7.2.4 Examples of Partial Wave Analysis

In general the scattering phase shifts can be found by solving the Schrödinger equation with the scattering asymptotic condition (equivalently solving the L-S integral equation). The phase shifts can then be extracted from the asymptotic form of the wavefunction $\psi_{\vec{k}}^{(+)}(\vec{r})$. This asymptotic form of $\psi_{\vec{k}}^{(+)}(\vec{r})$ can be written in several ways, the utility of a particular form depending on the specifics of the problem. That is we have that (page -1067-)

$$\psi_\ell^{(+)}(r) \underset{r\to\infty}{\sim} \frac{1}{2kr}\left(e^{i[kr-(\ell+1)\frac{\pi}{2}+2\delta_\ell]} + e^{-i[kr-(\ell+1)\frac{\pi}{2}]}\right)$$

$$\underset{r\to\infty}{\sim} \frac{e^{i\delta_\ell}}{kr}\cos[kr-(\ell+1)\tfrac{\pi}{2}+\delta_\ell]$$

$$\underset{r\to\infty}{\sim} \frac{e^{i\delta_\ell}}{kr}\sin[kr-\tfrac{\ell\pi}{2}+\delta_\ell]$$

$$\underset{r\to\infty}{\sim} \frac{e^{i\delta_\ell}}{kr}\left[\sin(kr-\tfrac{\ell\pi}{2})\cos\delta_\ell + \cos(kr-\tfrac{\ell\pi}{2})\sin\delta_\ell\right].$$

Now we have that

$$j_\ell(\rho) \underset{\rho\to\infty}{\sim} \frac{\sin(\rho - \frac{\ell\pi}{2})}{\rho}$$

and $$n_\ell(\rho) \underset{\rho\to\infty}{\sim} -\frac{\cos(\rho-\frac{\ell\pi}{2})}{\rho}$$

So the above can be written as

$$\psi_\ell^{(+)}(r) \underset{r\to\infty}{\sim} e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right]$$

$$\underset{r\to\infty}{\sim} e^{i\delta_\ell}\cos\delta_\ell\left[j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr)\right].$$

All of these forms are equivalent. As well, instead of solving the L-S equation

$$\psi_\ell^{(+)}(r) = j_\ell(kr) + 4\pi\int_0^\infty dr'\, r'^2\, G_+^\ell(r,r')\, U(r')\, \psi_\ell^{(+)}(r')$$

we can solve the Schrödinger equation directly. Recall that this is

$$(\nabla^2 + k^2)\,\psi_{\vec{k}}^{(+)}(\vec{r}) = U(r)\,\psi_{\vec{k}}^{(+)}(\vec{r}).$$

Substituting $\psi_{\vec{k}}^{(+)}(\vec{r}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\,\psi_\ell^{(+)}(r)\,P_\ell(\hat{k}\cdot\hat{r})$

we find the radial Schrödinger equation as usual for $\psi_\ell^{(+)}(r)$

$$\sum_{l=0}^{\infty} (2l+1) i^l \left[\nabla^2 + k^2 - U(r)\right]\left[\psi_l^{(+)}(r) P_l(\hat{k}\cdot\hat{r})\right] = 0$$

which becomes

$$0 = \sum_{l=0}^{\infty} (2l+1) i^l \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\psi_l^{(+)}(r)\right) - \frac{l(l+1)}{r^2}\psi_l^{(+)}(r) + \left(k^2 - U(r)\right)\psi_l^{(+)}(r)\right] P_l(\hat{k}\cdot\hat{r}) .$$

Since the $P_l(\hat{k}\cdot\hat{r})$ are independent, this is true for each l

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\psi_l^{(+)}(r)\right) - \left(\frac{l(l+1)}{r^2} + U(r)\right)\psi_l^{(+)}(r)\right] = -k^2 \psi_l^{(+)}(r) .$$

The general solutions to this equation must be matched up to the above scattering asymptotic forms that $\psi_l^{(+)}(r)$ is required to have. The phase shifts δ_l can be found from this.

1) Example: Hard Sphere Scattering

Suppose the potential describes a hard sphere of radius a:

$$U(r) = \infty \; ; \; r < a$$

$$U(r) = 0 \; ; \; r > a .$$

The Schrödinger equation becomes

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\psi_\ell^{(+)}(r)\right) - \frac{\ell(\ell+1)}{r^2}\psi_\ell^{(+)}(r)\right]$$

$$= -k^2\psi_\ell^{(+)}(r), \text{ for } r > a.$$

And for $r \leq a$ we have the boundary condition $\psi_\ell^{(+)}(r=a) = 0$, for a hard sphere.

