8.2. Fermi's Golden Rule

As time increases, we certainly expect the probability of transition to grow as well. Indeed, from our Dirac perturbation formula (page -1143- ), we might expect that for approximately equal initial and final energies $E_f,E_i$ ($\lambda \to \infty$) may grow like $t^{-1}$ so that the rate of transition

$$ R_{fi} = \frac{P_{fi}(t,\infty)}{1 + t} $$

grows in time!

However, it is precisely the transition to the states $E_f \neq E_i$ that can occur below the energy resolution limit that tempts this growth of the rate so that in fact the transition rate $R_{fi}$ is constant in time. This property is known as Fermi's Golden Rule for time-dependent perturbation theory.

To be more precise, as well as to determine the conditions for the validity of the rule, let us consider the example of scattering from a time-independent potential $V_0$, i.e. $H = H(\vec{r})$. We will consider the
Transition probability to scatter from some initial energy eigenstate \( |E_i\rangle \) to some final energy eigenstate \( |E_f\rangle \) is the sum of 2 plane waves \( |\psi_i\rangle \) also, but now of different energy \( E_f \) describing the incoming particle as well as target and \( |\psi_f\rangle \) the scattered particle which with different energy \( E_f \) and different state of the target particle. So the incoming particle interacts with the target exchanging energy and scatters. The transition probability is given to lowest order by the Dirac perturbation theory expression

\[
\rho_i(\epsilon, \lambda) = \sum_{E_f} \frac{1}{\epsilon} \int_{\lambda} \left| \langle \psi_f | H' | \psi_i \rangle \right|^2 e^{i \frac{(E_f - E_i)}{\hbar} t} dt
\]

just as previously. Defining the transition frequency

\[
\omega_i = \frac{E_f - E_i}{\hbar}
\]

and using the fact that \( H' \) is time independent yields
\[
\tilde{P}_f(t, t_0) = \frac{1}{h^2} \left| \langle \phi_f | H' | \phi_c \rangle \right|^2 \left[ \frac{\sin(\frac{1}{2} \omega_{fc}(t - t_0))}{\frac{1}{2} \omega_{fc}} \right]^2
\]

Since

\[
\int_{t_0}^{t} dt \, e^{i \omega_{fc} t} = \frac{e^{i \omega_{fc} t} - e^{i \omega_{fc} t_0}}{i \omega_{fc}}
\]

\[
= e^{i \omega_{fc} \frac{t + t_0}{2}} \left[ \frac{e^{i \omega_{fc} \frac{t - t_0}{2}} - e^{i \omega_{fc} \frac{t - t_0}{2}}}{2i} \right]
\]

\[
= e^{i \omega_{fc} \frac{t + t_0}{2}} \frac{\sin \omega_{fc} \frac{t - t_0}{2}}{\frac{1}{2} \omega_{fc}}
\]
Plotting this diffraction function

\[ (t-t_0)^2 \]

\[ \left| \frac{\sin \frac{1}{2} \omega_f (t-t_0)} {\frac{1}{2} \omega_f} \right|^2 \]

we see that it is strongly peaked about \( \omega_f = 0 \), i.e. \( E_f = E_i \) in the range \( t \to t_0 \) more sharply peaked. It decreases rapidly for increasing \( |\omega_f| \), vanishing at frequencies \( \omega_f = \pm \frac{2\pi n}{T-t_0} \) with \( n = 1, 2, 3, \ldots \) typical of diffraction patterns.

So defining the total time \( T \) over which the interaction occurs as

\[ T = t - t_0, \]

the transition rate is given by
\[ P_{fe} = \frac{P_{fe}(t, t_0)}{t - t_0} = \frac{P_{fe}(t, t_0)}{T} \]

\[ = \frac{1}{\hbar^2} \langle \psi_f | H | \psi_i \rangle^2 \cdot T \left[ \frac{\sin(\frac{1}{2} \omega_{fe} T)}{(\frac{1}{2} \omega_{fe} T)} \right]^2 \]

\[ = \frac{1}{\hbar^2} \langle \psi_f | H | \psi_i \rangle^2 \cdot f(T, \omega_{fe}) \]

The function \( f(t, \omega_{fe}) = T \left[ \frac{\sin(\frac{1}{2} \omega_{fe} T)}{(\frac{1}{2} \omega_{fe} T)} \right]^2 \)

has the properties:

1) \( f(T, 0) = T \)

2) \( f(T, \omega_{fe}) \to 0 \) as \( T \to \infty \) for any \( \omega_{fe} \neq 0 \).

This implies that \( f(T, \omega_{fe}) \) for large \( T \)

is very peaked at \( \omega_{fe} = 0 \).

3) \( \int_{-\infty}^{\infty} d\omega_{fe} f(T, \omega_{fe}) = T \int_{-\infty}^{\infty} d\omega_{fe} \left[ \frac{\sin(\frac{1}{2} \omega_{fe} T)}{(\frac{1}{2} \omega_{fe} T)} \right]^2 \]

\( (\text{let } s = \frac{1}{2} \omega_{fe} T) = 2 \int_{-\infty}^{\infty} ds \frac{\sin^2 s}{s^2} = 2\pi \)
Thus, these properties imply that
\[ f(T, \omega_{fi}) \xrightarrow{T \to \infty} 2\pi \delta(\omega_{fi}) = 2\pi \delta(E_f - E_i), \]
expressing the conservation of energy in a scattering process. Thus, for sufficiently large $T$ (to be quantified later), we showed that
\[ R_{fi} = \frac{2\pi}{\hbar} |\langle \Psi_f | H | \Psi_i \rangle|^2 \delta(E_f - E_i), \]
a form of Fermi's Golden Rule, the transition rate is a constant in time.

