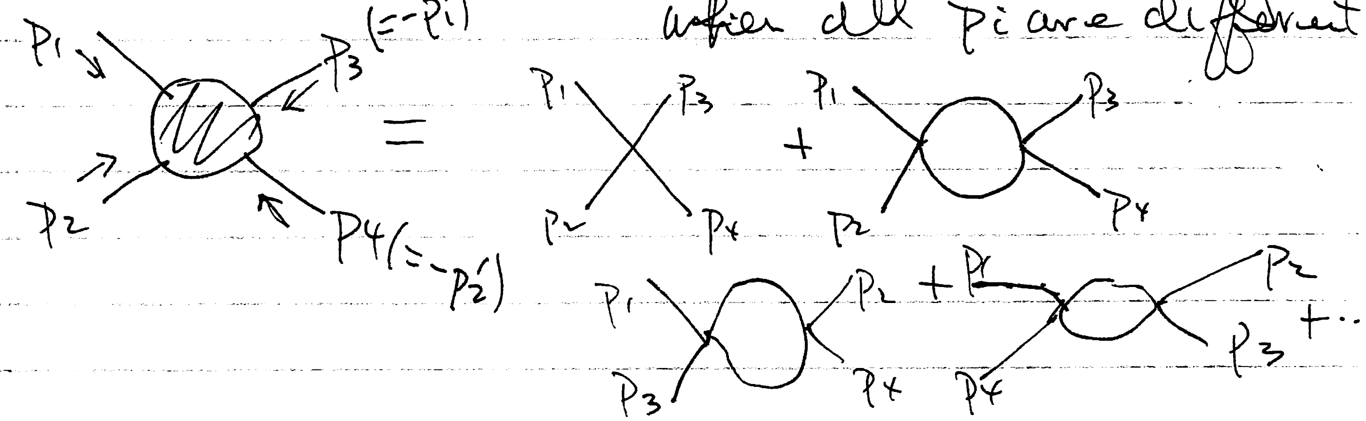
LSZ Reduction Formula
(without reduction)

Consider $\lambda\varphi^4$ theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4$$

We can calculate S_{fi} for $2\to 2$ scattering when all p_i are different



$$S_{2\to 2} = \langle p_3, p_4 | p_1, p_2\rangle$$

$$= (2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\Big[(-i\lambda) + \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{[(p_1+p_2+k)^2-m^2]} \times \frac{i}{[k^2-m^2]}$$

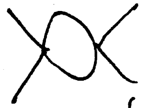
$$+ \text{2 other permutations}\Big]$$

Now instead of just the momentum space plane wavefunction for each external line, 1 in this case, write it as

$$1 = 1\,(-i)(p_1^2-m^2)\,\frac{i}{p_1^2-m^2}$$

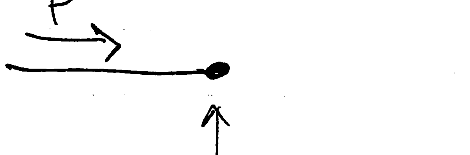
with the momentum for that external line, but now only take $p_1^2 \to m^2$ after cancelling factors, so

$$S_{2\to 2} = [-i(p_1^2-m^2)][-i(p_2^2-m^2)][-i(p_3^2-m^2)][-i(p_4^2-m^2)]$$

$$\times (2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \times \Big[$$  +  + perm

Same graphs new rules for external lines (incoming or outgoing) $\Big]_{\forall\, p_i^2 = m^2}$

External Lines (incoming or outgoing)



$$\longrightarrow \frac{i}{p^2-m^2}$$

just like any propagator

$$\phi(t,\vec{x}) = U^{-1}(t,0)\,\phi^{IP}(t,\vec{x})\,U(t,0)$$
$$U(t,t_0) = e^{iH_0^s t}\, e^{-iH(t-t_0)}\, e^{-iH_0^s t_0}$$