The general solution to the Schrödinger equation is given by (Schiff page 84 with $\rho = kr$) the spherical Bessel functions for $r \geq a$.

$$\psi_\ell^{(+)}(r) = A_\ell j_\ell(kr) + B_\ell n_\ell(kr) ,$$

The boundary condition at $r=a \Rightarrow$

$$\psi_\ell^{(+)}(r=a) = 0 = A_\ell j_\ell(ka) + B_\ell n_\ell(ka)$$

$$\Rightarrow \quad \frac{B_\ell}{A_\ell} = -\frac{j_\ell(ka)}{n_\ell(ka)} \quad \text{and so}$$

$$\boxed{\psi_\ell^{(+)}(r) = A_\ell\left[j_\ell(kr) - \frac{j_\ell(ka)}{n_\ell(ka)} n_\ell(kr)\right], \text{for } r \geq a}$$

Now we match this to the asymptotic form of $\psi_\ell^{(+)}(r)$

$$\psi_\ell^{(+)}(r) \underset{r\to\infty}{\sim} e^{i\delta_\ell}\cos\delta_\ell\left[j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr)\right]$$

to find the exact results

$$\boxed{\tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}}$$

and

$$\boxed{A_\ell = e^{i\delta_\ell}\cos\delta_\ell \, .}$$

Consider the s-partial wave ($l=0$) scattering. For $l=0$

$$j_0(\rho) = \frac{\sin\rho}{\rho} \quad \text{and} \quad n_0(\rho) = -\frac{\cos\rho}{\rho}$$

Thus

$$\tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = -\tan(ka)$$

$\Rightarrow \quad \delta_0 = -ka \mod \pi$. With the condition that $\delta_0 \to 0$ as $k \to 0$ we have uniquely

$$\boxed{\delta_0 = -ka}.$$

The s-wavefunction is simply, for $r \geq a$,

$$\psi_0^{(+)}(r) = e^{i\delta_0}\cos\delta_0\left[j_0(kr) - \tan\delta_0\, n_0(kr)\right]$$

$$= e^{i\delta_0}\frac{1}{kr}\left[\cos\delta_0 \sin kr + \sin\delta_0 \cos kr\right]$$

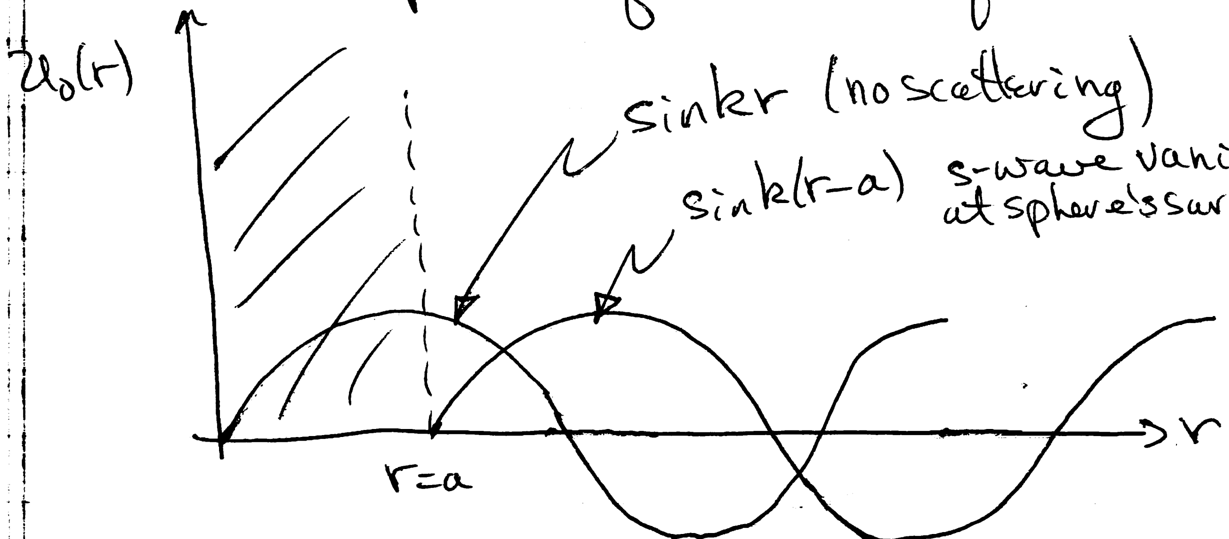
$$\equiv \frac{e^{i\delta_0}}{kr}u_0(r) \quad , \text{with}$$

$$u_0(r) = \sin kr \cos\delta_0 + \cos kr \sin\delta_0$$

$$= \sin(kr + \delta_0)$$

$$\boxed{u_0(r) = \sin k(r-a)}.$$

Recall, for no scattering $\delta_0 = 0$ and $u_0(r) = \sin kr$, the hard or rigid sphere shifts the phase of the wavefunction



Similar results hold for $l \geq 1$ - partial waves.

