CHAPTER 1.
SPECIAL RELATIVITY AND QUANTUM MECHANICS

§1.1 PARTICLES AND FIELDS

The two great structures of theoretical physics, the theory of special relativity and quantum mechanics, have been combined in the framework of relativistic quantum field theory to provide a language for the description of the most fundamental interactions of the elemental constituents of matter. From our present vantage point we see that the justification for such a formalism is that it is extremely successful in explaining all known elementary particle experiments up to the present experimentally accessible energies of approximately 500 GeV; that is down to distances comparable to the de Broglie-Einstein wavelength for such energies $\lambda =$ hc/E $=2.5 \times 10^{-16}$ cm $=$ 0.0025 fermi. These energies are achieved at high energy laboratories such as Fermilab, CERN, SLAC and DESY. It is quite remarkable that quantum mechanics, a theory formulated in order to explain atomic phenomena involving energies around 10 eV, that is to describe experiments dealing with atomic distances comparable to the Bohr radius $a_{Bohr} = \hbar^2/me^2 = 0.5 \times 10^{-8}$ cm , can, when combined with special relativity, provide a correct description of nature at distance scales a billion times shorter!

The origins of quantum field theory (QFT) , naturally, lie in those of quantum mechanics. Max Planck in explaining the blackbody radiation spectrum and Albert Einstein in analysing the photoelectric effect both concluded that electromagnetic radiation although classically appearing as a wave actually consisted of a superposition of many particles called photons. Photons obey the special relativistic relation between energy and momentum for massless particles; namely $E^2 = \vec{p}^{\,2} c^2$ with the de Broglie relation for the momentum $\vec{p} = \hbar\vec{k}$ with $\vec{k}$ the wave vector of the photon. Hence the energy of a collection of N monochromatic photons is simply $E = N\hbar\omega$, where the photon's angular frequency $\omega = |\vec{k}|c$. In addition the electomagnetic wave has two independent directions of polarization orthogonal to the direction of propagation of the wave. Hence we infer from this another property of the individual photons, that is, there are two states of polarization possible for a photon. So we find that the elemental constituents of electromagnetic radiation are particles called photons each being completely characterized by its momentum $\vec{p} = \hbar \vec{k}$ and its direction of polarization, which is orthogonal to the momentum. Calling the polarization vectors $\vec{\epsilon}_1(\vec{k})$ and $\vec{\epsilon}_2(\vec{k})$, we have that $\vec{\epsilon}_r \cdot \vec{\epsilon}_s = \delta_{rs}$ and $\vec{\epsilon}_r(\vec{k}) \cdot \vec{k} = 0$ for r=1,2 . In addition we can choose $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ such that $(\vec{\epsilon}_1(\vec{k}), \vec{\epsilon}_2(\vec{k}), \frac{\vec{k}}{|\vec{k}|})$ form a right handed Cartesian coordinate system basis.

With the development of quantum mechanics in the 1920's came the means to mathematically describe the state of a system containing photons. Suppose we consider space to be free of charges so that the photons are non-interacting (or alternatively, consider these photons to be in a charge free volume with periodic boundary conditions so that their wave-vectors are integer multiples of 2π over the volume's linear dimension in each direction). Then the states of this system can be described as having zero photons, one single photon, two photons, and so on. According to the postulates of quantum mechanics these states of the system are represented by vectors in Hilbert space. Each state and hence each vector is completely characterized by the values of all the simultaneously observable properties of the system. As we have seen each photon is completely characterized by its momentum $\hbar \vec{k}$ (once its momentum is specified its energy is known from the special relativistic energy formula $E = +\sqrt{\vec{p}^{\,2} c^2}$) and its polarization $\vec{\epsilon}_r(\vec{k})$. Thus we can denote the single photon states of the system by the Dirac ket-vector $|\vec{k}, r>$. That is $|\vec{k}, 1>$ is the state vector of the system when it contains a single photon of momentum $\hbar \vec{k}$ and polarization $\vec{\epsilon}_1(\vec{k})$. Likewise $|\vec{k}, 2>$ is the state vector of the system when it contains a single photon of momentum $\hbar \vec{k}$ and polarization $\vec{\epsilon}_2(\vec{k})$. The multi- photon states of the system can be built up as direct products of the one photon states. That is a state, $|(\vec{k}_1, r_1), (\vec{k}_2, r_2), ..., (\vec{k}_N, r_N)>$ with N photons with momentum $\vec{k}_1, \vec{k}_2, ..., \vec{k}_N$ and polarization $\vec{\epsilon}_{r_1}(\vec{k}_1), \vec{\epsilon}_{r_2}(\vec{k}_2), ..., \vec{\epsilon}_{r_N}(\vec{k}_N)$, respectively, is given by

$$|(\vec{k}_1, r_1), ..., (\vec{k}_N, r_N)> = |\vec{k}_1, r_1> |\vec{k}_2, r_2> ... |\vec{k}_N, r_N> . \quad (1.1.1)$$

In particular the no photon state, called the vacuum, is denoted $|0>$, with obvious interpretation.

Since each photon carries momentum $\hbar \vec{k}$, the momentum of the N photon state
$|(\vec{k}_1, r_1), ..., (\vec{k}_N, r_N)>$ is just the sum of the individual photon's momentum

$\sum_{i=1}^{N} \hbar\vec{k}_i$. Hence the momentum operator on the N photon state is

$$\vec{P}|(\vec{k}_1, r_1), (\vec{k}_2, r_2), ..., (\vec{k}_N, r_N) >= (\sum_{i=1}^{N} \hbar\vec{k}_i)|(\vec{k}_1, r_1), ..., (\vec{k}_N, r_N) > . \tag{1.1.2}$$

Similarly the energy of each photon is $\hbar\omega$,so the energy of the N photon state is simply the sum $\sum_{i=1}^{N} \hbar\omega_{k_i}$ where recall that $\omega_{k_i} = c|\vec{k}_i|$ Hence the Hamiltonian operating on the photon state yields

$$H|(\vec{k}_1, r_1), ..., (\vec{k}_N, r_N) >= (\sum_{i=1}^{N} \hbar\omega_{k_i})|(\vec{k}_1, r_1), ...(\vec{k}_N, r_N) > . \tag{1.1.3}$$

