9.2. Exchange Effects In Elastic Scattering

Consider the elastic scattering of two identical spin zero particles which interact via the potential $V(|\vec{r}_1-\vec{r}_2|)$. This 2-body problem reduces in the CM system to an effective central potential scattering problem. At first if we ignore any symmetry effects arising from the indistinguishability of the particles we have asymptotically that

$$\psi(\vec{r}) \underset{r\to\infty}{\sim} e^{ikz} + \frac{f(\theta)}{r} e^{ikr}$$

with $\vec{r} = \vec{r}_1 - \vec{r}_2$, the relative coordinate, and the incident particles where chosen to move along the $\pm\hat{z}$ directions. For identical particles though we know that the wavefunction must be symmetric under the interchange of the particles

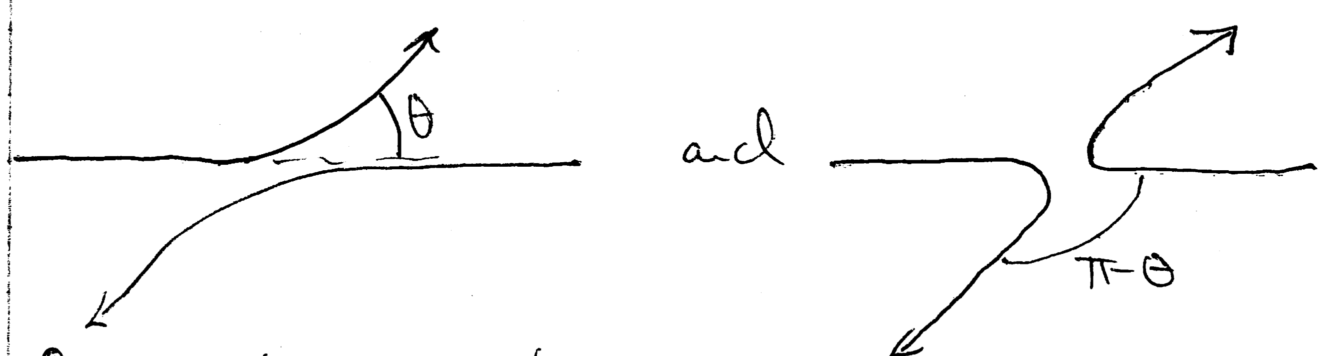
$$\psi(\vec{r}) = \psi(-\vec{r}) \qquad (\text{i.e. } \vec{r}_1 \leftrightarrow \vec{r}_2)$$

Then $\vec{r} \to -\vec{r} \Rightarrow r \to r$ but $\theta \to \pi - \theta$, and $z \to -z$

So we symmetrize the wavefunction

$$\psi(\vec{r}) \underset{r\to\infty}{\sim} N\left[e^{ikz} + e^{-ikz} + \left(f(\theta)+f(\pi-\theta)\right)\frac{e^{ikr}}{r}\right]$$

with N a normalization factor. Pictorially we see that we cannot tell the difference between



for identical particles, these are recorded as the same event.

The elastic cross-section is given by

$$d\sigma_{elastic} = \frac{\vec{J}^{sc.}_{out} \cdot \hat{r}\, d\Omega}{J_{in}}$$

with $J_{in} = N\frac{\hbar k}{m}$ and the scattered flux into $d\Omega$ is

$$\vec{J}^{sc.}_{out} \cdot \hat{r}\, d\Omega = N\frac{\hbar k}{m}\left|f(\theta)+f(\pi-\theta)\right|^2 d\Omega$$

Hence

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta) + f(\pi-\theta)|^2$$

$$= \underbrace{|f(\theta)|^2 + |f(\pi-\theta)|^2}_{\text{find this if particles were distinguishable}} + \underbrace{2\mathrm{Re}\left[f^*(\theta) f(\pi-\theta)\right]}_{\text{additional term due to symmetrization}}$$

additional term due to symmetrization — <u>enhancement</u>! near $\theta = \pi/2$.

The total cross section is

$$\sigma_{el} = \frac{1}{2}\int d\Omega \, \frac{d\sigma_{el}}{d\Omega}$$ with the $\frac{1}{2}$

needed so as not to <u>double</u> count when summing over all solid angles.

For elastic scattering of 2 identical spin $\frac{1}{2}$ particles we have 2 different cases

1) The 2 particles form a total spin singlet state $S=0$. Recall this was the <u>anti</u>-symmetric spin wavefunction

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$; it changes sign under exchange. Since the total wavefunction must change sign under exchange, this implies that the coordinate space wavefunction is symmetric under interchange. This is just like the 2 identical boson case

$$\Rightarrow \left.\frac{d\sigma_{el}}{d\Omega}\right|_{S=0} = |f(\theta) + f(\pi-\theta)|^2 .$$

2) The 2 spin $\frac{1}{2}$ particles form the total spin $S=1$ state, the triplet. The spin wavefunctions were

$|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$, $|\downarrow\downarrow\rangle$,

<u>symmetric</u> under interchange. Hence the coordinate space wavefunction must now <u>be</u> antisymmetric under exchange. Thus we have in this case

$$\left.\frac{d\sigma_{el}}{d\Omega}\right|_{S=1} = |f(\theta) - f(\pi-\theta)|^2$$

Note $\left.\left.\frac{d\sigma_{el}}{d\Omega}\right|_{S=1}\right|_{\theta=\frac{\pi}{2}} = 0$, here $\theta = \frac{\pi}{2}$ is

in the C-M frame, in the lab frame $\Rightarrow \theta_{lab} = \frac{\pi}{4}$. The cross section is diminished around $\theta = \frac{\pi}{2}$ due to (S=1) Pauli-exclusion principle.

For elastic scattering of 2 spin $\frac{1}{2}$ particles without polarizing the beams, we must average over the 2 polarizations. Thus there is 1 singlet state and 3 triplet states $\Rightarrow$

$$d\sigma_{el} = \frac{1}{4}\, d\sigma\Big|_{S=0} + \frac{3}{4}\, d\sigma\Big|_{S=1} .$$