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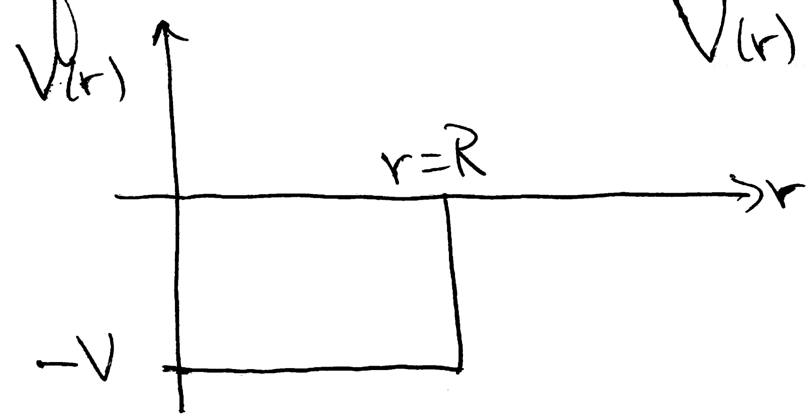
2) Example: Scattering Off An Attractive Square Well Potential

Suppose the potential is an attractive square well

$$V(r) = -V\,\Theta(R-r)$$
with $V > 0$.



For s-waves ($\ell=0$) the radial Schrödinger equation (page -1048-) becomes (p.-1105- also)

$$\frac{d^2}{dr^2}u_0(r) + \left(k^2 - U(r)\right)u_0(r)$$

with $\psi_0^{(+)}(r) = \dfrac{e^{i\delta_0}u_0(r)}{kr}$. For $r>R$ $U(r)=0$ and this becomes

$$\frac{d^2u_0(r)}{dr^2} + k^2u_0(r) = 0$$

$\Rightarrow$

$$\boxed{u_0(r) = A\sin kr + B\cos kr, \text{ for } r>R}$$

For $r<R$ and $U(r) = -\frac{2mV}{\hbar^2} = \mathcal{U}$ this yields

$$\frac{d^2u_0(r)}{dr^2} + (k^2+\mathcal{U})u_0(r) = 0$$

$$\boxed{\Rightarrow u_0(r) = \hat{A}\sin\hat{k}r + \hat{B}\cos\hat{k}r, \text{ for } r<R \quad \text{with } \hat{k} = \sqrt{k^2+\mathcal{U}}}$$

Since $\psi_{\ell=0}^{(+)}(r)$ is to be finite at $r=0$ we require that $u_0(r=0)=0$

$\Rightarrow$ $\boxed{\hat{B} = 0}$.

Thus

$$\boxed{u_0(r) = \begin{cases} \hat{A} \sin \hat{k} r & , r < R \\ A \sin kr + B \cos kr & , r > R \end{cases}}$$.

Now we have that the asymptotic form of $u^{(+)}_{l=0}(r)$ as $r \to \infty$ from page-1103

$\Rightarrow$

$$u_0(r) \underset{r \to \infty}{\sim} \sin kr \cos \delta_0 + \cos kr \sin \delta_0$$

Thus we identify

$$\boxed{\begin{aligned} A &= \cos \delta_0 . \\ B &= \sin \delta_0 \end{aligned}}$$

and

$$\boxed{u_0(r) = \begin{cases} \hat{A} \sin \hat{k} r & , r < R \\ \sin(kr + \delta_0) & , r > R . \end{cases}}$$

We still have the B.C. at $r=R$ to apply.

1) $$u_0(r)\Big|_{r=R^+} = u_0(r)\Big|_{r=R^-}$$

$$\Rightarrow \boxed{\sin(kR+\delta_0) = \hat{A}\sin\hat{k}R}$$

2) $$\frac{du_0(r)}{dr}\Big|_{r=R^+} = \frac{du_0(r)}{dr}\Big|_{r=R^-}$$

$$\Rightarrow \boxed{k\cos(kR+\delta_0) = \hat{k}\hat{A}\cos\hat{k}R}$$

Dividing the two relations (i.e. equating the logarithmic derivatives

$$\frac{d\ln u_0(r)}{dr}\Big|_{r=R^+} = \frac{d\ln u_0(r)}{dr}\Big|_{r=R^-}\ )$$

we find $$\boxed{k\cot(kR+\delta_0) = \hat{k}\cot\hat{k}R}$$

This is a transcendental equation for δ_0.