In this discrete energy case, we have that
\[
| P_{fi}(t, t_0) | = T | R_{fi} | \quad \text{if} \quad E_i = E_f \quad \omega_{fi} = 0
\]
\[
= \frac{T^2}{\hbar^2} |\langle \Psi_f | H | \Psi_i \rangle|^2.
\]
Thus, for $P_{fi}(t, t_0) < 1$, we must have that
\[
\frac{1}{\hbar^2} |\langle \Psi_f | H | \Psi_i \rangle|^2 < \frac{1}{T^2}.
\]
However, we must also have \( \omega_i T \) large so that \( f(T, \omega_i) \) can be approximated by \( \frac{2 \pi}{T} \omega_i \), in order to obtain Fermi's Golden Rule. Hence the region of validity for Fermi's Golden Rule is when the transition rates are small. That is just the condition for the validity for the use of first order perturbation theory. This was just equivalent to approximating \( \beta \) (or \( \Delta \)) to by \( \beta(0) \) in the Schrödinger equation for \( \Delta I_0 \). Thus there is very little depletion in the initial state, i.e., the transition rates are small.

When one of the particles in the final state has energy in the continuum, we can observe more carefully the cancellation of the \( T^2 \) growth of the \( \omega_i = 0 \) channel against the oscillation of the \( \omega_i = 0 \) but \( E \) in the continuum channels, resulting in a constant \( R_i \) rate and Fermi's Golden Rule. As well this continuum case describes very important realistic examples of scattering experiments.
In particular we will be applying the golden rule to the cases of

1) Elastic and Inelastic Scattering in which the detector observes a scattered particle with energy \( \frac{\hbar^2 k_f^2}{2m} \) which is in the continuum.

\[ k_i \rightarrow k_f \]

2) Photo emission in which the detector observes an emitted photon with energy \( \gamma \) which is in the continuum.

\[ \gamma \rightarrow \text{detector} \]

In reality all detectors have a finite energy resolution \( \Delta E \) hence we cannot measure the probability of detecting the system in the distinct state \( \lvert \Psi \rangle \) if the energy of this state lies in the continuum. All physical predictions involve an integration over a group of final states depending on the particular measurement being made. The detector signals when it detects a particle with momentum in the domain \( D \) of momentum space.
centered about $E_f = \hbar^2 k^2 / 2m$ so that the energy is in the interval $\Delta$ centered about $E_f = \hbar^2 k^2 / 2m$ for a massive (non-relativistic) particle or $E_f = \hbar k / c$ for a photon. Hence the detector records a series of peaks (excitations of the target)

![Graph showing peaks of energy distribution]

where the detector energy resolution $\Delta$ is small enough to resolve the individual peaks while large enough to contain an entire peak. Thus we need to count the number of states in $\Delta$. This is easily done by placing the system in a box of side length $L$ and volume $V = L^3$. The normalized final state plane wave is then

$$\psi_\mathbf{k} (\mathbf{r}) = (\mathbf{r} | \psi_\mathbf{k}) = \frac{1}{\sqrt{V}} e^{i \mathbf{k} \cdot \mathbf{r}}$$

with energy $E_f = \hbar^2 k^2 / 2m$. Implying periodic boundary conditions on the plane...
wave, the components of \( k \) are given by

\[
k_i = \frac{2\pi n_i}{L}, \quad n_i = 0, \pm 1, \pm 2, \ldots
\]

\( i = 1, 2, 3. \)

Thus, \( \Delta n_i = \frac{L}{2\pi} \Delta k_i \) and the number of states in \( D_f \) is simply

\[
\Delta n = \Delta n_x \Delta n_y \Delta n_z = \frac{L^3}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z
\]

\[
= \frac{\Omega}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z
\]

As \( L \to \infty \), the energy levels become continuously close and \( \Delta n \to \frac{\Omega}{(2\pi)^3} \), but for \( L \) large we write this as

\[
\Delta n = \frac{\Omega}{(2\pi)^3} \Delta^3 k
\]

In addition, there may be additional degeneracy associated with each energy eigenvalue labelled by other quantum numbers (besides the momentum). As usual, denote the degree of degeneracy by \( g(E) \) (for photons for example, always just 2, for each energy), thus
\[ dn = \frac{\Omega}{(2\pi)^3} g(E) d^3k \] is the number of states in \( Df \) when \( k \) is restricted to lie in the domain \( Df \) centered about \( k_f \).

The transition rate to any of these states is given by (recall they are centered about \( E_f \))

\[ \mathcal{S} R_{fc}(E) = \int \frac{dn \cdot R_{fc}(E)}{Df} \]

where

\[ R_{fc}(E) = \frac{1}{h^2} |\langle \phi_f | H \gamma_c | \Phi_c \rangle|^2 f(T, \frac{E - E_c}{\Delta}) \]

is the transition rate to final states \( |\Phi_c \rangle \) with energy \( E \) instead of \( E_f \). Of course \( E_f - \Delta \leq E \leq E_f + \Delta \) in the integral. Hence we have

\[ \mathcal{S} R_{fc}(E_f) = \frac{\Omega}{(2\pi)^3} \int d^3k \ g(E) R_{fc}(E) \]

\[ E_f - \Delta \leq E \leq E_f + \Delta \]

\[ \mathcal{S} R_{fc} \left( E_f \right) = \frac{\Omega}{(2\pi)^3} \int d^2k \int dk^2 \ g(E) R_{fc}(E) \]

\[ E_f - \Delta \leq E < E_f + \Delta \]
This we write as
\[ \Delta R_{f_i}(E_f) = \int dE \frac{\partial n(E)}{\partial E} R_{f_i}(E) \]

with \[ \frac{\partial n(E)}{\partial E} \] the density of final states (sometimes \( \rho(E) = n(E) \) is used). \[ \frac{\partial n(E)}{\partial E} \approx 8 \mathcal{R}_f g(E) \frac{Q}{(2\pi)^3 h^2} \frac{dk}{dE} \]

Thus we find
\[ \Delta R_{f_i}(E_f) = \int dE \frac{\partial n(E)}{\partial E} \frac{1}{h^2} K_{q_f} l^{H_f} |j_{\psi} > \big|^{2} f_{\psi} \frac{E-E_i}{h} \]