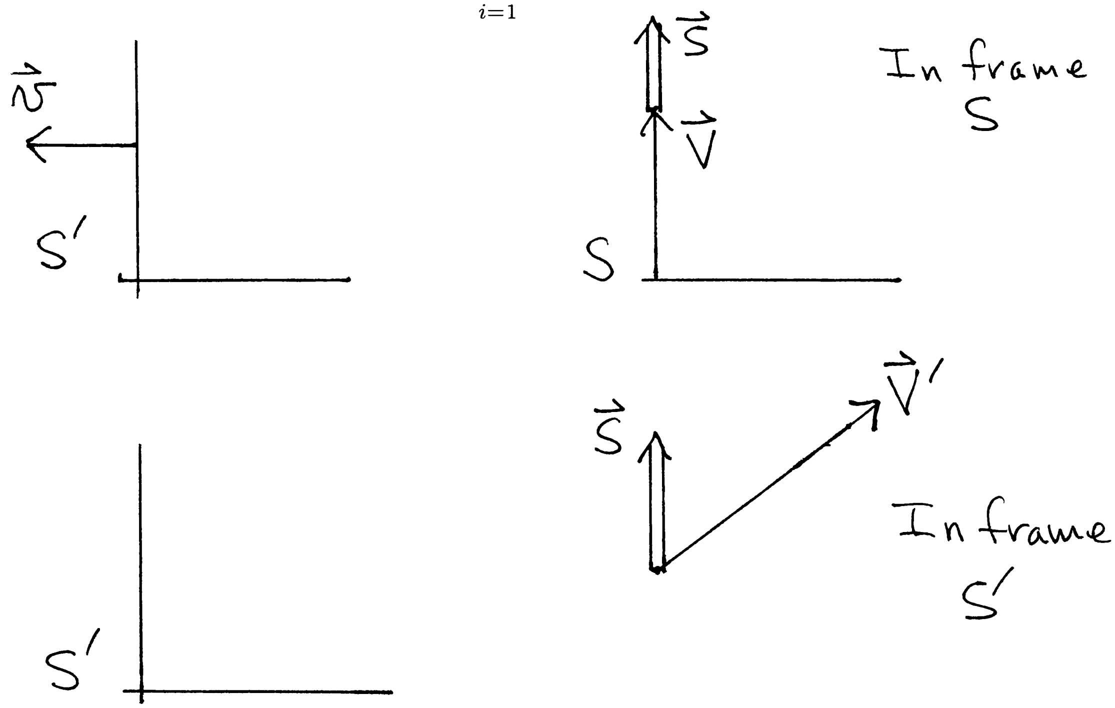

Figure 1.1.1

Besides the energy and momentum of the state we can further specify its polarization which is related it to the angular momentum of the photon. For a zero mass particle the angular momentum projected onto the direction of motion has a frame independent or intrinsic meaning and hence is an inherent property of the particle. Of course for a massive particle this is not so since we can always go to the rest frame of the particle then its spin is parallel or anti-parallel to nothing; we can even reverse its direction (start with spin parallel to the direction of motion, transform to another frame moving with +2v and the spin is anti-parallel to the direction of the particle's motion in this frame). Suppose we have a particle with spin projected along the direction of motion in one frame S. Let S' be a frame moving perpendicular to this direction with speed v. Hence the motion of the particle in S' appears skewed; spin and momentum are no longer parallel. But if the particle has velocity $V \gg v$ the angle between the spin and the particle's direction of motion is small. If the mass of the particle is zero its velocity is c, V=c, and the angle between spin and the direction of motion is zero. Hence the spin and the particle's direction of motion are aligned in all frames.

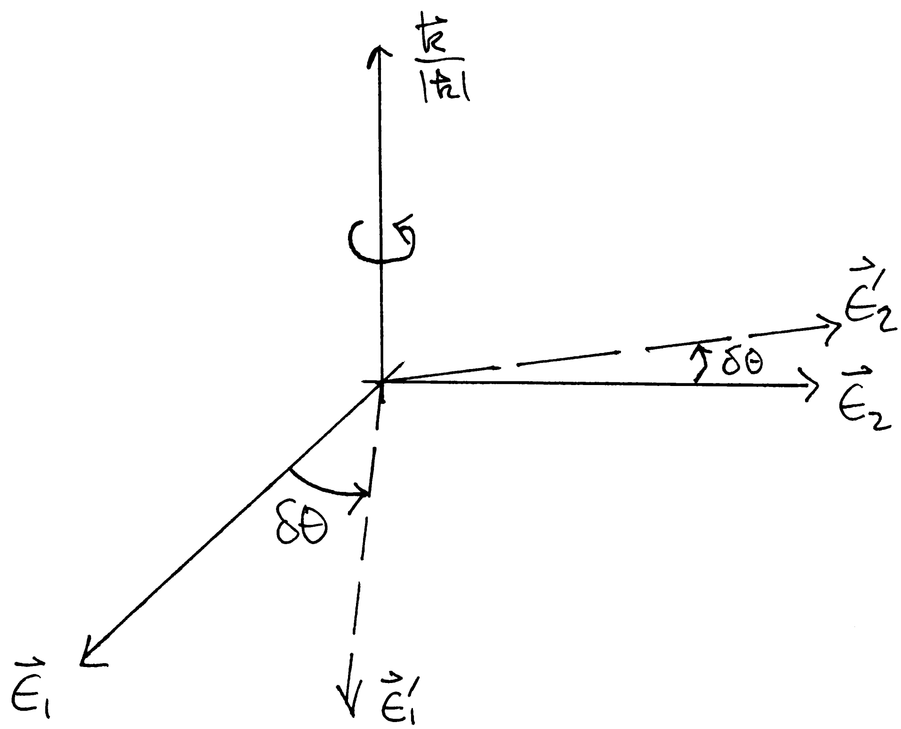

Figure 1.1.2

As we know there are right handed and left handed circularly polarized states for the photons, these imply a spin of +1 or -1, in units of $\hbar$, when projected along the $\vec{k}$ direction, parallel or anti-parallel to the $\vec{k}/|\vec{k}|$ direction. The photon is said to be a spin 1 object (the spin 0 state is missing due to the transversality of the photon). Its intrinsic angular momentum when projected along the direction of motion is $\pm\hbar$. That is

$$\vec{J} \cdot \vec{k}/|\vec{k}||k, \pm >= \pm\hbar|\vec{k}, \pm > \tag{1.1.4}$$

where $|\vec{k}, \pm >$ denote the right handed and left handed circularly polarized states. To elucidate this point further consider rotations of the plane polarization vectors about the $\vec{k}$ axis. First since $\vec{\epsilon}_r(\vec{k})$ is a vector we find that under infinitesimal rotations through angle $\delta\theta$

$$\begin{aligned} \vec{\epsilon}_1{}' &= \vec{\epsilon}_1 + \delta\theta\vec{\epsilon}_2 \\ \vec{\epsilon}_2{}' &= \vec{\epsilon}_2 - \delta\theta\vec{\epsilon}_1. \end{aligned} \tag{1.1.5}$$

