma}}_{\dot{\beta}}(\sigma^\nu)_{\gamma\dot{\gamma}}\delta_\alpha^{\ \beta}\delta_{\dot{\alpha}}^{\ \dot{\beta}} \\ &= S_\alpha^{\ \gamma}S^{*\dot{\gamma}}_{\dot{\alpha}}(\sigma^\nu)_{\gamma\dot{\gamma}}, \end{aligned}$$

or more simply written

$$\Lambda^{\mu\nu}\sigma_\mu = S\sigma^\nu S^\dagger. \tag{1.2.86}$$

So for every element $\pm S$ of SL(2,C) there is an element Λ of the Lorentz group, the mapping of SL(2,C) into $L^\uparrow_-$ is 2 to 1 since $\pm S \rightarrow \Lambda$.

We can use the SL(2,C) matrices to define the spinor representations of the Lorentz group. The spinor transformation laws are given by

$$\begin{aligned} \psi'_\alpha(x') &\equiv S_\alpha^{\ \beta}\psi_\beta(x) \\ \psi'^\alpha(x') &\equiv \psi^\beta(x)(S^{-1})_\beta^{\ \alpha} \end{aligned} \tag{1.2.87}$$

where $S_\alpha^{\ \beta}(S^{-1})_\beta^{\ \gamma} = \delta_\alpha^{\ \gamma}$ and ψ_α and ψ^α are two different two component complex spinors transforming (as we will see) as the $(\frac{1}{2}, 0)$ representation of the Lorentz group, the ψ are called Weyl spinors. Similarly, we can use $S^\dagger$ and $(S^\dagger)^{-1}$ to define two more different Weyl spinors, the complex conjugates of ψ, denoted $\bar{\psi}$, which transform as $(0, \frac{1}{2})$ representations of the Lorentz group

$$\bar{\psi}'_{\dot{\alpha}}(x') \equiv \bar{\psi}_{\dot{\beta}}(x)(S^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}$$

$$\bar{\psi}'^{\dot{\alpha}}(x') \equiv (S^{\dagger -1})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}(x) \tag{1.2.88}$$

where, for the adjoint matrices $(S^\dagger)^{\dot{\alpha}}{}_{\dot{\beta}}$, $\dot{\alpha}$ labels the rows and $\dot{\beta}$ labels the columns. As with tensors, higher rank spinors transform just like products of the basic rank 1 spinors, for example,

$$\psi'_{\alpha_1 \cdots \alpha_n}(x') = S_{\alpha_1}{}^{\beta_1} \cdots S_{\alpha_n}{}^{\beta_n} \psi_{\beta_1 \cdots \beta_n}(x)$$

$$\psi'_{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_m}(x') = S_{\alpha_1}{}^{\beta_1} \cdots S_{\alpha_1}{}^{\beta_m} \psi_{\beta_1 \cdots \beta_n \dot{\beta}_1 \cdots \dot{\beta}_m}(x)(S^\dagger)^{\dot{\beta}_1}{}_{\dot{\alpha}_1} \cdots (S^\dagger)^{\dot{\beta}_m}{}_{\dot{\alpha}_m}. \tag{1.2.89}$$

Since S is special, i.e. det S = 1, we have

$$(S^{-1})_\alpha{}^\beta = \begin{pmatrix} S_2{}^2 & -S_1{}^2 \\ -S_2{}^1 & S_1{}^1 \end{pmatrix}_{\alpha\beta}$$

$$= \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}S_\delta{}^\gamma \tag{1.2.90}$$

or in matrix notation

$$S^{-1} = -\epsilon S^T \epsilon. \tag{1.2.91}$$

Further ϵ is an anti-symmetric invariant second rank spinor, that is

$$\epsilon'_{\alpha\beta} = S_\alpha{}^\gamma S_\beta{}^\delta \epsilon_{\gamma\delta}$$

or again in matrix form

$$\epsilon' = S\epsilon S^T = SS^{-1}\epsilon = \epsilon.$$

Since the indices can be confusing, let's write this out explicitly

$$\epsilon'_{12} = S_1{}^1 S_2{}^2 \epsilon_{12} + S_1{}^2 S_2{}^1 \epsilon_{21} = -\det S = -1 = \epsilon_{12}.$$