\( f_{\psi} (T, \frac{E-E_i}{h}) \) is sharply peaked about \( E = E_i \), and since \( E_f - \Delta \leq E \leq E_f + \Delta \), the integral is dominated by the contribution from \( E \approx E_i \). Since \( \frac{\partial n(E)}{\partial E} \) and \( K_{q_f} l^{H_f} |j_{\psi} > \big|^{2} f_{\psi} \frac{E-E_i}{h} \) are slowly varying functions of \( E \), we can evaluate them at \( E = E_f = E_i \), and so pull them out of the integral.
\[ \mathcal{E}_{\text{b.c.}}(E_t) = \left[ \frac{\partial n(E_t)}{\partial E_t} \frac{i}{h^2} |\langle \psi_f | H | \psi_i >|^2 \right] \int_{E_t=E_i}^{E_t + \Delta} \int_{E_i - \Delta}^{E_i + \Delta} dE_f(T, \frac{E-E_i}{h}) \times \]

The integral can be performed, recall
\[ \int_{E_i - \Delta}^{E_i + \Delta} dE_f(T, \frac{E-E_i}{h}) = \int_{\frac{1}{2h}(E-E_i)T}^{\frac{1}{2h}(E-E_i)T} \left[ \frac{\sin \frac{1}{2h}(E-E_i)T}{\frac{1}{2h}(E-E_i)T} \right]^2 \]

(letting \( s = \frac{1}{2h}(E-E_i)T \))
\[ = 2h \int \frac{(E_t-E_i)T + \Delta T}{2h} ds \frac{\sin^2 s}{s^2} \]

Now if \( \Delta T >> h \) this becomes
\[ = 2h \int_{-\infty}^{+\infty} ds \frac{\sin^2 s}{s^2} = 2\pi h \]

Thus we find
For $T \Delta \gg h$

$$
\delta R_{\psi i}(E_f) = \left[ \frac{2\pi}{\hbar} \frac{dn(E_f)}{dE_f} \right] \left[ \frac{1}{2} \left| \langle \psi_f | H' | \psi_i \rangle \right|^2 \right] \delta(E_f - E_i),
$$

\[ E_f = E_i. \]

This is Fermi's Golden Rule for the transition rate when one particle is in the final state continuum. Note this is independent of $T \Delta$. We can obtain this result directly from our previous form of Fermi's Golden Rule by summing over all states $\psi_i$:

$$
\delta R_{\psi i}(E_f) = \sum_{\psi_i} \delta(n(E_f) \left| \psi_i \right|^{2})
$$

$$
= \sum_{\psi_i} \int dE \frac{dn(E)}{dE} R_{\psi i}(E)
$$

$$
= \int dE \frac{dn(E)}{dE} \left( \frac{2\pi}{\hbar} \right) \left| \left\langle \psi_f | H' | \psi_i \right\rangle \right|^2 \delta(E_f - E_i)
$$

$$
= \frac{2\pi}{\hbar} \left[ \frac{dn(E_f)}{dE_f} \left| \left\langle \psi_f | H' | \psi_i \right\rangle \right|^2 \right] \delta(E_f - E_i).
$$
Fermi's Golden Rule is valid for $T \gg \hbar$ while in order to insure that probability was less than one we need

$$\frac{\hbar}{T} |\langle \Psi_f | H | \Psi_i \rangle| \ll \frac{1}{T}.$$ 

Combining these conditions results in

$$\frac{\hbar}{\Delta} \ll T \ll \frac{\hbar}{|\langle \Psi_f | H | \Psi_i \rangle|}.$$ 

These inequalities can be satisfied for some $T$ provided $|\langle \Psi_f | H | \Psi_i \rangle|$ is sufficiently small. Thus again Fermi's Golden Rule is valid for calculating small transition rates.

Example: Elastic Scattering of a spinless particle from a potential $V(\hat{R})$

The small (time-independent) Hamiltonian is

$$H = \frac{1}{2m} \hat{p}^2 + V(\hat{R}) \equiv H_o + H'$$

with $H_o = \frac{1}{2m} \hat{p}^2$ and $H' = V(\hat{R})$. 
The H$_2$ eigenstates are given by the normalized plane waves:

- **Initial state**: \( \Psi_i (\vec{r}) = \langle \vec{r} | \Psi_i \rangle = \frac{1}{\sqrt{2}} e^{i \vec{k}_i \vec{r}} \)
- **Final state**: \( \Psi_f (\vec{r}) = \langle \vec{r} | \Psi_f \rangle = \frac{1}{\sqrt{2}} e^{i \vec{k}_f \vec{r}} \)

with \( |\vec{k}_i| = |\vec{k}_f| = k \) for elastic scattering, with initial and final energies:

\[
E_i = \frac{k_i^2}{2m} = E_f.
\]

Since \( k \in \mathbb{R}^+ \), the final state plane wave has its energy in the continuum. Applying Fermi's Golden Rule, we first evaluate the matrix element:

\[
\langle \Psi_f | H | \Psi_i \rangle = \frac{1}{2} \int d^3r \ e^{-i\vec{t}_i \vec{r}} V(\vec{r}) e^{i\vec{t}_f \vec{r}}
\]

\[
= \frac{1}{2} \int d^3r \ e^{-i\vec{t}_i \vec{r}} V(\vec{r})
\]

where \( \vec{t} = \vec{t}_f - \vec{t}_i \) is the momentum transfer. Next, we must evaluate the density of states...
\[
\frac{\delta n(E_f)}{\delta E_f} = 8\Omega_f \frac{m}{(2\pi\hbar)^3} \frac{\hbar^2}{k^2} \frac{d\hbar k}{dE_f}
\]
\[
= 8\Omega_f \frac{\hbar^2}{(2\pi\hbar)^3} \frac{m\hbar^2}{\hbar^2}
\]
\[
= 8\Omega_f \frac{\hbar^2}{(2\pi\hbar)^3} \frac{m\hbar}{k}
\]

Hence we find from page -1167-

\[
\delta R_{fi}(E_f) = (8\Omega_f \frac{\hbar^2}{(2\pi\hbar)^3} \frac{m\hbar}{k}) \frac{2\pi}{\hbar} \int_0^{\pi} \sin(\theta) e^{-i\theta} d\theta
\]