As we have shown for low energy scattering $\delta_\ell \underset{k\to 0}{\approx} (ka)^{(2\ell+1)}$, and

so the S-wave gives the dominant contribution to the cross-section. The S-wave scattering length a is defined as

i.e. $a \gtrless 0$ here

$$\delta_0 \equiv -(ka) \quad \text{as } k \to 0 .$$

The LHS of the transcendental B.C. equation becomes

$$k\cot(kR+\delta_0) \underset{k\to 0}{\sim} k\cot k(R-a)$$

$$\underset{k\to 0}{\sim} k\,\frac{\cos k(R-a)}{\sin k(R-a)}$$

$$\underset{k\to 0}{\sim} \frac{1}{R-a} .$$

The RHS of the equation becomes

$$\hat{k}\cot\hat{k}R = \sqrt{k^2+U}\,\cot R\sqrt{k^2+U}$$

$$\underset{k\to 0}{\sim} \sqrt{U}\,\cot\sqrt{U}R .$$

Hence the B.C. implies

$$\frac{1}{R-a} \underset{k\to 0}{\sim} \sqrt{U}\cot\sqrt{U}R$$

$$\boxed{\Rightarrow a = R\left[1 - \frac{1}{R\sqrt{U}\cot\sqrt{U}R}\right] \quad \text{with } U = \frac{2mV}{\hbar^2}.}$$

So for $r > R$, $u_0(r) = \sin(kr+\delta_0)$

$$\underset{k\to 0}{\sim} \sin k(r-a)$$

$$\underset{k\to 0}{\sim} k(r-a).$$

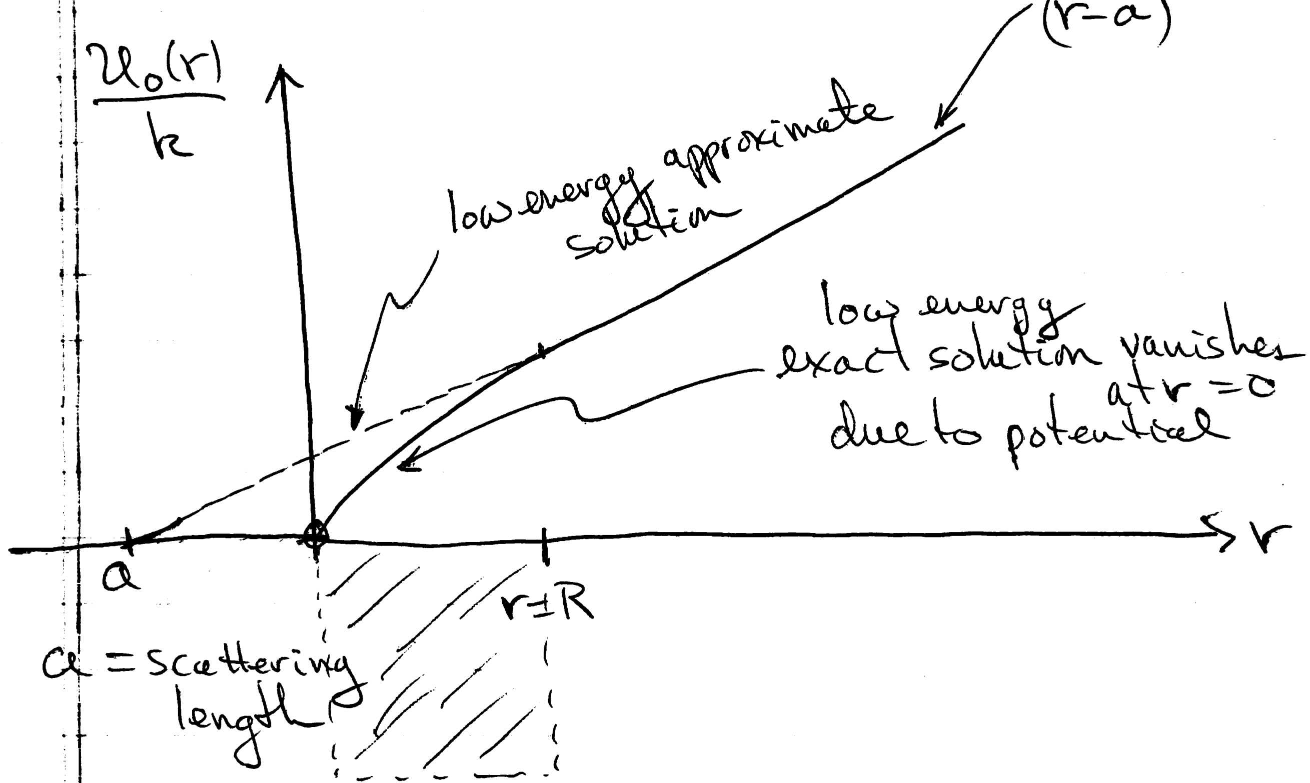
Hence $\frac{u_0(r)}{k} \underset{k\to 0}{\sim} r-a$, for $r > R$.