So indeed $\epsilon'_{\alpha\beta} = \epsilon_{\alpha\beta}$, ϵ is an invariant second rank spinor. Hence, we can use ϵ to lower and raise indices of the spinors analogous to the invariant metric tensor $g^{\mu\nu}$ which lowers and raises vector indices

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$$

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$$
$$\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}$$
$$\bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}. \tag{1.2.92}$$

As a consequence of (1.2.92) the transformation law for ψ^α, for instance, follows from that of ψ_α

$$\psi'^\alpha(x') = \epsilon^{\alpha\beta}\psi'_\beta(x') = \epsilon^{\alpha\beta}S_\beta{}^\gamma\psi_\gamma(x)$$
$$= \epsilon^{\alpha\beta}S_\beta{}^\gamma\epsilon_{\gamma\delta}\psi^\delta(x) = -\epsilon_{\delta\gamma}S_\beta{}^\gamma\epsilon_{\beta\alpha}\psi^\delta(x) = \psi^\delta(x)(S^{-1})_\delta{}^\alpha. \tag{1.2.93}$$

Thus, we can contract similar spinor indices to make Lorentz scalars

$$\psi'^\alpha(x')\psi'_\alpha(x') = (S^{-1})_\beta{}^\alpha S_\alpha{}^\gamma\psi^\beta(x)\psi_\gamma(x)$$
$$= \delta_\beta{}^\gamma\psi^\beta(x)\psi_\gamma(x) = \psi^\alpha(x)\psi_\alpha(x) \tag{1.2.94}$$

and similarly for $\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha}$. Also using the properties of the Pauli matrices we can make a four vector object whose vector index then contracts with another four vector index in order to make a scalar, for example

$$\psi'^\alpha(x')(\sigma^\mu)_{\alpha\dot\alpha}\partial'_\mu\bar\psi'^{\dot\alpha}(x') = (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}(\Lambda^{-1})^\nu{}_\mu\psi^\beta(x)(\sigma^\mu)_{\alpha\dot\alpha}\partial_\nu\bar\psi^{\dot\beta}(x).$$

But

$$(S^{-1})_\beta{}^\alpha(\sigma_\mu)_{\alpha\dot\alpha}(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\Lambda^{\mu\nu} = (S^{-1})_\beta{}^\alpha(\sigma_\mu)_{\alpha\dot\alpha}(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\left(\frac{1}{2}Tr[S\sigma^\nu S^\dagger\bar\sigma^\mu]\right)$$
$$= \frac{1}{2}(S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}(S\sigma^\nu S^\dagger)_{\delta\dot\delta}(\sigma_\mu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\delta\delta}$$
$$= (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}(S\sigma^\nu S^\dagger)_{\delta\dot\delta}\delta_\alpha{}^\delta\delta_{\dot\alpha}{}^{\dot\delta} = (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}(S\sigma^\nu S^\dagger)_{\alpha\dot\alpha}$$
$$= (\sigma^\nu)_{\beta\dot\beta}, \tag{1.2.95}$$

hence

$$\psi'^\alpha(x')(\sigma^\mu)_{\alpha\dot\alpha}\partial'_\mu\bar\psi'^{\dot\alpha}(x') = \psi^\alpha(x)(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha}(x). \tag{1.2.96}$$

As stated, $\psi \not\partial\bar\psi$ is a Lorentz invariant.

