7.2.3. The Born Approximation for the Partial Wave Analysis

The expression for the scattering amplitude $f^{(+)}(\vec{k},\vec{k})$ and phase shifts $\delta_\ell(k)$ we derived are exact

$$\frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -\int_0^\infty dr' r'^2 j_\ell(kr')U(r')\psi_\ell^{(+)}(r') .$$

The Born approximation replaced the exact wave function $\psi_{\vec{k}}^{(+)}(\vec{r})$ with the dominant term in its expansion, the plane wave $e^{i\vec{k}\cdot\vec{r}}$. In terms of the partial waves, this means in the Born approximation we replace the exact ℓth partial wave $\psi_\ell^{(+)}(r)$ with $j_\ell(kr)$ in the above formula for the phase shifts. Thus the above becomes

$$\frac{e^{i\delta_\ell^{Born}}\sin\delta_\ell^{Born}}{k} = -\int_0^\infty dr' r'^2 j_\ell(kr')U(r')j_\ell(kr') .$$

This simplifies at low energies since as $k \to 0$

$$j_\ell(kr') \underset{k\to 0}{\approx} \frac{(kr')^\ell}{(2\ell+1)!!}$$

with $(2\ell+1)!! = (2\ell+1)(2\ell-1)(2\ell-3)\cdots 1$.

Thus

$$\frac{e^{i\delta_\ell^{Born}}\sin\delta_\ell^{Born}}{k} \underset{k\to 0}{\approx} -\int_0^\infty dr'\, r'^2 \frac{(kr')^{2\ell}}{[(2\ell+1)!!]^2} U(r')$$

$\Rightarrow$

$$e^{i\delta_\ell^{Born}}\sin\delta_\ell^{Born} \underset{k\to 0}{\approx} -\frac{(k)^{(2\ell+1)}}{[(2\ell+1)!!]^2}\int_0^\infty dr'\, r'^{(2\ell+2)} U(r').$$

Since the RHS goes to zero as $k\to 0$, so must the LHS, hence $\delta_\ell^{Born} \ll 1$ as $k\to 0$. Expanding the LHS

$$e^{i\delta_\ell^{Born}}\sin\delta_\ell^{Born} \underset{k\to 0}{\approx} \delta_\ell^{Born} + O(\delta_\ell^{Born\,2})$$

we find

$$\delta_\ell^{Born}(k) \underset{k\to 0}{\approx} -\frac{k^{(2\ell+1)}}{[(2\ell+1)!!]^2}\int_0^\infty dr'\, r'^{(2\ell+2)} U(r')$$

$$\underset{k\to 0}{\approx} (ka)^{(2\ell+1)}$$

with

$$a^{(2\ell+1)} \equiv -\frac{1}{[(2\ell+1)!!]^2}\int_0^\infty dr'\, r'^{(2\ell+2)} U(r').$$

So we find that for $ka \to 0$ and $l' > l$ that

$$\delta_l^{Born} > \delta_{l'}^{Born}.$$

At low energy the larger angular momentum waves have smaller phase shifts. This result is due to the angular momentum barrier which keeps higher l states away from the region where $U(r)$ is greatest.

Although we have that $\delta_l^{Born} \underset{ka \to 0}{\approx} (ka)^{(2l+1)}$,

in fact we can derive such a low energy behavior for the exact phase shift not just in the Born approximation. So in general for an "a" independent of k we have that

$$\delta_l(k) \underset{(ka) \ll 1}{\approx} (ka)^{(2l+1)}.$$

This follows from the low energy behavior of the scattering Green function

$$4\pi G_+^{\ell}(r,r') = -ik\, j_\ell(kr_<)\, h_\ell(kr_>)$$
$$\underset{k\to 0}{\approx} -ik\left[\frac{(kr_<)^\ell}{(2\ell+1)!!}\right]\left[\frac{-i(2\ell-1)!!}{(kr_>)^{\ell+1}}\right]$$
$$\underset{k\to 0}{\approx} -\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell \frac{1}{(2\ell+1)},$$

independent of k. The L-S equation at low energy then becomes

$$\psi_\ell^{(+)}(r) = j_\ell(kr) + 4\pi\int_0^\infty dr'\, r'^2\, G_+^\ell(r,r')\, U(r')\, \psi_\ell^{(+)}(r')$$
$$\underset{k\to 0}{\approx} \frac{(kr)^\ell}{(2\ell+1)!!} - \frac{1}{(2\ell+1)}\int_0^\infty dr'\, r'^2\, U(r') \times$$
$$\times \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell \psi_\ell^{(+)}(r').$$