Hence the variation of $\vec{\epsilon}_r$ is

$$\begin{aligned} \delta\vec{\epsilon}_1 &= \vec{\epsilon}_1{}' - \vec{\epsilon}_1 = \delta\theta\vec{\epsilon}_2 \\ \delta\vec{\epsilon}_2 &= \vec{\epsilon}_2{}' - \vec{\epsilon}_2 = -\delta\theta\vec{\epsilon}_1. \end{aligned} \tag{1.1.6}$$

In the S' coordinate system we have a photon of polarization $\vec{\epsilon}_1{}'$, for instance, described by the state vector $|\vec{k}, 1' >$. In the S frame this photon's state vector $|\vec{k}, 1' >$ is just a superposition of the two photon polarization states in the unrotated frame

$$|\vec{k}, 1' >= |\vec{k}, 1 > +\delta\theta|\vec{k}, 2 > . \tag{1.1.7}$$

In quantum mechanics, rotations, as well as other symmetry transformations, are carried out by unitary operators. For rotations about the $\vec{k}$ axis this operator is

$$U(\delta\theta) = e^{-\frac{i}{\hbar}\delta\theta\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}} \tag{1.1.8}$$

where $\vec{J}$ is the angular momentum operator with components that obey the angular momentum commutation relations

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k. \tag{1.1.9}$$

Rotations are said to be generated by the angular momentum operator. The rotation operator then transforms the photon state in the initial frame into the photon state in the rotated frame

$$|\vec{k}, 1' >= U(\delta\theta)|\vec{k}, 1 > . \tag{1.1.10}$$

For infinitesimal rotations we can expand the exponential to first order only to find

$$
\begin{aligned}
|\vec{k},1' >&= \left(1-\frac{i}{\hbar}\delta\theta\vec{J}\cdot\frac{\vec{k}}{|k|}\right)|\vec{k},1>\\
&= |\vec{k},1> - \left(\frac{i}{\hbar}\delta\theta\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}\right)|\vec{k},1>, \qquad (1.1.11)
\end{aligned}
$$

comparing this with equation (1.1.7) for the rotated photon state we have that

$$
\begin{aligned}
|\vec{k},1' >&= |\vec{k},1> - \left(\frac{i}{\hbar}\delta\theta\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}\right)|\vec{k},1>\\
&= |\vec{k},1> +\delta\theta|\vec{k},2> \qquad (1.1.12)
\end{aligned}
$$

so

$$\left(\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}\right)|\vec{k},1>= +i\hbar|\vec{k},2> . \qquad (1.1.13)$$

Similarly we find

$$\left(\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}\right)|\vec{k},2>= -i\hbar|\vec{k},1> . \qquad (1.1.14)$$

The right handed and left handed circularly polarized photon states are just the eigenstates of the projected angular momentum and are superpositions of the plane polarization states. That is we define right handed and left handed polarization vectors (+ for right,− for left)

$$\vec{\epsilon}_{\pm}(\vec{k}) \equiv \mp\frac{1}{\sqrt{2}}(\vec{\epsilon}_1(\vec{k}) \pm i\vec{\epsilon}_2(\vec{k})), \qquad (1.1.15)$$

whose variation under rotations is found to be

$$\delta\vec{\epsilon}_{\pm} = \mp i\delta\theta\vec{\epsilon}_{\pm}. \qquad (1.1.16)$$

Thus the corresponding right handed and left handed circularly polarized photon states are defined by

$$|\vec{k},\pm >\equiv \pm\frac{1}{\sqrt{2}}(|\vec{k},1> \pm i|\vec{k},2>) \qquad (1.1.17)$$

and we find that

$$\left(\vec{J}\cdot\frac{\vec{k}}{|\vec{k}|}\right)|\vec{k},\pm>=\pm\hbar|\vec{k},\pm> . \tag{1.1.18}$$

The projection of the angular momentum along the direction of motion is called the helicity of the particle. The photon has helicity ± 1. We can use linearly polarized photon states $|\vec{k},r>$ or circularly polarized photon states $|\vec{k},\pm>$ or any superposition. For each value of $\vec{k}$ either pair form a basis for one photon states in Hilbert space.

As we have seen the N photon states are written as direct products of the one photon states and hence the total Hilbert space of states of our system is a sum over the N photon subspaces. The inner product of state vectors as well as the resolution of the identity in the whole space follows from the corresponding one photon state properties. Since the one photon states with different momentum and polarization are orthogonal we have that

$$<\vec{k}',r'|\vec{k},r>=\mu(\vec{k})^{-1}\delta_{rr'}\delta^3(\vec{k}-\vec{k}') \tag{1.1.19}$$

where $\mu(\vec{k})$ is the arbitrary normalization factor to be specified. Since the one particle subspace is spanned by $|\vec{k},r>$ we have the resolution of the identity on this subspace

$$1=\sum_{r=1}^{2}\int d^3k\mu(\vec{k})|\vec{k},r><\vec{k},r| \tag{1.1.20}$$

where $\mu(\vec{k})$ is the same normalization factor appearing in the inner product. We can check this by letting it operate on the state $|\vec{k},r>$

$$|\vec{k},r>=\sum_{s=1}^{2}\int d^3l\mu(\vec{l})(<\vec{l},s|\vec{k},r>)|\vec{l},s>$$

$$=\sum_{s=1}^{2}\int d^3l\mu(\vec{l})\left[\mu(\vec{l})^{-1}\delta_{rs}\delta^3(\vec{k}-\vec{l})\right]|\vec{l},s>$$

$$=|\vec{k},r> . \tag{1.1.21}$$

Typically $\frac{1}{\mu(\vec{k})}$ is taken to be factors such as 1, $(\sqrt{(2\omega_k)}), 2\omega_k$ times various powers of 2π. In these notes we will choose the Lorentz invariant convention that

$$\mu(\vec{k}) = \frac{1}{((2\pi)^3 2\omega_k)} \tag{1.1.22}$$

with $\omega_k = |\vec{k}|c$ or more generally for massive particles $\omega_k = \sqrt{\vec{k}^2 + m^2}$ (recall c=1). Hence for us the inner product is

$$< \vec{k}', r'|\vec{k}, r >= (2\pi)^3 2\omega_k \delta_{rr'} \delta^3(\vec{k} - \vec{k}') \tag{1.1.23}$$

and the one photon subspace identity is resolved to be

$$1 = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\vec{k}, r >< \vec{k}, r| \ . \tag{1.1.24}$$