Finally, let's consider infinitesimal Lorentz transformations

$$x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu \tag{1.2.97}$$

where now, for infinitesimal transformations, S differs from the identity by an infinitesimal matrix Σ

$$
\begin{aligned}
S_{\alpha}{}^{\beta} &= \delta_{\alpha}{}^{\beta} + \Sigma_{\alpha}{}^{\beta} \\
S^{*\dot\beta}_{\dot\alpha} &= \delta_{\dot\alpha}{}^{\dot\beta} + \Sigma^{*\dot\beta}_{\dot\alpha}.
\end{aligned} \tag{1.2.98}
$$

Since ϵ is invariant we have that

$$
\epsilon_{\alpha\beta} = S_{\alpha}{}^{\gamma} S_{\beta}{}^{\delta} \epsilon_{\gamma\delta} = (\delta_{\alpha}{}^{\gamma} + \Sigma_{\alpha}{}^{\gamma})(\delta_{\beta}{}^{\delta} + \Sigma_{\beta}{}^{\delta})\epsilon_{\gamma\delta}
$$

$$
= \left[\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} + \Sigma_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} + \Sigma_{\beta}{}^{\delta}\delta_{\alpha}{}^{\gamma}\right]\epsilon_{\gamma\delta} = \epsilon_{\alpha\beta} + \epsilon_{\gamma\beta}\Sigma_{\alpha}{}^{\gamma} + \epsilon_{\alpha\gamma}\Sigma_{\beta}{}^{\gamma} \tag{1.2.99}
$$

which implies that Σ is symmetric. With lowered indices we have

$$
\Sigma_{\beta\alpha} - \Sigma_{\alpha\beta} = 0. \tag{1.2.100}
$$

Now given $\omega^{\mu\nu}$ we desire $\Sigma_{\alpha\beta}$; using $\Lambda^{\mu\nu} = \frac{1}{2} Tr[S\sigma^{\nu} S^{\dagger}\bar\sigma^{\mu}]$ we find

$$
g^{\mu\nu} + \omega^{\mu\nu} = \frac{1}{2}\left(\delta_{\alpha}{}^{\beta} + \Sigma_{\alpha}{}^{\beta}\right)(\sigma^{\nu})_{\beta\dot\beta}\left(\delta_{\dot\alpha}{}^{\dot\beta} + \Sigma^{*\dot\beta}_{\dot\alpha}\right)(\bar\sigma^{\mu})^{\dot\alpha\alpha}
$$

$$
= \frac{1}{2}(\sigma^{\nu})_{\alpha\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha} + \frac{1}{2}\Sigma_{\alpha}{}^{\beta}(\sigma^{\nu})_{\beta\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha} + \frac{1}{2}(\sigma^{\nu})_{\alpha\dot\beta}\Sigma^{*\dot\beta}_{\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha}
$$

$$
= g^{\mu\nu} + \frac{1}{2}\Sigma_{\alpha}{}^{\beta}(\sigma^{\nu})_{\beta\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha} + \frac{1}{2}\Sigma^{*\dot\beta}_{\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha}(\sigma^{\nu})_{\alpha\dot\beta}. \tag{1.2.101}
$$

Thus, we must find a solution for

$$
\begin{aligned}
\omega^{\mu\nu} &= \frac{1}{2}\Sigma_{\alpha}{}^{\beta}(\sigma^{\nu})_{\beta\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha} + \frac{1}{2}\Sigma^{*\dot\beta}_{\dot\alpha}(\bar\sigma^{\mu})^{\dot\alpha\alpha}(\sigma^{\nu})_{\alpha\dot\beta} \\
&= \frac{1}{2} Tr\left[\Sigma\sigma^{\nu}\bar\sigma^{\mu} + \Sigma^{\dagger}\bar\sigma^{\mu}\sigma^{\nu}\right].
\end{aligned} \tag{1.2.102}
$$