Now what are these new graphs? They are the Green functions. Cryptically

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$$

Heisenberg pix

$$= \langle 0|\phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_n})|0\rangle \quad \left(x^0_{i_1} > x^0_{i_2} \cdots > x^0_{i_n}\right)$$

$$= {}_H\langle 0|U^{-1}(t_{i_1},0)\,\phi^{IP}(x_{i_1})\,U(t_{i_1},0)\,U^{-1}(t_{i_2},0)\,\phi^{IP}(x_{i_2})\,U(t_{i_2},0)$$
$$\cdots\phi^{IP}(x_{in})\,U(t_{in},0)|0\rangle_H$$

$$= {}_H\langle 0|U^{-1}(\infty,0)\,U(\infty,0)\,U^{-1}(t_{i_1},0)\,\phi^{IP}(x_{i_1})\,U(t_{i_1},t_{i_2})$$
$$\phi^{IP}(x_{i_2})\,U(t_{i_2},t_{i_3})\cdots\phi^{IP}(x_{in})\,U(t_{in},0)\,U^{-1}(-\infty,0)$$
$$U(-\infty,0)|0\rangle_H$$

$$\left(U^{-1}(t,0) = U(0,t)\right)$$
So
$$U(t_1,0)\,U^{-1}(t_2,0) = U(t_1,t_2)$$

$$= {}_H\langle 0|U^{-1}(\infty,0)\,U(\infty,t_{i_1})\,\phi^{IP}(x_{i_1})\,U(t_{i_1},t_{i_2})\,\phi^{IP}(x_{i_2})$$
$$\cdots\phi^{IP}(x_{in})\,U(t_{in},-\infty)\,U(-\infty,0)|0\rangle_H$$

$$e^{i\varphi} = \langle 0| \ldots U(\infty,-\infty)|0\rangle$$
$$= \langle 0|T e^{i\int_{-\infty}^{\infty} \mathcal{L}_I^{IP} d^4x}|0\rangle$$

But $|A(t)\rangle_{IP} = U(t,0)|A\rangle_H$

So
$$U(-\infty,0)|0\rangle_H = |0(t\to-\infty)\rangle_{IP} = |0\rangle$$

and $U(\infty,0)|0\rangle_H = |0(t\to+\infty)\rangle_{IP} = e^{i\varphi}|0\rangle$

Also For this time order $x_{i_1}^0 > x_{i_2}^0 > \cdots > x_{i_n}^0$ we have that

$$T\,\phi^{IP}(x_1)\cdots\phi^{IP}(x_n)\,U(\infty,-\infty)$$

$$= U(\infty,t_{i_1})\,\phi^{IP}(x_{i_1})\,U(t_{i_1},t_{i_2})\,\phi^{IP}(x_{i_2})\,U(t_{i_2},t_{i_3}) \cdot \phi^{IP}(x_{i_3})\cdots\phi^{IP}(x_{i_n})\,U(t_{i_n},-\infty)$$

Thus we find

$$\boxed{\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle = \frac{\langle 0|T\left(\phi^{IP}(x_1)\cdots\phi^{IP}(x_n)\,e^{i\int d^4x\,\mathcal{L}_I^{IP}(x)}\right)|0\rangle}{\langle 0|T\,e^{i\int d^4x\,\mathcal{L}_I^{IP}(x)}|0\rangle}}$$

The Gell-Mann – Low Formula

But now we can expand the interaction picture terms — they are just what we found in the S-matrix case — except instead of $a(\vec{p}), a^{\dagger}(\vec{p})$ we have

ϕ's for the ~~ex~~ external lines. So the Fourier transform is exactly the same as a propagators $\langle 0|T\phi^{IP}(x)\phi^{IP}(y)|0\rangle$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2-m^2}$$

So we find $\underline{\underline{2}}$ things really -

1) The perturbation expansion of

$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ in terms of

Feynman diagrams.

2) The general result that all S-matrix elements are given by the residues of the Green functions in momentum space.

called "amputation" of external legs of Green functions

$$\boxed{S_{2\to 2} = \left[-i(p_1^2 - m^2)\right] \cdots \left[-i(p_4^2 - m^2)\right] \times \langle 0 | T\, \tilde{\phi}(p_1)\, \tilde{\phi}(p_2)\, \tilde{\phi}(p_3)\, \tilde{\phi}(p_4) | 0 \rangle \quad \forall\, p_i^2 = m^2}$$

Hence we are interested in calculating Green's function, the n-point function will have all the scattering info about n-body scattering (i.e. $2 \to n-2$, $4 \to n-4$ etc.)