This choice of normalization is invariant under restricted Lorentz transformations as can be seen by re-writing the momentum integral in a manifestly invariant form. Consider

$$d^4k\delta(k^2 - m^2)\theta(k^o) \tag{1.1.25}$$

where $\theta(k^0)$ is the step function. This expression is invariant since d^4k as well as $k^2 = k_\lambda k^\lambda$ are; since restricted Lorentz transformations preserve the sign of k^0, it also is manifestly invariant. Writing the Dirac delta function as the sum of positive and negative energy delta functions

$$\begin{aligned} \delta(k^2 - m^2) &= \delta((k^0 - \omega_k)(k^0 + \omega_k)) \\ &= \frac{1}{2\omega_k}\delta(k^0 - \omega_k) + \frac{1}{2\omega_k}\delta(k^0 + \omega_k), \end{aligned} \tag{1.1.26}$$

with $\omega_k \equiv \sqrt{k^2 + m^2}$, the restricted Lorentz invariance is proved and we have

$$\frac{d^3k}{2\omega_k} = d^4k\delta(k^2 - m^2)\theta(k^0). \tag{1.1.27}$$

We can discuss more conveniently the multi- photon states by introducing operators that create and annihilate photons. When acting on a state vector the creation operator for a photon of momentum $\hbar\vec{k}$ and polarization

$\vec{\epsilon}_r(\vec{k})$, denoted, $a_r^\dagger(\vec{k})$, adds the photon to the state. Similarly the annihilation (or destruction) operator for a photon of momentum $\hbar\vec{k}$ and polarization $\vec{\epsilon}_r(\vec{k})$, denoted $a_r(\vec{k})$, removes that particular photon from the state when acting on it. $a_r(\vec{k})$ and $a_r^\dagger(\vec{k})$ are Hermitian conjugates of each other. Thus if we start with the no photon state, the vacuum state, and try to remove a photon from it we cannot and the operation gives zero

$$a_r(\vec{k})|0>\equiv 0. \tag{1.1.28}$$

On the other hand if we add a single photon to the vacuum state we obtain the one photon state

$$a_r^\dagger(\vec{k}) \equiv \ |\vec{k}, r> . \tag{1.1.29}$$

Starting with the one photon state we can destroy the one photon to return to the vacuum state

$$a_r(\vec{k})|\vec{k}', r' >= \alpha|0> \tag{1.1.30}$$

with α a constant of proportionality. So we find that

$$[a_r(\vec{k}), a_{r'}^\dagger(\vec{k}')]|0>= \alpha|0>\neq 0. \tag{1.1.31}$$

Assuming that the commutator is a c-number we can find α from the state normalization

$$< \vec{k}', r'|\vec{k}, r >=< 0|a_{r'}(\vec{k}')a_r^\dagger(\vec{k})|0>$$

or

$$\frac{1}{\mu(\vec{k})}\delta_{rr'}\delta^3(\vec{k}-\vec{k}') =< 0|[a_{r'}(\vec{k}'), a_r^\dagger(\vec{k})]|0> \tag{1.1.32}$$

since

$$a_{r'}(\vec{k}')|0>= 0.$$

Thus with $< 0|0>\equiv 1$ and $[a, a^\dagger]$ a c-number we have

$$[a_{r'}(\vec{k}'), a_r^\dagger(\vec{k})] = \frac{1}{\mu(\vec{k})}\delta_{rr'}\delta^3(\vec{k}-\vec{k}'). \tag{1.1.33}$$

With our conventions $\frac{1}{\mu(\vec{k})} = (2\pi)^3 2\omega_k$ this yields

$$[a_{r'}(\vec{k}'), a_r^\dagger(\vec{k})] = (2\pi)^3 2\omega_k \delta_{rr'}\delta^3(\vec{k}-\vec{k}'). \tag{1.1.34}$$

Further we can represent the two photon state by the action of $a^\dagger_r(\vec{k})$ twice

$$|(\vec{k}_1, r_1), (\vec{k}_2, r_2) >= a^\dagger_{r_1}(\vec{k}_1) a^\dagger_{r_2}(\vec{k}_2)|0> . \tag{1.1.35}$$

Since photons obey Bose-Einstein statistics we have that the two photon state is symmetric under the interchange of particles

$$|(\vec{k}_1, r_1), (\vec{k}_2, r_2) >= |(\vec{k}_2, r_2), (\vec{k}_1, r_1) > . \tag{1.1.36}$$

This implies that

$$[a^\dagger_{r'}(\vec{k}'), a^\dagger_r(\vec{k})] = 0 \tag{1.1.37}$$

taking the adjoint gives

$$[a_{r'}(\vec{k}'), a_r(\vec{k})] = 0. \tag{1.1.38}$$

Finally the general N photon state is given by $a^\dagger_r(\vec{k})$ acting N times on the vacuum

$$|(\vec{k}_1, r_1), ..., (\vec{k}_N, r_N) >= a^\dagger_{r_1}(\vec{k}_1)...a^\dagger_{r_N}(\vec{k}_N)|0> . \tag{1.1.39}$$

The structure of our Hilbert space of states is seen to be summarized by the creation and annihilation operator algebra

$$[a_{r'}(\vec{k}'), a^\dagger_r(\vec{k})] = (2\pi)^3 2\omega_k \delta_{rr'} \delta^3(\vec{k} - \vec{k}') \tag{1.1.40}$$

$$[a_{r'}(\vec{k}'), a_r(\vec{k})] = 0 = [a^\dagger_{r'}(\vec{k}'), a^\dagger_r(\vec{k})]. \tag{1.1.41}$$

(Due to our continuum normalization we must be careful about states with n photons of momentum $\vec{k}$ and polarization $\epsilon_r(\vec{k})$, $|n(\vec{k}, r)>$. These can be constructed by going over to discrete momentum notation by dividing up $\vec{k}$ space into cells of volume $\Delta\Omega_k$. Then

$$\int d^3k = \sum_k \Delta\Omega_k$$

$$\delta^3(\vec{k} - \vec{k}') = \frac{1}{\Delta\Omega_k} \delta_{kk'}. \tag{1.1.42}$$

If we define

$$a_{\vec{k},r} \equiv (\sqrt{\Delta\Omega_k})\ a_r(\vec{k}) \tag{1.1.43}$$

then

$$[a_{\vec{k}',r'}, a^\dagger_{\vec{k},r}] = \sqrt{(\Delta\Omega_{k'}\Delta\Omega_k)}(2\pi)^3 2\omega_k \delta_{rr'} \frac{1}{\Delta\Omega_k}\delta_{kk\prime}$$
$$= (2\pi)^3 2\omega_k \delta_{rr'}\delta_{kk'}. \quad (1.1.44)$$