Recall that the elastic cross-section is defined as

\[
J_{in} \delta_{el}(\Theta, \Phi) = \frac{\# \text{ of transitions } |\psi_i\rangle \rightarrow |\psi_f\rangle}{\text{unit time}} \bigg|_{E_f=E_i}
\]

\[
= \text{transition rate for } |\psi_i\rangle \rightarrow |\psi_f\rangle \text{ with } E_f=E_i
\]

\[
\Rightarrow \delta_{el}(\Theta, \Phi) = \frac{\delta R_{fi}(E_f)}{J_{in}}
\]
Letting \( \hat{\mathbf{r}}_i = \hbar \hat{\mathbf{z}} \), the incident flux is simply

\[
J_\text{in} = \hat{\mathbf{J}}_\text{in} \cdot \hat{\mathbf{z}}
= \frac{\hbar}{2\pi i} \left[ \psi_i^*(\mathbf{r}) \frac{\partial}{\partial \mathbf{r}} \psi_i(\mathbf{r}) - (\frac{\partial}{\partial \mathbf{r}} \psi_i(\mathbf{r}))^* \psi_i^*(\mathbf{r}) \right]
= \frac{1}{\alpha} \frac{\hbar k}{m}.
\]

Thus

\[
|\sigma| = \frac{1}{\pi} \frac{\hbar k}{m} \left( \frac{\hbar}{2\pi \hbar^3} \right) \frac{1}{\hbar} \frac{1}{2\pi} \times
| \int d^3r e^{-i\mathbf{q}_0 \cdot \mathbf{r}} V(\mathbf{r}) |^2
\]

\[
\Rightarrow
\sigma_{\text{el}}(\theta, \phi) = \frac{d\sigma_{\text{el}}}{d\Omega} = \left| -\frac{1}{4\pi} \int d^3r e^{-i\mathbf{q}_0 \cdot \mathbf{r}} \frac{2m}{\hbar^2} V(\mathbf{r}) \right|^2
\]

Precisely the same result we obtained for elastic scattering in the Born approximation (page 1034).
As well we can apply Fermi's Golden Rule to inelastic scattering processes

Example: Scattering of spin $\frac{1}{2}$ muons ($\mu^+$) from single electron ($e^-$) atom.

The full (time-independent) Hamiltonian is

$$H = H_0 + H'$$

where

$$H_0 = \frac{1}{2m_\mu} \vec{P}_\mu^2 + \frac{1}{2m_e} \vec{P}_e^2 - \frac{Ze^2}{r_e}$$

\(KE \text{ of } \mu^+\)

\(KE \text{ of } e^- \text{ and Coulomb interaction of bound } e^- \text{ and nucleus (Ze)}\)
\[ H' = Z \beta \mu \, e^2 \int d^3 r' \frac{\rho_N(r')}{|\vec{r}_\mu - \vec{r}'_e|} - \frac{Z \beta \mu \, e^2}{|\vec{r}_\mu - \vec{r}_e|} \]

Coulomb interaction of $\mu^\pm$ with nucleus

Coulomb interaction of $\mu^\pm$ with $e^-$

We assume the nucleus is at rest since it is much more massive than $e$ or $\mu$. For $\mu^-$, $Z_\mu = -1$, for $\mu^+$, $Z_\mu = +1$ and $\rho_N(r')$ is the nuclear charge density normalized so that $\int d^3 r' \rho_N(r') = 1$.

The initial and final states are eigenstates of $H_0$. For the incoming state we have the non-interacting $\mu^\pm$ plane wave and single $e^-$ atom

\[ \psi_i(\vec{r}_e, \vec{r}_\mu) = \frac{2}{\sqrt{\mathcal{D}}} f_i(\vec{r}_e) \, e^{i \vec{k}_i \cdot \vec{r}_\mu} \]

initial bound state $e^-$ wavefunction

incident $\mu^\pm$ plane wave with spinor $\psi$ function for spin $\pm \frac{1}{2}$ particle
With initial energy:

\[ E_i = \varepsilon_i + \frac{\hbar^2 k_i^2}{2m_\mu} \]

atomic bound state energy of \( e^- \)

Likewise the final outgoing state is comprised of the non-interacting \( \mu^+ \) plane wave and single \( e^- \) atom wavefunctions:

\[ \phi_f(x, y, z) = 2 \phi(x) \frac{1}{\sqrt{5v}} e^{i \frac{\hbar}{m_\mu} k \cdot \mathbf{r}} X(x) \]

final bound state \( e^- \) wavefunction

outgoing \( \mu^+ \) plane wave with spinor wavefunction

with final energy:

\[ E_f = \varepsilon_f + \frac{\hbar^2 k_f^2}{2m_\mu} \]

atomic bound state energy of \( e^- \)

KE of outgoing \( \mu^+ \)

For single electron Coulomb atoms:

The bound state energies are given by
$$E_n = -\frac{m_0 c^2 (2\alpha)^2}{2n^2},\; n=1,2,...$$

Overall energy conservation demands that (His time independent)
$$E_i = E_f$$
$$\Rightarrow \epsilon_i^0 + \frac{\hbar^2 k^2}{2m}\; = \; \epsilon_f^0 + \frac{\hbar^2 k^2}{2m}$$

According to Fermi's Golden Rule we must calculate the initial and final state matrix element of $H$
$$\langle \psi_f | H | \psi_i \rangle = \chi^\dagger(\sigma_f) \chi(\sigma_i) \frac{1}{2} \int d^3 r e^{i \frac{\pi}{\hbar}} F_\mu$$

$$\times 2\hbar e \left[ Z^2 e^2 f(\frac{P^f \cdot \vec{p}^i}{1 - \frac{\vec{p}^i \cdot \vec{p}^f}{m^2}}) - \frac{Z e^2}{1 - \frac{\vec{p}^f \cdot \vec{p}^i}{m^2}} \right]$$

$$\times 2\hbar e \left[ 2^2 \left( \frac{P^i \cdot \vec{p}^f}{1 - \frac{\vec{p}^f \cdot \vec{p}^i}{m^2}} \right) - \frac{Z e^2}{1 - \frac{\vec{p}^f \cdot \vec{p}^i}{m^2}} \right]$$