Recall that the asymptotic solution

$$\psi_0^{(+)}(r) \underset{r\to\infty}{\sim} \frac{e^{i\delta_0}u_0(r)}{kr}$$

$$\underset{r\to\infty}{\sim} e^{i\delta_0}\frac{\sin(kr+\delta_0)}{kr}$$

Thus we can plot the 2 solutions



Note we have chosen $a<0$ here; that is

$$R\sqrt{U}\cot\sqrt{U}R<1$$

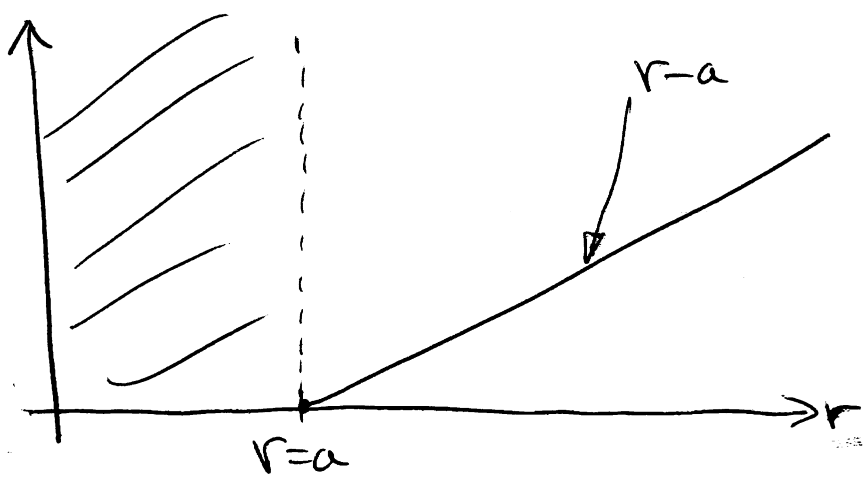
, we could have chosen $a>0$. Either choice is possible for attractive potentials, it depends on the range (R) and magnitude (U) of the potential. Thus we see that the scattering length is not really a length. It does not give the range of the potential.

The scattering length for the hard sphere recall was simply given by $\delta_0=-ka$. (page -1108-)

So the wavefunction was

$$u_0(r)=\sin k(r-a)\underset{k\to0}{\approx}k(r-a)$$

And so for the rigid sphere $\frac{u_0(r)}{k}$



and the scattering length a is positive.

The scattering length determines the scattering amplitude at low energies. In general as $k \to 0$, $f^{(+)}$ is dominated by S-waves

$$f^{(+)}(\vec{k},\vec{k}') = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l}\sin\delta_l}{k} P_l(\hat{k}\cdot\hat{r})$$

$$\underset{k\to 0}{\approx} e^{i\delta_0} \frac{\sin\delta_0}{k} \underbrace{P_0(\hat{k}\cdot\hat{r})}_{=1}$$

Now $\delta_0 \underset{k\to 0}{\approx} -ka \Rightarrow$

$$\underset{k\to 0}{\approx} e^{-ika} \frac{\sin(-ka)}{k}$$

$$\underset{k\to 0}{\approx} -a$$

Hence the low energy differential cross section is

$$\sigma(\theta,\varphi) \underset{k\to 0}{\approx} |a|^2$$

and the total elastic cross-section is

$$\sigma \underset{k\to 0}{\approx} 4\pi |a|^2 .$$

For the attractive square well above we had

$$a = R\left[1 - \frac{1}{R\sqrt{u}\cot\sqrt{u}R}\right]$$

So

$$\sigma(\theta,\varphi) \underset{k\to 0}{\approx} R^2\left[1 - \frac{1}{R\sqrt{u}\cot\sqrt{u}R}\right]^2 .$$

This is the lowest order term in an energy expansion of the cross-section, we can go on to find the next order corrections in energy to δ_0. Recall the transcendental B.C. equation for δ_0 (page -1112-)

$$k\cot(kR+\delta_0) = \bar{k}\cot\bar{k}R$$

$$= k\,\frac{\cos kR\cos\delta_0 - \sin kR\sin\delta_0}{\cos kR\sin\delta_0 + \sin kR\cos\delta_0}.$$

Multiplying this out $\Rightarrow$

$$k\cot\delta_0 = \frac{\bar{k}\cot\bar{k}R + k\tan kR}{1 - \frac{\bar{k}}{k}\cot\bar{k}R\tan kR}.$$

Since the RHS is an even function of k we can Taylor expand in powers of k^2

$$k\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0k^2 + \ldots$$

Where a is the s-wave scattering length and $\underline{r_0}$ is the s-wave effective range. This definition of the scattering length is the same as earlier since

$$k\cot\delta_0 \underset{\substack{k\to 0 \\ (\Rightarrow \delta_0 \to 0)}}{\approx} \frac{k}{\delta_0} = \frac{-1}{a} \Rightarrow \delta_0 = -ka.$$

More generally we use the above expansion as the definition of a and r_0. It is called the effective range approximation. It is a valid approximation as long as 1) $kR \ll 1$, which means the incident energy << inverse range of the potential and 2) $k^2 \ll U$, which means the incident energy << depth of the potential well. For small enough k, these will be true.