The n identical photon state is

$$|n(\vec{k},r)> = \frac{1}{\sqrt{n!}}(a^\dagger_{\vec{k},r})^n|0> . \quad) \quad (1.1.45)$$

Further all of the observables, like H and $\vec{P}$, can be constructed in terms of the creation and annihilation operators. To do this note that

$$a_r^\dagger(\vec{k})a_r(\vec{k})|(\vec{k}_1,r_1),...,(\vec{k}_N,r_N)>$$
$$= \left(\sum_{i=1}^N (2\pi)^3 2\omega_k \delta_{rr_i}\delta^3(\vec{k}-\vec{k}_i)\right)|(\vec{k}_1,r_1),...,(\vec{k}_N,r_N)> . \quad (1.1.46)$$

Suppose we integrate $\vec{k}$ over some volume, Ω_k, in $\vec{k}$ space and sum over polarizations r=1,2

$$\sum_{r=1}^2 \int_{\Omega_k} \frac{d^3k}{(2\pi)^3 2\omega_k} a_r^\dagger(\vec{k})a_r(\vec{k})|(\vec{k}_1,r_1),\cdots,(\vec{k}_N,r_N)>$$
$$= \left(\sum_{i=1}^N \delta(\vec{k}_i \subset \Omega_k)\right)|(\vec{k}_1,r_1),\cdots,(\vec{k}_N,r_N)> \quad (1.1.47)$$

where

$$\delta(\vec{k}\subset\Omega_k) = \begin{cases} 1 \; if \;\; k\subset\Omega_k \\ 0 \; if \;\; k\not\subset\Omega_k \end{cases} . \quad (1.1.48)$$

If $\vec{k}_i \subset \Omega_k$ we have a contribution to the sum, we are just counting the number of photons with momentum contained in the volume Ω_k. Hence the operator

$$N_{\Omega_k} \equiv \sum_{r=1}^2 \int_{\Omega_k} \frac{d^3k}{(2\pi)^3 2\omega_k} a_r^\dagger(\vec{k})a_r(\vec{k}) \quad (1.1.49)$$

is the photon counting or number operator for volume Ω_k. If we integrate $\vec{k}$ over all space then we find the total photon number operator N_∞

$$N_\infty = \sum_{r=1}^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_r^\dagger(\vec{k})a_r(\vec{k}) \quad (1.1.50)$$

such that

$$N_\infty|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) >= N|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) > . \tag{1.1.51}$$

The number operator allows us to express the Hamiltonian and the momentum operators as sums over momentum of the number operator for photons of momentum $\vec{k}$ and either polarization times the photon's energy, $\hbar\omega_k$, or its momentum, $\hbar\vec{k}$

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \hbar\omega_k a_r^\dagger(\vec{k}) a_r(\vec{k}) \tag{1.1.52}$$

$$\vec{P} = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \hbar\vec{k} a_r^\dagger(\vec{k}) a_r(\vec{k}). \tag{1.1.53}$$

Using (1.1.40, 1.1.41) directly we can verify that indeed these give the energy and momentum of the N photon state

$$H|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) >= (\sum_{i=1}^{N} \hbar\omega_{k_i})|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) > \tag{1.1.54}$$

$$\vec{P}|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) >= (\sum_{i=1}^{N} \hbar\vec{k}_i)|(\vec{k}_1, r_1), \cdots, (\vec{k}_N, r_N) > \tag{1.1.55}$$

in agreement with equations (1.1.2) and (1.1.3).

Since the creation and annihilation operators correspond to a single photon energy, their time evolution in the Heisenberg representation is simply oscillatory

$$a_r(\vec{k}, t) \equiv e^{-i\omega_k t} a_r(\vec{k}). \tag{1.1.56}$$

We can use the Hamiltonian operator above and the canonical commutation relations to verify that the Heisenberg equations of motion are satisfied for $a_r(\vec{k}, t)$

$$[H, a_r(\vec{k}, t)] = -i\hbar \frac{\partial}{\partial t} a_r(\vec{k}, t). \tag{1.1.57}$$

That is

$$[H, a_r(\vec{k}, t)] = \sum_{s=1}^{2} \int \frac{d^3l}{(2\pi)^3 2\omega_l} \hbar\omega_l e^{-i\omega_k t} [a_s^\dagger(\vec{l}) a_s(\vec{l}), a_r(\vec{k})]$$

$$= \sum_{s=1}^{2} \int \frac{d^3 l}{(2\pi)^3 2\omega_l} \hbar\omega_l e^{-i\omega_k t} (-1)(2\pi)^3 2\omega_l \delta_{rs} \delta^3(\vec{k} - \vec{l}) a_s(\vec{l})$$

$$= -\hbar\omega_k a_r(\vec{k}, t)$$

$$= -i\hbar \frac{\partial}{\partial t} a_r(\vec{k}, t). \tag{1.1.58}$$

Rather than work in momentum space we can Fourier transform our creation and annihilation operators to work in a coordinate representation. Since $a_r^\dagger(\vec{k})$ creates a photon with momentum $\vec{k}$ and polarization $\vec{\epsilon}_r(\vec{k})$, that is a plane wave, its wave function is $\vec{\epsilon}_r(\vec{k})e^{-i\vec{k}\cdot\vec{x}}$, while $a_r(\vec{k})$ annihilates a photon with momentum $\vec{k}$ and polarization $\vec{\epsilon}_r(\vec{k})$, also a plane wave, so its wave function is $\vec{\epsilon}_r(\vec{k})e^{+i\vec{k}\cdot\vec{x}}$. Hence, summing over these plane waves, we define the Heisenberg representation quantum field operator in space-time

$$\vec{A}(\vec{x}, t) \equiv \vec{A}^{(+)}(\vec{x}, t) + \vec{A}^{(-)}(\vec{x}, t) \tag{1.1.59}$$

with

$$\vec{A}^{(+)}(\vec{x}, t) \equiv \sum_{r=1}^{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \vec{\epsilon}_r(\vec{k}) a_r(\vec{k}) e^{+i\vec{k}\cdot\vec{x} - i\omega_k t} \tag{1.1.60}$$

$$\vec{A}^{(-)}(\vec{x}, t) \equiv \sum_{r=1}^{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \vec{\epsilon}_r(\vec{k}) a_r^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}. \tag{1.1.61}$$

