7.2.1. Scattering by Complex Potentials

As we have seen above, probability is conserved when scattering from real potentials. Hence we are led to elastic scattering. In general however, it could be that inelastic scattering may also occur, specifically processes in which particles are removed from the system. This type of inelastic scattering can be described by considering complex potentials. In this case the Hamiltonian is not Hermitian and probability will not be conserved, that is there will be a net probability flux. From above we will have that $|S_l(k)|^2 \neq 1$. The complex potential is one way of including the absorption of a beam of particles in a scattering process, but it is one particular type of inelastic process only. Other inelastic processes will need further generalizations of the interactions between particles to be described.

Since $|S_\ell(k)| \neq 1$, we now have

$$S_\ell(k) \equiv e^{-2\eta_\ell(k)} \, e^{2i\delta_\ell(k)}$$

where η_ℓ, δ_ℓ are real and η_ℓ is <u>non-negative</u>. This implies that $|S_\ell(k)| \leq 1$. To see this must be the case consider the scattering amplitude for elastic scattering

$$f(\vec{k},\vec{k}') = \sum_{\ell=0}^{\infty} (2\ell+1)\left[\frac{S_\ell(k)-1}{2ik}\right] P_\ell(\hat{k}\cdot\hat{k}') .$$

For $\vec{k}=\vec{k}'$, the forward scattering amplitude is

$$f(\vec{k},\vec{k}) = \sum_{\ell=0}^{\infty} (2\ell+1)\left[\frac{S_\ell(k)-1}{2ik}\right] \underbrace{P_\ell(1)}_{=1}$$

$$= \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1)\left[1 - S_\ell(k)\right] .$$

Taking the imaginary part yields

$$\mathrm{Im}\, f(\vec{k},\vec{k}) = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\left[1-\mathrm{Re}\, S_l(k)\right]$$

$$= \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\left[1-e^{-2\eta_l}\cos 2\delta_l\right].$$

The total elastic cross section is

$$\sigma_{elastic} = \int d\Omega\, |f(\vec{k},\vec{k}')|^2$$

$$= \int d\Omega \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)\,\times$$

$$\times \left[\frac{S_l(k)-1}{2ik}\right]^*\left[\frac{S_{l'}(k)-1}{2ik}\right]P_l(\hat{k}\cdot\hat{k}')P_{l'}(\hat{k}\cdot\hat{k}').$$

Recall that $\int d\Omega\, P_l(\hat{k}\cdot\hat{k}')P_{l'}(\hat{k}\cdot\hat{k}') = \frac{4\pi}{2l+1}\delta_{ll'}$,

So

$$\sigma_{elastic} = 4\pi\sum_{l=0}^{\infty}(2l+1)\left|\frac{S_l(k)-1}{2ik}\right|^2$$

$$= \frac{4\pi}{4k^2}\sum_{l=0}^{\infty}(2l+1)\left|1-S_l(k)\right|^2$$

$$= \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left|1-e^{-2\eta_l}e^{2i\delta_l}\right|^2$$

$$= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)\left[1 + e^{-4\eta_l} - e^{-2\eta_l} 2\cos 2\delta_l\right]$$

$$= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)\left[2\left(1 - e^{-2\eta_l}\cos 2\delta_l\right) - \left(1 - e^{-4\eta_l}\right)\right].$$

Thus we have that

$$\boxed{\sigma_{elastic} = \frac{4\pi}{k} \operatorname{Im} f(\vec{k},\vec{k}) - \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)\left(1 - e^{-4\eta_l}\right).}$$

Thus we have 3 cases

1) If $\eta_l = 0$ then $|S_l(k)| = 1$ and we recover $\sigma_{elastic} = \sigma_{Total} = \frac{4\pi}{k} \operatorname{Im} f(\vec{k},\vec{k})$

The optical theorem in the usual non-absorptive case, Where we used in general

$\sigma_{Total} = \sigma_{elastic} + \sigma_{inelastic}$, hence $\sigma_{inel.} = 0$.

2) If $\eta_\ell > 0$, then

$$\frac{4\pi}{k} \mathrm{Im}\, f(\vec{k},\vec{k}) = \sigma_{elastic} + \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1-e^{-4\eta_\ell}\right)$$

with each term on the RHS positive.

Then $\sigma_{inelastic}$, the total cross section for absorption of particles out of the incident beam is just

$$\sigma_{inelastic} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1-e^{-4\eta_\ell}\right)$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1-|S_\ell|^2\right).$$

And we obtain the generalized optical theorem

$$\sigma_{total} = \frac{4\pi}{k} \mathrm{Im}\, f(\vec{k},\vec{k})$$

$$= \sigma_{elastic} + \sigma_{inelastic},$$

3) If $\eta_\ell < 0$, then $\sigma_{inelastic} < 0$, yielding a physically nonsensical result.