Within the effective range approximation the scattering amplitude is

$$f^{(+)}(\vec{k},\vec{k}') \underset{k\to 0}{\approx} e^{i\delta_0}\frac{\sin\delta_0}{k}$$

$$\underset{k\to 0}{\approx} \frac{\sin\delta_0}{k(\cos\delta_0 - i\sin\delta_0)}$$

$$\underset{k\to 0}{\approx} \frac{1}{k\cot\delta_0 - ik}$$

$$\underset{k\to 0}{\approx} \frac{1}{\left(\frac{-1}{a}+\frac{r_0 k^2}{2}\right)-ik}$$

having used the effective range approx. in the last step. The differential elastic cross-section becomes

$$\boxed{\begin{aligned}\sigma(\theta,\varphi) &= |f^{(+)}(\vec{k},\vec{k}')|^2 \underset{k\to 0}{\approx} \frac{1}{(k\cot\delta_0)^2+k^2}\\ \begin{pmatrix}\text{using effective}\\ \text{range approx.}\end{pmatrix} &\underset{k\to 0}{\approx} \frac{1}{\left(-\frac{1}{a}+\frac{r_0k^2}{2}\right)^2+k^2}\end{aligned}}$$

This is the general form of the low energy elastic scattering differential cross section, valid for short range ($kR \ll 1$) and very deep ($k^2 \ll U$) potentials.

Notice that we can consider the conditions for the formation of a bound state using this formula. We found in section 7.2.2 that for a bound state to occur $f^{(+)}(\vec{k},\vec{k}')$ must develop a simple pole on the positive imaginary k axis. Let $k=i\kappa$ in the effective range approximation for $f^{(+)}$, indeed

$$f^{(+)}(\vec{k},\vec{k}')\Big|_{k=i\kappa} \approx \frac{1}{\kappa-\left[\frac{1}{a}+\frac{r_0\kappa^2}{2}\right]}$$

a pole occurs for solutions to the equation in which $\kappa>0$,

$$\boxed{\mathcal{K} = \frac{1}{a} + \frac{r_0 \mathcal{K}^2}{2}}\ .$$

Now for a bound state to just form, the energy must be infinitesimally negative $E_{Bound} = 0^-$, that is $\mathcal{K} = 0^+$. For this to happen the s-wave scattering length must diverge $a \to \infty$. Hence when a bound state of the potential just forms, that is just as $a \to \infty$, the elastic scattering differential cross-section diverges $\sigma(\theta,\varphi) \to \infty$. This then is a signal, i.e. when $\sigma(\theta,\varphi) \to \infty$, that a bound state has occurred!

Although we motivated the effective range approximation within an example, in fact we can show that it's true for s-wave scattering at low energies in general. That is for arbitrary potential as $k \to 0$

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_0 k^2$$

with a and r_0 k-independent constants.

Consider the Schrödinger equation for s-waves, the wave function is given by

$$\psi_{\vec{k}}^{(+)}(\vec{r}) = \frac{e^{i\delta_0}}{kr}\sin\delta_0 \, W(r) \left(= \psi_0^{(+)}(r) \underbrace{P_0(\hat{k}\cdot\hat{r})}_{1} \right)$$

with

$$\frac{d^2}{dr^2} W(r) + \left(k^2 - U(r)\right) W(r) = 0$$

(Note: $$\psi_0^{(+)}(r) = \frac{e^{i\delta_0} u_0(r)}{kr} = \frac{e^{i\delta_0} \sin\delta_0 W(r)}{kr}$$

So $u_0(r) = \sin\delta_0 \, W(r)$; $\sin\delta_0$ is an r-independent normalization constant hence u_0 and W obey the same Schrödinger (radial) equation). We require $\psi_0^{(+)}(r)$ to have the asymptotic form (page -1103-)

$$W(r) \underset{r\to\infty}{\sim} \frac{\sin(kr+\delta_0)}{\sin\delta_0}$$

i.e. $$\psi_0^{(+)}(r) \underset{r\to\infty}{\sim} \frac{e^{i\delta_0}}{kr}\sin(kr+\delta_0) .$$

For the case of no-scattering, $U=0$, the Schrödinger equation reduces to the free (Helmholtz) equation with wave function $\phi(r)$

$$\frac{d^2}{dr^2}\phi(r) + k^2\phi(r) = 0 .$$

Since $\phi(r)$ is to be finite we, in particular, require $\phi(r=0) = 1$ as a normalization condition. As well we require $\phi(r)$ for large r to agree with $W(r)$

$$\phi(r) \underset{r\to\infty}{\sim} \frac{\sin(kr+\delta_0)}{\sin\delta_0} .$$

These 2 conditions determine $\phi(r)$ to be

$$\phi(r) = \frac{\sin(kr+\delta_0)}{\sin\delta_0} \quad \text{for all } r \text{ and } k.$$

So defined $\phi(r)$ differs from $W(r)$ only where the potential is non-zero.