Multiplying by σ_{μ} and $\bar\sigma_{\nu}$ we have

$$
\begin{aligned}
(\sigma_{\mu})_{\gamma\dot\gamma}(\bar\sigma_{\nu})^{\dot\delta\delta}\omega^{\mu\nu} &= \frac{1}{2}\left[(\sigma_{\mu})_{\gamma\dot\gamma}(\bar\sigma_{\nu})^{\dot\delta\delta} - (\sigma_{\nu})_{\gamma\dot\gamma}(\bar\sigma_{\mu})^{\dot\delta\delta}\right]\omega^{\mu\nu} \\
&= 2\Sigma_{\gamma}{}^{\delta}\delta_{\dot\gamma}{}^{\dot\delta} + 2\Sigma^{*\dot\delta}_{\dot\gamma}\delta_{\gamma}{}^{\delta}.
\end{aligned} \tag{1.2.103}
$$

Using $\Sigma_{\alpha}{}^{\alpha} = 0 = \Sigma^{*\dot\alpha}_{\dot\alpha}$, we find

$$
\Sigma_{\gamma}{}^{\delta} = \frac{1}{8}\left[(\sigma_{\mu})_{\gamma\dot\gamma}(\bar\sigma_{\nu})^{\dot\gamma\delta} - (\sigma_{\nu})_{\gamma\dot\gamma}(\bar\sigma_{\nu})^{\dot\gamma\delta}\right]\omega^{\mu\nu} \tag{1.2.104}
$$

and similarly

$$\Sigma^{*\dot{\delta}}_{\dot{\gamma}} = \frac{1}{8}\left[(\bar{\sigma}_\nu)^{\dot{\delta}\gamma}(\sigma_\mu)_{\gamma\dot{\gamma}} - (\bar{\sigma}_\mu)^{\dot{\delta}\gamma}(\sigma_\nu)_{\gamma\dot{\gamma}}\right]\omega^{\mu\nu}. \tag{1.2.105}$$

The above commutators of the Pauli matrices arise frequently and so we define the matrices

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{i}{2}\left[(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta} - (\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\right] = \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \equiv \frac{i}{2}\left[(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\sigma^\nu)_{\alpha\dot{\beta}} - (\bar{\sigma}^\nu)^{\dot{\alpha}\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}\right] = \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}_{\ \dot{\beta}}. \tag{1.2.106}$$

Thus we secure

$$\Sigma_\alpha^{\ \beta} = \frac{-i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^{\ \beta}$$

$$(\Sigma^\dagger)^{\dot{\beta}}_{\ \dot{\alpha}} = \frac{+i}{4}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}. \tag{1.2.107}$$

¿From our definitions of $(\sigma^\mu)_{\alpha\dot{\alpha}}$ we see that

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta} = g^{\mu\nu}\delta_\alpha^{\ \beta} - i(\sigma^{\mu\nu})_\alpha^{\ \beta}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\sigma^\nu)_{\alpha\dot{\beta}} = g^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}} - i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}. \tag{1.2.108}$$

The infinitesimal spinor transformations can now be obtained

$$\psi'_\alpha(x') = S_\alpha^{\ \beta}\psi_\beta(x) = \psi_\alpha(x) - \frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x)$$

$$\equiv \psi_\alpha(x) - \frac{1}{2}\omega_{\mu\nu}(D^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x) \tag{1.2.109}$$

and

$$\bar{\psi}'_{\dot{\alpha}}(x') = \bar{\psi}_{\dot{\beta}}(x)(S^\dagger)^{\dot{\beta}}_{\ \dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}(x) + \frac{i}{4}\omega_{\mu\nu}\bar{\psi}_{\dot{\beta}}(x)(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}$$

$$\equiv \bar{\psi}_{\dot{\alpha}}(x) - \frac{1}{2}\omega_{\mu\nu}(\bar{D}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}. \tag{1.2.110}$$

Hence, the spinor representations are given by

$$(D^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{+i}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta}$$

$$(\bar{D}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \equiv \frac{-i}{2}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}. \tag{1.2.111}$$