So

$$\langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle \quad \text{(Heisenberg pix)}$$

for all n are central to theory.

Consider

$$Z[J] \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n$$

$$J(x_1) \cdots J(x_n) \langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

Shorthand

$$Z[J] = \langle 0 | T e^{i\int d^4x\, J(x)\phi(x)} | 0 \rangle$$

This is known as the generating <u>functional</u> for Green's functions.

$$\left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

functional derivative $\quad \frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)$

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1) Perturb. formula for $Z[J]$ — the Gell-Mann — Low formula

$$Z[J] = \frac{\langle 0 | T e^{i\int d^4x (\mathcal{L}_I^{IP} + J\phi^{IP})} | 0 \rangle}{\langle 0 | T e^{i\int d^4x\, \mathcal{L}_I^{IP}} | 0 \rangle}$$

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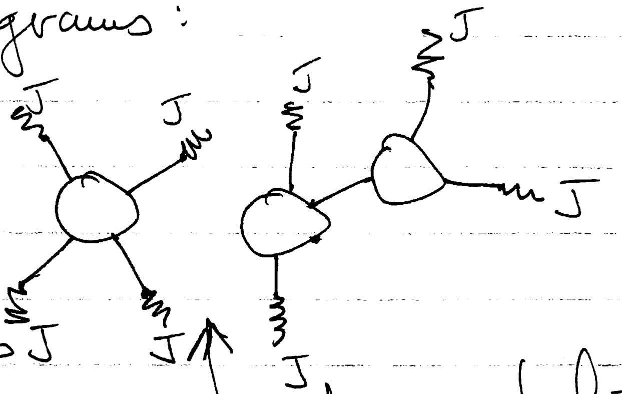
2) Connected diagrams:

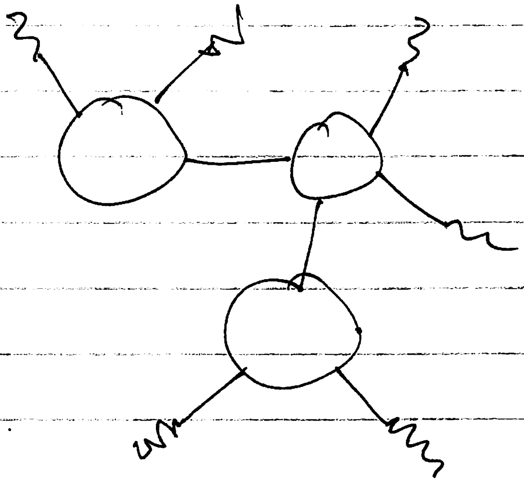
$$Z[J] = \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}$$



not connected = disconnected

$$= e^{Z^c[J]}$$

$$Z^c[J] = \sum_{\substack{\underline{\text{connected}} \\ \text{Feynman} \\ \text{diagrams} \\ \underline{\text{only}}}}$$



= called connected Green functions (or sometimes truncated functions)

3) Legendre transform connected functions to get "the effective action"

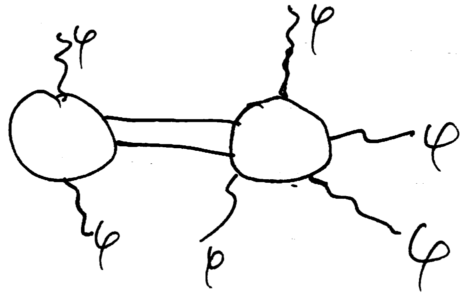
$$Z^c[J] = \Gamma[\varphi] + i\int d^4x\, J(x)\,\varphi(x)$$

where
$$\frac{\delta Z^c[J]}{\delta J(x)} = i\varphi(x)$$

likewise
$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -iJ(x).$$

One can show

1) $$\Gamma[\varphi] = \sum_{\substack{\text{1-particle}\\ \text{irreducible}\\ \text{diagrams}\\ \text{only}}}$$



2) $$\Gamma[\varphi] = \underbrace{i\int d^4x\,\mathcal{L}(x)}_{\text{tree}} + \underbrace{O(\hbar)}_{\text{one-loop etc.}}$$

$\hbar^n \leftarrow$ counts loops of diagrams