Writing $kx \equiv k^o x^o - \vec{k}\cdot\vec{x} = \omega_k t - \vec{k}\cdot\vec{x}$ we have simply

$$\vec{A}(x) = \sum_{r=1}^{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \vec{\epsilon}_r(\vec{k}) [a_r(\vec{k}) e^{-ikx} + a_r^\dagger(\vec{k}) e^{+ikx}]. \tag{1.1.62}$$

$\vec{A}^{(+)}$ is the positive frequency part of $\vec{A}$ and it annihilates photons with wave function

$$\vec{u}_r(x) = \vec{\epsilon}_r(\vec{k}) e^{-ikx} \tag{1.1.63}$$

and $\vec{A}^{(-)}$ is the negative frequency part of $\vec{A}$ and it creates photons with wave function

$$\vec{v}_r(x) = \vec{u}_r^*(x) = \vec{\epsilon}_r^{\,*}(\vec{k}) e^{+ikx} = \vec{\epsilon}_r(\vec{k}) e^{+ikx}. \tag{1.1.64}$$

The fact that $a_r(\vec{k}), a_r^\dagger(\vec{k})$ are quantum operators implies the same for $\vec{A}(x)$. The quantization rules, that is the commutation relations for $a_r(\vec{k})$ and $a_r^\dagger(\vec{k})$, imply commutation relations for the space-time fields also. Recall that the momentum space commutation rules look just like those of quantum mechanics for generalized coordinates and momenta. That is define

$$(2\pi)^3 2\omega_k Q_r(\vec{k}) \equiv (\sqrt{(\hbar/2)})[a_r(\vec{k}) + a_r^\dagger(\vec{k})] \tag{1.1.65}$$

$$P_r(\vec{k}) \equiv -i(\sqrt{(\hbar/2)})[a_r(\vec{k}) - a_r^\dagger(\vec{k})]. \tag{1.1.66}$$

Then the commutation relations (1.1.40, 1.1.41) imply that

$$[P_r(\vec{k}), Q_s(\vec{l}\,)] = -i\hbar\delta_{rs}\delta^3(\vec{k} - \vec{l}\,) \tag{1.1.67}$$

$$[Q_r(\vec{k}), Q_s(\vec{l}\,)] = 0 = [P_r(\vec{k}), P_s(\vec{l}\,)]. \tag{1.1.68}$$

(In the discrete representation

$$Q_r(\vec{k}) = \frac{1}{\sqrt{\Delta\Omega_k}} Q_{\vec{k},r} \tag{1.1.69}$$

$$P_r(\vec{k}) = \frac{1}{\sqrt{\Delta\Omega_k}} P_{\vec{k},r} \tag{1.1.70}$$

so that

$$[P_{\vec{k},r}, Q_{\vec{l},s}] = -i\hbar\delta_{rs}\delta_{kl}. \quad) \tag{1.1.71}$$

Note that the Hamiltonian becomes in term of Q_r and P_r

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3}[\frac{1}{4}P_r^2(\vec{k}) + (2\pi)^6\omega_k^2 Q_r^2(\vec{k})] - H_0 \tag{1.1.72}$$

with the constant

$$H_0 = \sum_{r=1} \int d^3k(\frac{1}{2}\hbar\omega_k)\delta_{rr}\delta^3(\vec{k} - \vec{k})$$

which is just that of an infinite collection of harmonic oscillators. The space-time quantum field can now be written as

$$\vec{A}(x) = \sum_{r=1}^{2} \int d^3k \sqrt{\frac{2}{\hbar}}\vec{\epsilon}_r(\vec{k}) \left[Q_r(\vec{k})\cos kx + \frac{1}{(2\pi)^3 2\omega_k} P_r(\vec{k})\sin kx\right]. \tag{1.1.73}$$

Treating $\vec{A}(x)$ as a generalized coordinate in our space-time representation, then we define its canonically conjugate momentum field as the "velocity" field

$$\vec{\Pi}(x) \equiv -\dot{\vec{A}}(x)$$

$$= \sum_{r=1}^{2} \int d^3k \sqrt{2\hbar}\ \vec{\epsilon}_r(\vec{k})\omega_k \left[Q_r(\vec{k}) \sin kx - \frac{1}{(2\pi)^3 2\omega_k} P_r(\vec{k}) \cos kx \right]. \quad (1.1.74)$$

The commutation relations for these Heisenberg fields at equal time become

$$[\Pi^i(\vec{x},t), A^j(\vec{y},t)] = +i \int \frac{d^3k}{(2\pi)^3} \left[\sum_{r=1}^{2} \epsilon_r^i(\vec{k}) \epsilon_r^j(\vec{k}) \right] \cos\left[\vec{k}\cdot(\vec{x}-\vec{y}) \right]$$

$$= +i \int \frac{d^3k}{(2\pi)^3} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= +i\delta_{tr}^{ij}(\vec{x}-\vec{y}) \quad (1.1.75)$$

where the transverse Dirac delta function is defined by

$$\delta_{tr}^{ij}(\vec{x}-\vec{y}) \equiv \int \frac{d^3k}{(2\pi)^3} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \left(\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla_x^2} \right) \delta^3(\vec{x}-\vec{y})$$

so that

$$\partial_i \delta_{tr}^{ij}(\vec{x}) = 0, \quad (1.1.76)$$

that is δ_{tr}^{ij} is divergenceless as is $\vec{A}$ and $\vec{\Pi}$. These are the form of the quantization rules in coordinate space.

Since the relativistic energy-momentum relation holds for photons, $\omega_k^2 - \vec{k}^2 = k^2 = 0$, the wave functions obey the relativistic dispersion relation so that

$$\partial^2 \vec{u}_r(x) = -k^2 \vec{u}_r(x) = 0. \quad (1.1.77)$$

The Heisenberg equations of motion describing the time evolution of the momentum space operators are converted into a partial differential equation, the wave equation, describing the time evolution of the space-time quantum field operators

$$\partial^2 \vec{A}(x) = 0. \quad (1.1.78)$$

In addition to this, recall that the polarization vectors are perpendicular to the momentum of the photon so the wave function is transverse,

$$\vec{\nabla} \cdot \vec{u}_r(x) = -i\vec{k} \cdot \vec{u}_r(x) = 0. \tag{1.1.79}$$