We must check that these matrices indeed obey the Lorentz algebra as did the tensor representation matrices. After some tedious Pauli matrix algebra we find

$$[\sigma^{\mu\nu},\sigma^{\rho\sigma}]_\alpha^{\ \beta} = \frac{-1}{4}\left[(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu),(\sigma^\rho\bar\sigma^\sigma - \sigma^\sigma\bar\sigma^\rho)\right]_\alpha^{\ \beta}$$
$$= -2i\left(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} + g^{\nu\sigma}\sigma^{\mu\rho} - g^{\nu\rho}\sigma^{\mu\sigma}\right)_\alpha^{\ \beta}. \qquad (1.2.112)$$

Thus the spinor representation obeys the Lorentz algebra

$$[D^{\mu\nu},D^{\rho\sigma}]_\alpha^{\ \beta} = \left(g^{\mu\rho}D^{\nu\sigma} - g^{\mu\sigma}D^{\nu\rho} + g^{\nu\sigma}D^{\mu\rho} - g^{\nu\rho}D^{\mu\sigma}\right)_\alpha^{\ \beta}, \qquad (1.1.113)$$

and ψ_α is the $(\frac{1}{2},0)$ spinor representation of the Lorentz group. Similarly the commutation relation for $\bar\sigma^{\mu\nu}$ can be worked out and we find that the complex conjugate dotted spinors $\bar\psi_{\dot\alpha}$ are the $(0,\frac{1}{2})$ representation of the Lorentz group with the $\bar D^{\mu\nu}$ obeying the Lorentz algebra.

As with tensors, we find the intrinsic variations of the spinor fields are given by

$$\bar\delta\psi_\alpha = \psi'_\alpha(x) - \psi_\alpha(x) = \delta\psi_\alpha - \delta x^\mu\partial_\mu\psi_\alpha$$
$$\bar\delta\bar\psi_{\dot\alpha} = \bar\psi'_{\dot\alpha}(x) - \bar\psi_{\dot\alpha}(x) = \delta\bar\psi_{\dot\alpha} - \delta x^\mu\partial_\mu\bar\psi_{\dot\alpha}. \qquad (1.2.114)$$

For Poincare' transformations

$$x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu + \epsilon^\mu$$

we find that

$$\bar\delta\psi_\alpha = \frac{1}{2}\omega_{\mu\nu}\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta_\alpha^{\ \beta} - (D^{\mu\nu})_\alpha^{\ \beta}\right] - \epsilon^\mu\partial_\mu\psi_\alpha(x)$$
$$\equiv -\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x) + i\epsilon^\mu P_\mu\psi_\alpha(x) \qquad (1.2.115)$$

and

$$\bar\delta\bar\psi_{\dot\alpha} = \frac{1}{2}\omega_{\mu\nu}\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta^{\dot\beta}_{\ \dot\alpha} - (\bar D^{\mu\nu})^{\dot\beta}_{\ \dot\alpha}\right]\psi_{\dot\beta}(x) - \epsilon^\mu\partial_\mu\bar\psi_{\dot\alpha}(x)$$
$$\equiv -\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\dot\beta}_{\ \dot\alpha}\bar\psi_{\dot\beta}(x) + i\epsilon^\mu P_\mu\bar\psi_{\dot\alpha}(x). \qquad (1.2.116)$$

As with tensors, the P^μ and $M^{\mu\nu}$ obey the defining commutation relations of the Poincare' group, equation (1.2.51). Note that in the rest frame for $\psi(x)$, assuming it describes a massive particle with rest frame four momentum $p^\mu = (m,0,0,0)$,

$$J^i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk} = \frac{i}{2}\epsilon_{ijk}D^{jk} = -\frac{1}{4}\epsilon_{ijk}\sigma^{jk}$$

$$= +\frac{1}{2}\sigma^i, \tag{1.2.117}$$

hence the third component of the intrinsic angular momentum J^3 has eigenvalues $\pm\frac{1}{2}$ and the particle has spin $\frac{1}{2}$. Similarly finding $\vec{K}$ we have that $\vec{N}\cdot\vec{N} = \frac{1}{2}(\frac{1}{2}+1)$ and $\vec{N}^\dagger \cdot \vec{N}^\dagger = 0$, so ψ_α is the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Similarly we find that $\bar{\psi}_{\dot{\alpha}}$ is the $(0, \frac{1}{2})$ representation of the Lorentz group.