Since the spinor wavefunctions are orthonormal
$$\chi^\dagger(\sigma_f) \chi(\sigma_i) = \delta_{\sigma_f \sigma_i}$$
as well, the $e^-$ bound state wavefunctions
are orthonormal
\[ \int d^3 r \: \psi_f^*(\vec{r}) \psi_i(\vec{r}) = \delta_{fi} \]
hence we find
\[ \langle \psi_f | H' | \psi_i \rangle = \delta_{fi} \frac{1}{\sqrt{2}} \left\{ \int d^3 \vec{r} \: \frac{e^{-i \hat{g} \cdot \vec{r}}}{|\vec{r}_\mu - \vec{r}_\ell|} + e^{-i \hat{g} \cdot \vec{r}} \int d^3 \vec{r} \: \frac{e^{-i \hat{g} \cdot \vec{r}}}{|\vec{r}_\mu - \vec{r}_\ell|} \right\} \]
where the momentum transfer \( \vec{q} = \vec{r}_f - \vec{r}_i \),

Hence we must evaluate the Fourier transform of the Coulomb potential
\[ \int d^3 \vec{r} \: \frac{e^{-i \hat{g} \cdot \vec{r}_\mu}}{|\vec{r}_\mu - \vec{r}_\ell|} = \int d^3 \vec{s} \: \frac{e^{-i \hat{g} \cdot (\vec{s} + \vec{r}_\ell)}}{|\vec{s}|} \]
where \( \vec{s} = \vec{r}_\mu - \vec{r}_\ell \)
\[ \int d^3 \vec{s} \: \frac{e^{-i \hat{g} \cdot \vec{s}}}{|\vec{s}|} = e^{i \hat{g} \cdot \vec{r}_\ell} \int d^3 \vec{s} \: \frac{e^{-i \hat{g} \cdot \vec{s}}}{|\vec{s}|} \]
But \[ \int d^3 \frac{e^{-i \vec{q} \cdot \vec{r}}}{|\vec{r} - \vec{r}'|} = \frac{4\pi}{\vec{q}^2}, \] hence

\[ \int d^3 r e^{-i \vec{q} \cdot \vec{r}} \frac{e^{-i \vec{q} \cdot \vec{r}'}}{|\vec{r} - \vec{r}'|} = \frac{4\pi}{\vec{q}^2} e^{-i \vec{q} \cdot \vec{r}}. \]

Substituting into the matrix element yields

\[ \langle \Psi \mid H' \mid \Psi \rangle = \delta_{\Psi_0 \Psi_0'} \left( \frac{Z e^2}{\hbar} \right) \frac{4\pi}{\vec{q}^2} \times \]

\[ x \left[ \int d^3 r \ e^{-i \vec{q} \cdot \vec{r}} \rho_N(r) \right. \]

\[ \left. - \int d^3 r \ e^{-i \vec{q} \cdot \vec{r}} \ 2 f(\vec{r}) \abar f(\vec{r}) e^{-i \vec{q} \cdot \vec{r}} \right] \]

Defining

\[ F_N(q) = \int d^3 r \ e^{-i \vec{q} \cdot \vec{r}} \rho_N(r) = \text{Nuclear (charge) form factor} \]

\[ F_f(q) = \int d^3 r \ e^{-i \vec{q} \cdot \vec{r}} \ 2 f(\vec{r}) \abar f(\vec{r}) e^{-i \vec{q} \cdot \vec{r}} = \text{Atomic form factor} \]

This becomes
\[
\langle \psi | H' | \psi_i \rangle = \mathcal{S}_{\psi_i, \mathcal{F}i} \left( \frac{2\pi e^2}{\hbar^2} \right) \frac{4\pi}{\hat{q}^2} \left[ Z F_{\psi}(q) \delta_{\psi \mathcal{F}} - F_{\psi}(q) \right]
\]

Recall that \( \rho_\mathcal{F} \) is normalized to 1 so

\[
F_N(0) = \int d^3 r / \rho_N(r) = 1
\]

and

\[
F_{\mathcal{F}i}(0) = \int d^3 r e^2 \mathcal{F}_i(r) \mathcal{F}_i(r) = \delta_{\mathcal{F}i}.
\]

The next quantity we need is the golden rule in the density of final states,

\[
\frac{dN(E_f)}{dE_f} = S \Omega_f g(E_f) \frac{\gamma_f}{2\pi \hbar^3} k_f^2 \frac{d \mathbf{k}_f}{dE_f}
\]

Now the final state energy is

\[
E_f = E_f^0 + \frac{\hbar^2 k_f^2}{2m_f}, \text{ with the continuum.}
\]

Thus

\[
\frac{dE_f}{d \mathbf{k}_f} = \frac{\hbar^2 k_f}{m_f}, \text{ Since the } \mu^+ \text{ can be spin up or down, } \sigma_f = \pm 1, \text{ } E_f \text{ has an additional 2-fold spin degeneracy, so we must sum over these spin states in } \frac{dN}{dE}.
\]
Putting this altogether we get
\[ \frac{\partial n(E_f)}{\partial E_f} = \delta \Omega_f \sum_{\sigma_f = \pm 1} \frac{\rho}{2\pi i} \frac{k_f^2}{\hbar^2} \frac{m_{\mu \hbar} \Sigma_f}{r_f^2} \]

\[ = \delta \Omega_f \frac{\rho}{2\pi i \hbar^3} m_{\mu \hbar} k_f \sum_{\sigma_f = \pm 1} \]