Consequently the space-time field is constrained to be divergenceless

$$\vec{\nabla} \cdot \vec{A}(x) = 0. \tag{1.1.80}$$

(Thus the necessity of the transverse $\delta-$function in the quantization rules.) These equations just have the form of the Maxwell equations in free space in the Coulomb gauge. In fact introducing electric and magnetic (quantum) fields $\vec{E}$ and $\vec{B}$, respectively, as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{1.1.81}$$

we find that

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{1.1.82}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.1.83}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1.1.84}$$

$$\begin{aligned} \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \\ &= -\nabla^2 \vec{A} + \frac{\partial^2 \vec{A}}{\partial (ct)^2} - \frac{\partial}{\partial (ct)}\left(\frac{\partial \vec{A}}{\partial (ct)}\right) \\ &= \partial^2 \vec{A} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{1.1.85}$$

These are just the Maxwell equations in vacuum. Further we can calculate the Hamiltonian and momentum operators in terms of these fields to find expressions similar to the classical energy and Poynting vector ones

$$H = \frac{1}{2}\int d^3x(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) - H_0 \tag{1.1.86}$$

$$\vec{P} = \frac{1}{2}\int d^3x \vec{E} \times \vec{B} \tag{1.1.87}$$

where H_0 is given in equation(1.1.72)

$$H_0 = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{2}\omega_k [(2\pi)^3 2\omega_k \delta_{rr} \delta^3(\vec{k} - \vec{k})]$$

$$= \sum_{r=1}^{2} \sum_{k} \frac{1}{2}\omega_k. \tag{1.1.88}$$

H_o is just the infinite zero point energy of our infinite collection of quantum harmonic oscillators. Since we defined the vacuum as having zero energy we subtract off this zero point constant energy from the Maxwell form of the Hamiltonian.

The dynamics of the quantum vector potential can be made to look more Lorentz covariant by introducing the 4-vector potential $A^\mu(x), \mu = 0, 1, 2, 3$

$$A^\mu(x) = (\phi(x), \vec{A}(x)) \tag{1.1.89}$$

where $\phi(x)$ is the quantum scalar potential, which will be zero in the Coulomb gauge. We also define with the anti-symmetric covariant field strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1.1.90}$$

With the definitions of the electric and magnetic fields now given as

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{1.1.91}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.1.92}$$

the field strength tensor becomes

$$F^{\mu\nu} = \begin{bmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{bmatrix}. \tag{1.1.93}$$

Our dynamical equations for the vector potential can now be derived from an action principle with Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.1.94}$$

and action

$$\Gamma_o = \int d^4x\mathcal{L} \tag{1.1.95}$$

along with the Coulomb gauge subsidiary condition

$$\vec{\nabla}\cdot\vec{A} = 0. \tag{1.1.96}$$

Thus the action can be written as

$$\Gamma_0 = \frac{1}{2}\int d^4xA_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu \tag{1.1.97}$$

where it is assumed that $A^\mu(x) \to 0$ as $x \to \infty$ so that total space-time divergences can be thrown away. The variation of the action with respect to A^μ yields the Euler-Lagrange equations of motion for the field

$$\frac{\partial\mathcal{L}}{\partial A^\mu} - \partial^\lambda\frac{\partial\mathcal{L}}{\partial(\partial^\lambda A^\mu)} = 0 \tag{1.1.98}$$

which are

$$\partial_\lambda F^{\lambda\mu} = \partial^2 A^\mu - \partial^\mu\partial_\lambda A^\lambda = 0. \tag{1.1.99}$$

Now the $\mu = 0$ component of this is the equation of motion for the scalar potential ϕ

$$\partial^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\partial_\lambda A^\lambda) = 0. \tag{1.1.100}$$

However we must impose the Coulomb gauge condition on this equation which implies

$$\partial_\lambda A^\lambda = \frac{1}{c}\frac{\partial\phi}{\partial t} - \vec{\nabla}\cdot\vec{A} = \frac{1}{c}\frac{\partial\phi}{\partial t} \tag{1.1.101}$$

hence (1.1.100) reduces to Poisson's equation $\nabla^2\phi = 0$ with the above boundary condition that $\phi(x) \to 0$ as $x \to \infty$. This yields the solution $\phi(x) \equiv 0$. Further $\partial_\lambda A^\lambda = 0$ so that the spatial components of (1.1.99) become the desired wave equation, $\partial^2\vec{A}(x) = 0$.

Although the action is manifestly Lorentz invariant the choice of the Coulomb gauge destroys the invariance. However the physical quantities, as we have seen for the Hamiltonian and momentum, are expressible in terms of $F^{\mu\nu}$ and hence are gauge and therefore Lorentz invariant. Later in the course we will use manifestly Lorentz invariant gauge subsidiary conditions rather than the Coulomb gauge. Recall that a gauge transformation of the potentials is given by

$$\vec{A}' = \vec{A} - \vec{\nabla}\Lambda \tag{1.1.102}$$

$$\phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t} \tag{1.1.103}$$

or in 4-vector notation

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda \tag{1.1.104}$$

where $\Lambda = \Lambda(x)$ is an arbitrary function of space-time. Clearly $\vec{E}$ and $\vec{B}$ are gauge invariant since under gauge transformations $F^{\mu\nu}$ is invariant

$$F'^{\mu\nu} = \partial^{\mu}A'^{\nu} - \partial^{\nu}A'^{\mu} = F^{\mu\nu} + \partial^{\mu}\partial^{\nu}\Lambda - \partial^{\nu}\partial^{\mu}\Lambda = F^{\mu\nu}. \tag{1.1.105}$$

As stated above all observables can be written in terms of the covariant field strength tensor $F^{\mu\nu}$. Hence any theory formulated in terms of the potentials A^{μ} must be locally gauge invariant if the observables are to be. We will study this invariance requirement in great detail later.

As we have seen we can formulate the dynamics of the photon field in terms of a Lagrangian and action principle (plus gauge fixing condition). In addition to time evolution of the field we also have the fact that the field is a quantum operator and obeys commutation relations (1.1.75). These also can be formulated in terms of quantization rules for the Lagrangian. The action is given in equation (1.1.97) applying the Coulomb gauge condition and the fact that $\phi = 0$ reduces it to

$$\Gamma_0 = \frac{1}{2}\int d^4x \partial_{\mu}\vec{A}\cdot\partial^{\mu}\vec{A}. \tag{1.1.106}$$

This is just to the action calculated from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\vec{A}\cdot\partial^{\mu}\vec{A}. \tag{1.1.107}$$

The momentum canonically conjugate to the fields is defined as

$$\Pi_i(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}^i}(x) = \dot{A}^i(x). \tag{1.1.108}$$

Hence the canonical quantization procedure can be formulated as

$$[\Pi^i(\vec{x},t), A^j(\vec{y},t)] = +i\delta^{ij}_{tr}(\vec{x}-\vec{y}), \tag{1.1.109}$$

with the canonical momentum defined above the transversality condition is built into the fields.