For finite Poincare' transformations

$$x'^\mu = \Lambda^{\mu\nu} x_\nu + a^\mu \tag{1.2.118}$$

we again exponentiate the generators to obtain

$$\psi'_\alpha(x) = S_\alpha^{\ \beta} \psi_\beta(\Lambda^{-1}(x-a))$$

$$= \left[e^{+ia^\mu P_\mu} e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}\right]_\alpha^{\ \beta} \psi_\beta(x) \tag{1.2.119}$$

and

$$\bar{\psi}'_{\dot{\alpha}}(x) = \bar{\psi}_{\dot{\beta}}(\Lambda^{-1}(x-a))(S^\dagger)^{\dot{\beta}}_{\ \dot{\alpha}}$$

$$= \left[e^{+ia^\mu P_\mu} e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}\right]^{\dot{\beta}}_{\ \dot{\alpha}} \bar{\psi}_{\dot{\beta}}(x) \tag{1.2.120}$$

with

$$\Lambda^{\mu\nu} = \frac{1}{2} Tr[S\sigma^\nu S^\dagger \bar{\sigma}^\mu] \tag{1.2.121}$$

and $\omega^{\mu\nu}(\Lambda)$ given by equation (1.2.57)

$$e^{\frac{1}{2}\omega_{\mu\nu}(\Lambda)(x^\mu\partial^\nu - x^\nu\partial^\mu)} x^\rho = (\Lambda^{-1})^\rho_{\ \sigma} x^\sigma. \tag{1.2.122}$$

Thus we have found all representations of the Poincare' group.

Finally, let's relate our two-component Weyl spinors to the usual Dirac four-component spinors. We can realize the Clifford algebra defining the 4x4 Dirac matrices $\gamma^{\mu a}_{\ \ b}$ by using the Pauli matrices, this representation being referred to as the Weyl basis (or representation) or the chiral basis (or representation). Defining the Dirac matrices as

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{1.2.123}$$

that is,

$$\gamma^{0a}{}_{b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{1a}{}_{b} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{2a}{}_{b} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{3a}{}_{b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab} . \tag{1.2.124}$$

Thus, the γ^μ obey the defining Dirac anti-commutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbf{1}. \tag{1.2.125}$$

Also, we can define an additional matrix γ_5

$$\gamma_5 \equiv +i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \begin{pmatrix} -\sigma^0 & 0 \\ 0 & +\bar{\sigma}^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} . \tag{1.2.126}$$

In this basis the 4 component complex Dirac spinor, denoted ψ_D^a, is given in terms of two Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$

$$\psi_D^a \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix}_a . \tag{1.2.127}$$

Under a Lorentz transformation the Dirac spinor transforms as

$$\psi_D'^a(x') = L^a{}_b \psi_D^b(x) \tag{1.2.128}$$

where

$$L^a{}_b = \begin{pmatrix} S_\alpha{}^\beta & 0 \\ 0 & (S^{\dagger -1})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}_{ab} . \tag{1.2.129}$$

Further since $\Lambda^{\mu\nu}\sigma_\mu = S\sigma^\nu S^\dagger$ and $\Lambda^{\mu\nu}\bar\sigma_\mu = S^{\dagger -1}\bar\sigma^\nu S^{-1}$ we have

$$\Lambda^{\mu\nu}\gamma^a_{\mu b} = L^a{}_c \gamma^{\nu c}{}_d (L^{-1})^d{}_b . \tag{1.2.131}$$