Applying Fermi's Golden Rule for the case of a final particle in the continuum we find (page 1167)

\[ \delta R_{\epsilon_f}(E_f) = \frac{2\pi}{\hbar} \frac{\partial n(E_f)}{\partial E_f} |K_{\epsilon_f} \phi_{\epsilon_i}|^2 \]

\[ = \frac{2\pi}{\hbar} \left( \delta \Omega_f \frac{\rho}{2\pi i \hbar^3} m_{\mu \hbar} k_f \sum_{\sigma_f = \pm 1} \right) \times \]

\[ \times \left( \delta_{\epsilon_{\phi}, \epsilon_{\phi}} \left| \frac{2e^2}{\hbar} \frac{4\pi i}{q^2} \left[ \lambda E_{\phi}(q) \delta_{\phi} - F_{\phi}(q) \right] \right|^2 \right) \]

\[ = \delta \Omega_f \frac{1}{\rho} \frac{4}{(\hbar c)^2} \left( \frac{e^2}{\hbar c} \right)^2 \frac{m_{\mu c^2}}{\hbar} k_f \times \]

\[ \times \left| \lambda E_{\phi}(q) \delta_{\phi} - F_{\phi}(q) \right|^2 \]
$\sum_{i} \delta_{\tau_i} = 1$ and $\sum_{\tau_i=\pm1} \delta_{\tau_i} = 1$.

Since the incident muons have spin, we can prepare the initial state as polarized or unpolarized depending on the experimental situation. To be concrete, let's take our initial beam of muons as unpolarized, so we can average over the spins. Since there are $2s+1 = 2$ spin states, we sum over $\tau_i = \pm 1$ and divide by 2 for the average

$$\frac{1}{2} \sum_{\tau_i=\pm1} \delta_{\tau_i}.$$

So doing, we can define the initial spin averaged differential cross section as the spin averaged transition rate divided by the incident flux

$$d\sigma = \frac{1}{2} \sum_{\tau_i=\pm1} \delta_{\tau_i} \frac{\delta R_{fi}(E_f)}{J_{in}}.$$

From our expression for $\delta R_{fi}$ we see it is independent of $\tau_i$, hence the initial spin average just gives

$$\frac{1}{2} \sum_{\tau_i=\pm1} \delta_{\tau_i} = 1.$$
So the spin averaged cross-section is
\[ \frac{d\sigma_{fi}}{J_{in}} = \frac{8R_{fi}(E_f)}{J_{in}} \]

(Since \( H \) is independent of spin, the average over the initial \( \mu^\pm \) spin and sum over the final \( \mu^\pm \) spin just yields a factor of \( 1 \))

\[ \frac{1}{2} \sum_{\sigma_i = \pm 1} \sum_{\sigma_f = \pm 1} \delta_{\sigma_i \sigma_f} = 1 \]
hence we could have neglected spin altogether (in the wavefunction as well as in the density of states). If the initial beam of muons were spin-polarized, i.e., say \( \sigma_i = +1 \), the mu's stay in that spin state since \( H \) is independent of spin, again the cross-section would simply be

\[ \frac{d\sigma_{fi}}{J_{in}} = \frac{8R_{fi}(E_f)}{J_{in}} \]

As usual for plane waves the incident flux is
\[ J_{in} = \frac{\hbar k_i}{m_\mu} \frac{1}{\Omega} \], so
we find
\[ \frac{d\Gamma_{fi}}{d\Omega_f} \bigg|_{\Omega_f = 0} = 0 \frac{\alpha^2}{(\pi \hbar c)^2} \left( \frac{M_{\mu} C^2}{\hbar c} \right) \left( \frac{\alpha^2}{(\pi \hbar c)^2} \right) \left( \frac{e^2}{\hbar c} \right)^2 \frac{m_{\mu} C^2}{\hbar} k_f \]

\[ \left| z F_\alpha(q) \delta_{fi} - F_{fi}(q) \right|^2 \]

\[ \Rightarrow \]

\[ \frac{d\Gamma_{fi}}{d\Omega_f} \bigg|_{\Omega_f = 0} = \frac{4 \alpha^2}{(\pi \hbar c)^2} \left( \frac{M_{\mu} C^2}{\hbar c} \right) \left| z F_\alpha(q) \delta_{fi} - F_{fi}(q) \right|^2 \frac{k_f}{k_i} \]

Note, as required, all volume factors cancel. Since \[ \alpha = \frac{1}{137} \], the transition rate is small, and so Fermi's Golden Rule is a sensible approximation. Our result in general, it applies to elastic as well as inelastic scattering.

Let's apply the above formula to the specific case of elastic scattering off atomic hydrogen (Z = 1)

Elastic scattering means \[ k_f = k_i = k \]
The momentum transfer $\vec{q} = \vec{k}_f - \vec{k}_i$ has magnitude
$$ q^2 = \vec{q}^2 = 4k^2 \sin^2 \frac{\Theta}{2} $$
with
$$ \cos \Theta = k_f \cdot k_i. $$
Since $k_f = k_i$ we have that $E_f = E_i \Rightarrow E_0 = E_i$. The initial and final energy levels of the H-atom are the same. Let's choose $i$ to be the H-ground state

$$ A_i(\vec{r}_e) = 2_{1,1,0}(\vec{r}_e) = \frac{1}{\sqrt{15}} \left( \frac{2}{a_0} \right)^{3/2} e^{-r/a_0} $$
$$ = A_f(\vec{r}_e) $$

with $a_0 = \text{Bohr radius}$ and

$$ E_i = -\frac{\text{MeC}^2 a^2}{2} = E_f. $$

The atomic elastic scattering form factor becomes

$$ F_{ii}(q) = \int d^3r_e \ e^{-i\vec{q}\cdot\vec{r}_e} |A_i(\vec{r}_e)|^2 $$
$$ = \frac{1}{8\pi} \left( \frac{2}{a_0} \right)^3 \int d^3r_e e^{-i\vec{q}\cdot\vec{r}_e} e^{-2r_e/a_0} $$
\[-1184-\]

\[
= \frac{1}{8\pi} \left( \frac{2}{a_0} \right)^3 \int_0^{2\pi} d\phi e \int_0^\infty dr e r^2 e^{-\frac{2re+1}{a_0}} \int r \cos \theta e^{-i \theta} e \sin \frac{1}{2} \theta e^{-\frac{i}{q} r e} \sin \frac{1}{2} \theta e^{-\frac{i}{q} r e} \]

\[
= \frac{1}{4 \pi} \left( \frac{2}{a_0} \right)^3 \frac{i}{q} \int_0^\infty dr e r^2 [e^{-\left(\frac{2}{a_0} + i\frac{q}{b}\right) r} - e^{-\left(\frac{2}{a_0} - i\frac{q}{b}\right) r}]\]