There is no reason we should not describe all relativistic particles in a similar manner. For instance consider spin 0, neutral particles with mass m. They are described solely by their momentum $\hbar\vec{k}$ and mass m , having energy $E^2 = \vec{p}^{\,2}c^2 + m^2c^4$ and no spin. The totality of free particle states is then given by

$$\begin{aligned} |0> ----&vacuum\ state \\ |\vec{k}> ----&1\ particle\ state \\ |\vec{k}_1,\vec{k}_2>=|\vec{k}_1>|\vec{k}_2> ----&2\ particle\ state \\ &... \\ &... \\ &... \\ |\vec{k}_1,\vec{k}_2,...,\vec{k}_N>=|\vec{k}_1>...|\vec{k}_N> ----&N\ particle\ state \\ &... \\ &... \\ &... \end{aligned} \quad . \tag{1.1.110}$$

The inner product in the Hilbert space is given as

$$<\vec{k}'|\vec{k}>=(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \tag{1.1.111}$$

with

$$\omega_k \equiv \sqrt{\vec{k}^2+m^2}. \tag{1.1.112}$$

Again we can introduce creation and annihilation operators for these scalar particles $a^\dagger(\vec{k})$ and $a(\vec{k})$, respectively, so that

$$a(\vec{k})|0>\equiv 0 \tag{1.1.113}$$

while

$$a^\dagger(\vec{k})|0> \equiv |\vec{k}> \tag{1.1.114}$$

$$a^\dagger(\vec{k}_1)...a^\dagger(\vec{k}_N)|0> \equiv |\vec{k}_1, ..., \vec{k}_N> \tag{1.1.115}$$

and so on.

Hence the operators obey the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{l})] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}) \tag{1.1.116}$$

$$[a(\vec{k}), a(\vec{l})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{l})]. \tag{1.1.117}$$

The number operator, Hamiltonian and momentum operator are given by

$$N_\infty = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger(\vec{k}) a(\vec{k}) \tag{1.1.118}$$

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \hbar\omega_k a^\dagger(\vec{k}) a(\vec{k}) \tag{1.1.119}$$

$$\vec{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \hbar\vec{k} a^\dagger(\vec{k}) a(\vec{k}) \tag{1.1.120}$$

with

$$N_\infty|\vec{k}_1, ..., \vec{k}_N> = N|\vec{k}_1, ..., \vec{k}_N> \tag{1.1.121}$$

$$H|\vec{k}_1, ..., \vec{k}_N> = (\sum_{i=1}^{N} \hbar\omega_{k_i})|\vec{k}_1, ..., \vec{k}_N> \tag{1.1.122}$$

$$\vec{P}|\vec{k}_1, ..., \vec{k}_N> = (\sum_{i=1}^{N} \hbar\vec{k}_i)|\vec{k}_1, ..., \vec{k}_N> . \tag{1.1.123}$$

Since these particles have no spin their wave functions are just plane waves $u(x) = e^{-ikx}$ and $v(x) = u^*(x) = e^{+ikx}$. Thus we can Fourier transform the creation and annihilation operators to obtain a quantum field operator for a scalar, spin 0, mass m particle

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} [a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{+ikx}]. \tag{1.1.124}$$

Because $k^2 = m^2$ we find that the time evolution of the field is given by the Klein- Gordon equation

$$(\partial^2 + m^2)\Phi(x) = 0. \tag{1.1.125}$$

The momentum space commutation relations become equal time commutation relations for the field $\Phi(x)$ and its canonically conjugate momentum $\Pi(x) = \dot{\Phi}(x)$

$$\delta(x^0 - y^0)[\Pi(x), \Phi(y)] = \delta(x^0 - y^0)[\dot{\Phi}(x), \Phi(y)] = -i\delta^4(x - y) \tag{1.1.126}$$

(there is no transversality condition to satisfy here).

As before all of this can be further summarized by specifying the Lagrangian for the scalar field from which the equations of motion follow as Euler-Lagrange equations. That is the action is given by

$$\Gamma_0 = \int d^4x \mathcal{L} \tag{1.1.127}$$

with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2. \tag{1.1.128}$$

The Euler-Lagrange equations describe the dynamics

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0 = -(\partial^2 + m^2)\Phi. \tag{1.1.129}$$

The momentum field cannonically conjugate to the field $\Phi(x)$ is defined in analogy to classical mechanics as

$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}}(x) = \dot{\Phi}(x). \tag{1.1.130}$$

The canonical quantization procedure then specifies the equal time commutation relations between the field "coordinate" and the canonically conjugate field "momentum"

$$\delta(x^0 - y^0)[\Pi(x), \Phi(y)] = -i\delta^4(x - y). \tag{1.1.131}$$

Similarly we can quantize fermions, particles with spin $\frac{1}{2}$. However, by the Pauli principle or Fermi-Dirac statistics, the multi-particle states are anti-symmetric under interchange of particles. Hence for consistency reasons their creation and annihilation operators will obey anti-commutation relations rather than commutation relations. In what follows we will develop the field theory for these anti-commuting operators.

Finally the space-time field operator approach to describe the relativistic quantum mechanics of particles allows us to manifestly maintain Lorentz invariance by summarizing the dynamics of our fields by means of the quantum field Lagrangian. In what we have discussed so far the particles have been noninteracting or free. The number operator for each particle state (oscillator mode) commuted with H, hence it did not change in time. Thus there was no absorption or emission of particles. In fact we did not even have their scattering here. We must introduce interactions among the particles so that the field operators not only create and annihilate single particles but can pair produce particles, allow for particle- anti-particle annihilation and so on. The field operators then take into account the effects of the cloud of virtual particles surrounding any interacting particle. The dynamics for non-interacting particles was summarized in their free field equations

$$\partial^2 \vec{A} = 0, \qquad for\ photons \tag{1.1.132}$$

$$(\partial^2 + m^2)\Phi = 0, \qquad for\ spinless\ scalars. \tag{1.1.133}$$

If they are to interact they must couple to other fields, that is there must appear sources on the right hand side of these field equations. Basically they will become coupled non-linear partial differential equations. We will not be able to solve them exactly only approximately. Perturbation theory will assume the interactions are small and so the solutions will be written as a power series expansion in their strength. However before delving into this in detail let's step back a bit and review special relativity and quantum mechanics since, as we have seen, they play such a crucial role in the formulation of field theory.