For left, ψ_{DL}, and right, ψ_{DR}, handed spinors we have

$$\psi_{DL} \equiv \frac{1}{2}(1-\gamma_5)\psi_D$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar\chi^1 \\ \bar\chi^2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \tag{1.2.132}$$

and

$$\psi_{DR} \equiv \frac{1}{2}(1+\gamma_5)\psi_D$$
$$= \begin{pmatrix} 0 \\ 0 \\ \bar\chi^1 \\ \bar\chi^2 \end{pmatrix} . \tag{1.2.133}$$

Thus, we have that ψ_{DL} corresponds to our $(\frac{1}{2}, 0)$ spinor ψ_α while ψ_{DR} corresponds to our $(0, \frac{1}{2})$ spinor $\bar\chi^{\dot\alpha}$. If the Dirac spinor is a Majorana spinor, ψ_M, that is ψ_D is self charge conjugate, then

$$\psi^C_M = C\bar\psi^T_M \equiv \psi_M \tag{1.2.134}$$

with the charge conjugation matrix C given by $(C\gamma^\mu C^{-1} = -\gamma^{\mu T})$

$$C = i\gamma^2\gamma^0$$
$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\bar\sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.2.135}$$

and the conjugate Dirac spinor is defined as

$$\bar\psi_D \equiv \psi^\dagger_D \gamma^0 . \tag{1.2.136}$$

The Majorana condition (1.2.134) implies that

$$C\bar{\psi}_M^T = i\gamma^2\gamma^0\gamma^0\psi_M^*$$

$$= \begin{pmatrix} 0 & i\sigma^2 \\ i\bar{\sigma}^2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi} \\ \chi \end{pmatrix}$$

$$= \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \equiv \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{1.2.137}$$

or $\psi = \chi, \bar{\psi} = \bar{\chi}$. Hence we find that a 4 component Majorana spinor is made up of a 2 component Weyl spinor and its complex conjugate

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{1.2.138}$$

Needless to say the Weyl representation for the Dirac γ matrices is not the only way we could have reralized the Dirac algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}. \tag{1.2.139}$$

After all, this remains invariant under unitary transformations U, $U^\dagger = U^{-1}$

$$\hat{\gamma}^\mu \equiv U^\dagger\gamma^\mu U,$$

and so (1.2.139) becomes

$$\hat{\gamma}^\mu\hat{\gamma}^\nu + \hat{\gamma}^\nu\hat{\gamma}^\mu = 2g^{\mu\nu}\mathbf{1}. \tag{1.2.140}$$

Further we can use these unitary transformations to define linear combinations of the Weyl spinor components to form a four-component complex spinor

$$\hat{\psi}_D \equiv U^\dagger\psi_D. \tag{1.2.141}$$

Under a Poincare' transformation we have

$$\hat{\psi}'_D(x') = U^\dagger\psi'_D(x') = U^\dagger L\psi_D(x) = U^\dagger LUU^\dagger\psi_D(x)$$

$$= U^\dagger LU\psi_D(x) = \hat{L}\hat{\psi}_D(x) \tag{1.2.142}$$

where $\hat{L} \equiv U^\dagger L U$. As before we have for the hatted transformations

$$\Lambda^{\mu\nu}\hat{\gamma}_\mu = \Lambda^{\mu\nu}U^\dagger\gamma_\mu U = U^\dagger L\gamma^\nu L^{-1}U = U^\dagger LUU^\dagger\gamma^\nu UU^\dagger L^{-1}U$$
$$= \hat{L}\hat{\gamma}^\nu\hat{L}^{-1}. \quad (1.2.143)$$

Thus, all relations go through as before with all quantities replaced by their hatted values.