Now \[
\int_0^\infty ds \ e^{-bs} = -\frac{d}{db} \int_0^\infty ds e^{-bs} = -\frac{d}{db} \frac{1}{b} \]

\[
= \frac{1}{b^2} \quad \Rightarrow
\]

\[
F_{ii}(q) = \frac{1}{4} \left( \frac{2}{a_0} \right)^3 \frac{i}{q} \left[ \frac{1}{(\frac{2}{a_0} + i\frac{q}{b})^2} - \frac{1}{(\frac{2}{a_0} - i\frac{q}{b})^2} \right]
\]

\[
= \frac{1}{4} \left( \frac{2}{a_0} \right)^3 \frac{i}{q} \left[ \frac{(\frac{2}{a_0} - i\frac{q}{b})^2 - (\frac{2}{a_0} + i\frac{q}{b})^2}{(\frac{4}{a_0^2} + q^2)^2} \right]
\]

\[
= \left( \frac{2}{a_0} \right)^4 \frac{1}{\left(\frac{4}{a_0^2} + q^2\right)^2}
\]

\[
= \left( \frac{2}{a_0} \right)^4 \frac{16}{a_0^4} \frac{1}{\left(1 + \left(\frac{q}{a_0}\right)^2\right)^2}
\]
\[ F_{\text{ic}}(q) = \frac{1}{(1 + \frac{qao}{6})^2} \]

For hydrogen the nucleus is just a single proton and so a point-like structure at the atomic scale, hence we have \( p_0(\mathbf{r}) = 8^3(\mathbf{r}) \) for hydrogen, this yields

\[ F_{\text{N}}(q) = \int d^3 \mathbf{r} \ e^{-i\mathbf{q}\cdot\mathbf{r}} p_0(\mathbf{r}) = \int d^3 \mathbf{r} \ e^{-i\mathbf{q}\cdot\mathbf{r}} 8^3(\mathbf{r}) = 1 \]

for the nuclear form factor.

Thus for elastic scattering \((8_{\text{sc}} = 8_{\text{ic}} = 1)\)

The differential cross-section is

\[ \frac{d\sigma_{\text{el}}}{d\Omega} = \frac{4\pi^2}{q^4} \left( \frac{m_e c^2}{\hbar c} \right)^2 \left| 1 - \frac{1}{(1 + \frac{qao}{6})^2} \right|^2. \]

Now

\[ \frac{4\pi^2}{q^4} \left| \frac{m_e c^2}{\hbar c} \right|^2 = \left( \frac{2e^2 m_e c^2}{q^2 \hbar c} \right)^2 = \left( \frac{2e^2 m_e}{q^2 \hbar^2} \right)^2 \]

\[ = \left( \frac{2e^2 m_e}{4\hbar^2 k^2 \sin^2 \frac{\theta}{2}} \right)^2 = \left( \frac{e^2}{4 \frac{k^2}{2m_e} \sin^2 \frac{\theta}{2}} \right)^2 \]
Recall that \( \frac{t^2 \hbar^2}{2m} = E \mu \), the incident and outgoing KE of the muon. So

\[
\frac{4 \alpha^2}{q^4} \left( \frac{m \mu c^2}{\hbar} \right)^2 = \left( \frac{e^2}{4 E \mu \sin^2 \theta} \right)^2
\]

\[
\equiv \frac{d\sigma_{\text{Rutherford}}}{d\Omega}, \quad \text{this is just}
\]

the Rutherford cross-section, that obtained from Coulomb scattering of a muon from a fixed point charge (page 1047). So we find

\[
\frac{d\sigma_{\text{el}}}{d\Omega} = \frac{d\sigma_{\text{Rutherford}}}{d\Omega} \left( 1 - \frac{1}{(1 + (q8)^2)^2} \right)^2
\]
Note that as $q \to 0$ we have in general

$$F_{ii}(q) = \int d^3r e^{-i\mathbf{q} \cdot \mathbf{r}} |\varphi_i(\mathbf{r})|^2$$

$$\propto \int d^3r |\varphi_i(\mathbf{r})|^2 + O(q^2)$$

$$\propto 1 + O(q^2)$$, hence specifically

$$\frac{d\sigma_{el}}{d\omega} \propto \left| \frac{d\sigma_{total}}{d\omega} \right|^2 \left( 1 - \left( \frac{q}{2 \omega (a_0^2)} \right)^2 \right)$$

$$\propto O(a^2)$$

The incident muon sees an essentially neutral H-atom $F_{ii} \approx F_0 \approx 1$, so there is very little low-energy elastic scattering. As the momentum transfer increases, the muon starts to probe the structure of the H-atom, it sees the difference between $F_0$ and $F_{ii}$. For very large $q^2$ the muon misses the electron cloud completely and probes into the H-atom to see only the point-like H-nucleus (i.e. the e-cloud has changed to spread through volume $a_0^3$). Hence, we expect the cross-section to describe scattering
So far for low energy

\[ \frac{d\sigma}{dE} \underset{q \to 0}{\sim} \left( \frac{1}{4} a_0^4 q^4 \right) \frac{d \sigma \text{Rutherford}}{d \Omega} \]

\[ \underset{q \to 0}{\sim} \frac{4 \alpha^2}{q^4} \left( \frac{M u c^2}{\hbar c} \right)^2 \left( \frac{1}{4} a_0^4 q^4 \right) \]

\[ \frac{d \sigma}{d E} \underset{q \to 0}{\sim} \left( \frac{a_0^2 M u c^2}{\hbar c} \right)^2 \text{ acostant} \]