In the low energy limit, $k\to 0$, the Schrödinger equations become

$$\frac{d^2 W(r)}{dr^2} = U(r)\,W(r)$$

and
$$\frac{d^2\phi(r)}{dr^2} = 0 \quad , \text{ for } k\to 0.$$

Hence, since $\phi(0) = 1$, we have

$$\phi(r) = 1 - \frac{r}{a} \quad \text{for } k \to 0$$

where a is a k-independent constant of integration. This form of $\phi(r)$ agrees with the low-energy limit of the exact solution

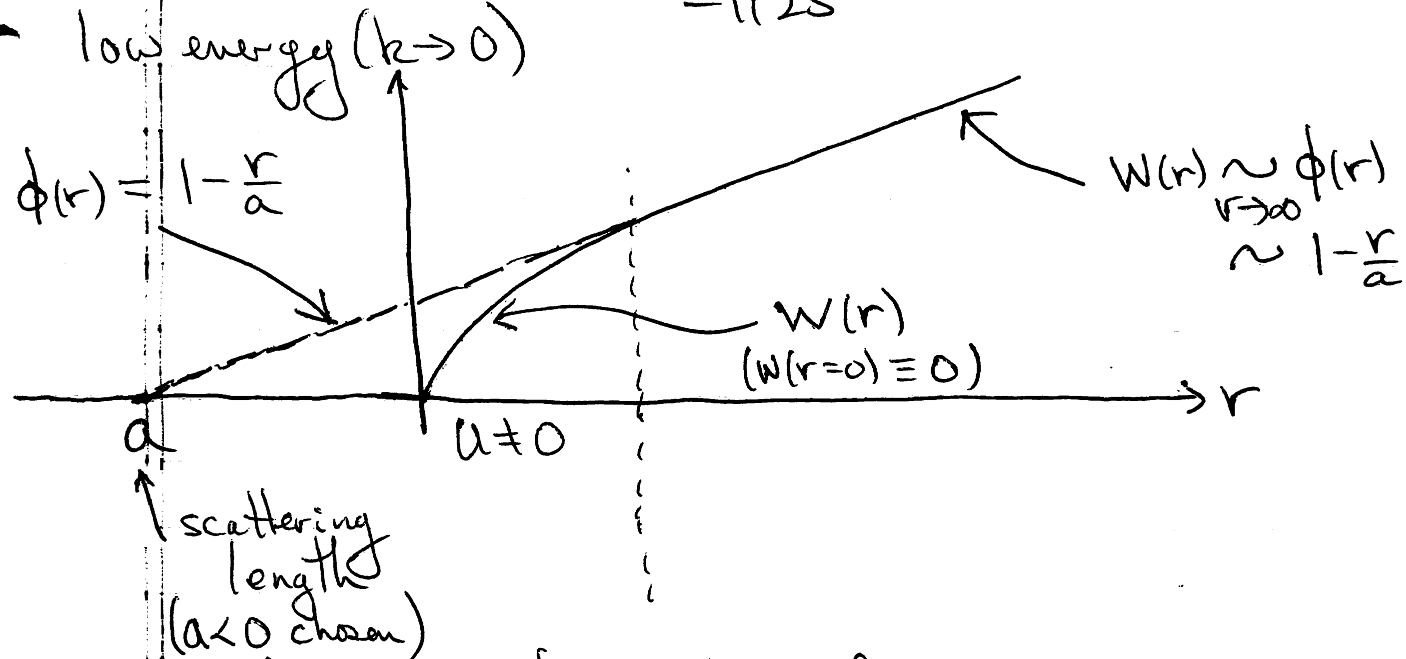
$$\phi(r) = \frac{\sin(kr+\delta_0)}{\sin\delta_0} = \frac{\sin kr \cos\delta_0 + \cos kr \sin\delta_0}{\sin\delta_0}$$

$$\underset{k\to 0}{\sim} \frac{kr}{\delta_0} + 1$$

So $1 - \frac{r}{a} = \frac{kr}{\delta_0} + 1 \Rightarrow a = -\frac{\delta_0}{k}$ as $k \to 0$.