Hence

$$\left(\frac{\psi_\ell^{(+)}(r)}{k^\ell}\right) \underset{k\to 0}{\approx} \frac{r^\ell}{(2\ell+1)!!} - \frac{1}{(2\ell+1)}\int_0^\infty dr'\, r'^2\, U(r') \times$$
$$\times \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell \left(\frac{\psi_\ell^{(+)}(r')}{k^\ell}\right).$$

As $k \to 0$, $\left(\frac{\psi_\ell^{(+)}(r)}{k^\ell}\right)$ obeys an integral equation with no k dependence, so the solution $\phi_\ell^{(+)}(r)$ is independent of k

$$\left(\frac{\psi_\ell^{(+)}(r)}{k^\ell}\right) \underset{k\to 0}{\simeq} \phi_\ell^{(+)}(r) \; ;$$

that is

$$\psi_\ell^{(+)}(r) \underset{k\to 0}{\simeq} k^\ell \, \phi_\ell^{(+)}(r) \, .$$

$\phi_\ell^{(+)}(r)$ obeys the k-independent integral equation

$$\phi_\ell^{(+)}(r) = \frac{r^\ell}{(2\ell+1)!!} - \frac{1}{(2\ell+1)}\int_0^\infty dr' \, r'^2 \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell \times$$

$$\times \, U(r') \, \phi_\ell^{(+)}(r') \, .$$

All this implies that the low-energy phase shifts are given by

$$\frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -\int_0^a dr' r'^2 j_\ell(kr') U(r') \psi_\ell^{(+)}(r')$$

$$\underset{k\to 0}{\approx} -\int_0^\infty dr' r'^2 \left[\frac{(kr')^\ell}{(2\ell+1)!!}\right] U(r') \left[k^\ell \phi_\ell^{(+)}(r')\right],$$

Simplifying this yields

$$e^{i\delta_\ell}\sin\delta_\ell \underset{k\to 0}{\approx} -\frac{k^{(2\ell+1)}}{(2\ell+1)!!} \underbrace{\int_0^\infty dr' r'^{(\ell+2)} U(r') \phi_\ell^{(+)}(r')}_{\text{independent of } k}.$$

So as $k \to 0$, the RHS vanishes like $k^{(2\ell+1)}$ hence the LHS must also go to 0 as $k \to 0 \Rightarrow \delta_\ell(k) \ll 1$ as $k \to 0$.

Expanding, as in the Born approximation,

$$e^{i\delta_\ell}\sin\delta_\ell \underset{k\to 0}{\approx} \delta_\ell$$

$\Rightarrow$

$$\boxed{\delta_\ell \underset{k\to 0}{\simeq} (ka)^{(2\ell+1)} \quad \text{with } a \text{ now given by} \quad a^{(2\ell+1)} \equiv -\frac{1}{(2\ell+1)!!}\int_0^\infty dr'\, r'^{(\ell+2)}\, U(r')\, \phi_\ell^{(+)}(r')}$$

For $k \to 0$ and $\ell' > \ell$ we have $\delta_\ell > \delta_{\ell'}$, only the small angular momentum values contribute to the scattering.

Since $S_\ell(k) \equiv e^{2i\delta_\ell}$, δ_ℓ is only defined modulo π, that is δ_ℓ and $\delta_\ell + \pi$ yield the same $S_\ell(k)$.

Defining $\delta_\ell \to 0$ as $k \to 0$ removes this freedom and fixes δ_ℓ uniquely.