There are several common choices for the four- component Dirac quantities. We have first defined the Weyl (or chiral) representation, in brief review in obvious notation

$$\gamma^\mu_{Weyl} \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (1.2.144)$$

That is

$$\gamma^0_{Weyl} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\gamma^i_{Weyl} = \begin{pmatrix} o & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (1.2.145)$$

The Weyl basis Dirac spinor, now denoted ψ_{Weyl}, is given as

$$\psi_{Weyl} \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix}. \quad (1.2.146)$$

Under Poincare' transformations

$$\psi'_{Weyl}(x') = L\psi_{Weyl}(x) \quad (1.2.147)$$

with

$$L = \begin{pmatrix} S & 0 \\ 0 & S^{\dagger -1} \end{pmatrix}. \quad (1.2.148)$$

Left handed and right handed chiral spinors are defined by

$$\psi_{Weyl\ L} \equiv \frac{1}{2}(1 - \gamma_{5\ Weyl})\psi_{Weyl}(x) = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$
$$\psi_{Weyl\ R} \equiv \frac{1}{2}(1 + \gamma_{5\ Weyl})\psi_{Weyl}(x) = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (1.2.149)$$

with

$$\gamma_{5\ Weyl} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}.$$

Another common representation is that of Dirac

$$\gamma^{\mu}_{Dirac} \equiv U^{\dagger}\gamma^{\mu}_{Weyl}U \tag{1.2.150}$$

with

$$U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$U^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.2.151}$$

Hence

$$\gamma^{\mu}_{Dirac} = \frac{1}{2}\begin{pmatrix} (\sigma+\bar{\sigma})^{\mu} & (\sigma-\bar{\sigma})^{\mu} \\ (\bar{\sigma}-\sigma)^{\mu} & -(\sigma+\bar{\sigma})^{\mu} \end{pmatrix} \tag{1.2.152}$$

that is

$$\gamma^{0}_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\gamma^{i}_{Dirac} = \begin{pmatrix} 0 & \sigma^{i} \\ \bar{\sigma}^{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}. \tag{1.2.153}$$

Writing out all the components in order to be explicit, we have

$$\gamma^{0}_{Dirac} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^{1}_{Dirac} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^{2}_{Dirac} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^{3}_{Dirac} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.2.154}$$

The γ_5 matrix becomes

$$\gamma_{5\ Dirac} \equiv U^\dagger \gamma_{5\ Weyl} U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.2.155}$$

Note the γ matrices in all representations obey $\gamma^{0\dagger} = \gamma^0, \gamma^{i\dagger} = -\gamma^i, \gamma_5^\dagger = \gamma_5$. The Dirac four component spinors (or bi-spinors as they are sometimes called) in the Dirac representation are

$$\begin{aligned} \psi_{Dirac} \equiv U^\dagger \psi_{Weyl} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} (\psi + \bar{\chi}) \\ (-\psi + \bar{\chi}) \end{pmatrix}. \end{aligned} \tag{1.2.156}$$

Hence, the chiral spinors are given by

$$\begin{aligned} \psi_{Dirac\ L} &= \frac{1}{2}(1 - \gamma_{5\ Dirac})\psi_{Dirac} = U^\dagger \psi_{Weyl\ L} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ -\psi \end{pmatrix} \\ \psi_{Dirac\ R} &= \frac{1}{2}(1 + \gamma_{5\ Dirac})\psi_{Dirac} = U^\dagger \psi_{Weyl\ R} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\chi} \\ \bar{\chi} \end{pmatrix}. \end{aligned} \tag{1.2.157}$$

Another common representation is the Majorana representation in which all the γ matrices have pure imaginary matrix elements. In this basis we have

$$\gamma^\mu_{Majorana} \equiv U^\dagger \gamma^\mu_{Dirac} U \tag{1.2.158}$$

with the unitary transformation matrix also being hermitian and given as

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sigma^2 \\ \sigma^2 & -1 \end{pmatrix} = U^\dagger. \tag{1.2.159}$$