From the condition that $\psi_{\vec{k}}^{(+)}(\vec{r}=0)$ be finite, we have that

$W(r=0) = 0$ as a boundary condition. Hence we can again plot the low energy behavior of the free and exact solutions as we did in the examples



According to Bethe and Schwinger we can consider the integral

$$I=\int_0^\infty dr\,\frac{d}{dr}\left\{\left[W(r,k{=}0)\overleftrightarrow{\frac{d}{dr}}W(r)\right]-\left[\phi(r,k{=}0)\overleftrightarrow{\frac{d}{dr}}\phi(r)\right]\right\}$$

(where $f\overleftrightarrow{\frac{d}{dr}}g\equiv f\frac{dg}{dr}-\frac{df}{dr}g$).

Since it is a total divergence we can evaluate it simply,

$$I=\left[W(r,k{=}0)\overleftrightarrow{\frac{d}{dr}}W(r)\right]\Bigg|_{r=0}^{\infty}-\left[\phi(r,k{=}0)\overleftrightarrow{\frac{d}{dr}}\phi(r)\right]\Bigg|_{r=0}^{\infty}$$

Since $W(r)\underset{r\to\infty}{\sim}\phi(r)$ for all k, the upper limit of the integrand vanishes

$$I = \left[\phi(r,k=0)\overleftrightarrow{\frac{d}{dr}}\phi(r)\right]\Big|_{r=0} - \left[W(r,k=0)\overleftrightarrow{\frac{d}{dr}}W(r)\right]\Big|_{r=0}$$

Since $\psi^{(+)}_{\ell=0}(r)$ is finite at $r=0$, we have $W(r=0)=0$ for all k, hence the second term above vanishes,

$$I = \left[\phi(r,k=0)\overleftrightarrow{\frac{d}{dr}}\phi(r)\right]\Big|_{r=0}$$

By definition $\phi(r=0)=1$ for all k while

$$\phi(r) = \frac{\sin(kr+\delta_0)}{\sin\delta_0}$$ implies

$$\frac{d\phi(r)}{dr} = k\,\frac{\cos(kr+\delta_0)}{\sin\delta_0}$$ and so

for all k $$\frac{d\phi(r)}{dr}\Big|_{r=0} = k\cot\delta_0 .$$

Now as $k\to 0$ we had $\phi(r) \underset{k\to 0}{\approx} 1-\frac{r}{a}$

$$\Rightarrow \quad \frac{d\phi(r,k=0)}{dr}\Big|_{r=0} = -\frac{1}{a}$$

So
$$I = \left[\phi(r,k=0)\frac{d\phi(r)}{dr} - \frac{d\phi(r,k=0)}{dr}\phi(r)\right]\Big|_{r=0}$$

$$\boxed{I = k\cot\delta_0 + \frac{1}{a}}$$

On the other hand we can use Schrödinger's equation to evaluate the integral

$$I = \int_0^\infty dr \left\{\left[W(r,k=0)\frac{d^2}{dr^2}W(r) - \frac{d^2W(r,k=0)}{dr^2}W(r)\right]\right.$$

$$\left. - \left[\phi(r,k=0)\frac{d^2}{dr^2}\phi(r) - \frac{d^2\phi(r,k=0)}{dr^2}\phi(r)\right]\right\}$$

using the Schrödinger equation

$$= \int_0^\infty dr \left\{ \left[W(r,k=0)(-k^2 + \cancel{U(r)})W(r) \right.\right.$$

$$\left. - \cancel{U(r)\,W(r,k=0)}\,W(r)\right]$$

$$\left. - \left[\phi(r,k=0)(-k^2\phi(r))\right]\right\}$$

$$\boxed{I = k^2\int_0^\infty dr\left[\phi(r,k=0)\phi(r) - W(r,k=0)W(r)\right]}$$

Thus we obtain the exact result

$$\boxed{k\cot\delta_0 + \frac{1}{a} = k^2\int_0^\infty dr\left[\phi(r,k=0)\phi(r) - W(r,k=0)W(r)\right]}$$

This is true for all k and all potentials.

Now let $k \to 0$, we define

$$\frac{1}{2}r_0 \equiv \int_0^\infty \left[(\phi(r,k=0))^2 - (W(r,k=0))^2\right]$$

and obtain

$$\boxed{k\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0 k^2 \quad, \text{as } k \to 0.}$$

r_0 is the effective range, it depends only on the $k=0$ solutions to the Schrödinger equation (i.e. it is independent of k). Since $W(r,k=0)$ and $\phi(r,k=0)$ differ only where $U \neq 0$, r_0 has contributions only from coordinates r where $U(r) \neq 0$. Thus r_0 is a measure of the range of the potential.