The incident proton just sees a barrier at the origin. \[ \]
from the point (proton) charge, this is precisely
the Rutherford cross section result
\[ \frac{d\sigma_{el}}{d\Omega} \rightarrow \frac{d\sigma_{Rutherford}}{d\Omega} \]

Next consider elastic scattering from a large \( Z \) nucleus at high \( q \). Since \( Z \) is large the \( ZF_{11} \) factor will overpower the \( F_{11} \) atomic form factor, thus
\[ \frac{d\sigma_{el}}{d\Omega} = \frac{d\sigma_{Rutherford}}{d\Omega} \cdot Z^2 \cdot |F_{0}(q)|^2 \]

with
\[ F_{0}(q) = \int d^3 r e^{-iq\cdot\mathbf{r}} \rho_{N}(r) \] (for spherically symmetric nuclei \( \rho_{N}(r) = \rho_{N}(r) \))
\[ = 4\pi \int_0^\infty r^2 \rho_{N}(r) \left( \frac{\sin qr}{qr} \right) \]

and as always, \( F_{0}(0) = 1 \). Thus we find
Finally let's consider inelastic scattering of $\mu^+$ from atomic hydrogen.

For inelastic scattering the initial and final states of the hydrogen atom are different $\delta E_f = 0$. Hence from page 1182 we have

$$\frac{d\sigma_f}{d\Omega} = \frac{4\alpha^2}{(q^2)^2} \left( \frac{M_\mu c^2}{\hbar c} \right)^2 |F_i(q)|^2 \left( \frac{k_f}{k_i} \right), \quad f \neq i.$$

Defining $q_{el}^2 = 4k_i^2 \sin^2 \frac{\theta}{2}$ and recalling the Rutherford cross section...
\[
\frac{d\sigma_{\text{Rutherford}}}{d\Omega} = \frac{4L^2}{q^4} \left( \frac{m_e c^2}{\hbar c} \right)^2
\]

we can write the inelastic cross section as

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\text{Rutherford}}}{d\Omega} \left( \frac{g_{ef}}{g^2} \right)^2 \left( \frac{k_f}{k_i} \right) \left| F_e^{(q)} \right|^2
\]

Now the momentum transfer is

\[
g^2 = q^2 = (k_f - k_i)^2
\]

\[
g^2 = k_f^2 + k_i^2 - 2k_f k_i \cos \Theta.
\]

We can use the conservation of total energy to relate \(k_f\) to \(k_i\)

\[
E_i = E_i^0 + \frac{\hbar^2 k_i^2}{2m_e} = E_f = E_f^0 + \frac{\hbar^2 k_f^2}{2m_e}
\]

\[
\Rightarrow k_f^2 - k_i^2 = \frac{2m_e (E_f^0 - E_i^0)}{\hbar^2}
\]

\[
= \frac{2m_e}{\hbar^2} \left( -\frac{m_e c^2 \Delta^2}{2} \right) \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)
\]

Choosing the initial state of the H-atom to be the ground state \(n_i = 1\),
we have
\[ 2_f (\vec{r}, \vec{p}) = 2_{100} (\vec{r}, \vec{p}) = R_{10} (\vec{r}) \gamma_0 (\theta, \phi) \]

while we choose the final state to be \( n_f = n \neq 1 \) so that
\[ 2_f (\vec{r}, \vec{p}) = \sum_l \sum_m a_{nlm} R_{nl} (\vec{r}) Y_l^m (\theta, \phi) \quad l = 0 \quad m = -l \]

The muon energies are related by
\[ k_f^2 - k_i^2 = \frac{\mu m e c^2 a^2}{\hbar^2 n^2} (n^2 - 1) \]

It remains to evaluate the atomic (inelastic) form factor
\[ F_{fi} (q) = \int d^3 r e^{-i \vec{q} \cdot \vec{r}} 2_f (\vec{r}) 2_i (\vec{r}) \]
\[ = \sum_l \sum_m a_{nlm} \int d^3 r e^{-i \vec{q} \cdot \vec{r}} R_{nl} (\vec{r}) R_{10} (\vec{r}) \times Y_l^m (\theta, \phi) \gamma_0 (\theta, \phi) \]
\[ = \frac{1}{4\pi} \]
Choosing $\hat{q} \cdot \hat{r} = \cos \theta$, we can expand the plane wave in terms of the angular momentum basis

$$e^{i\hat{q} \cdot \hat{r}} = \sum_{l'=0}^{\infty} \frac{\sqrt{4\pi}}{\sqrt{2l'+1}} Y_{l'}^0(\theta, \phi) \sum_{l} \frac{1}{\sqrt{2l+1}} i^{l'} j_{l'}(q \rho r) P_l(\cos \theta) \sqrt{2l+1} Y_{l}^0(\theta, \phi)$$

Substituting above $F(q)$ gives

$$F(q) = \sum_{m=-l}^{l} \sum_{l'=0}^{\infty} \frac{\sqrt{4\pi}}{\sqrt{2l'+1}} \sum_{l=0}^{\infty} i^{l'} \sum_{m=-l}^{l} \sum_{m=-l}^{l} \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi \text{e}^{i m^\ast \theta} j_m(q \rho r) R_{n\mu}(\rho r) R_{10}(\rho r) \times$$

$$\{ d \Phi \}$$

$$= \delta_{l l'} \delta_{m m'}$$
Hence

\[ F_{\text{ext}}(r) = \sum_{l=0}^{n_l} \frac{A_{n_l} \alpha^2 l^2}{2l+1} \int_0^\infty r^2 j_l(qr) \times \frac{R_{\text{nl}}(r)}{R_{\text{l0}}(r)} \, dr \]

8.3. Semi-Classical Treatment of Electromagnetic Radiation:
The interaction of Bound States of charged particles with Photons

Up to present we have dealt with the interaction of matter with matter, we have not included the interaction of matter with electromagnetic radiation. For example, the emission or absorption of photons by atoms. A consistent quantum mechanical treatment of the matter as well as electromagnetic radiation would require the generalization of our quantum mechanical postulates to the special relativistic Maxwell equations governing the time evolution of the electromagnetic field as well as the generalization of the