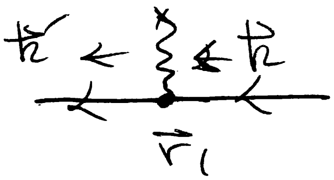
$$-4\pi f^{(+)}_{Born}(\vec{k},\vec{k}') = $$



$$= \int d^3r_1\, e^{-i\vec{k}'\cdot\vec{r}_1}\, U(\vec{r}_1)\, e^{+i\vec{k}\cdot\vec{r}_1}$$

$$= \int d^3r_1\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}_1}\, U(\vec{r}_1)$$

as we found earlier.

7.6.3. Examples of Scattering from Potentials

1) Scattering from a potential barrier

$$V(\vec{r}) = V\,\Theta(a-r)$$

$$V>0\,, \quad r \geq 0.$$

So $U(\vec{r}) = \frac{2m}{\hbar^2} V\,\Theta(a-r)$ and

the Fourier transform is simply that of the step function

$$\tilde{U}(\vec{q}) = \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, U(\vec{r})$$

$$= \frac{2mV}{\hbar^2}\int_0^a dr\, r^2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\, e^{-iqr\cos\theta}$$

$$= \frac{2mV}{\hbar^2}\int_0^a dr\, r^2 (2\pi)\left(\frac{1}{-iqr}\right)\underbrace{\left(e^{-iqr} - e^{+iqr}\right)}_{=-2i\sin(qr)}$$

$$= \frac{8\pi mV}{\hbar^2 q}\underbrace{\int_0^a dr\, r\sin(qr)}_{=-\frac{d}{dq}\int_0^a dr\cos(qr)}$$

$$= -\frac{8\pi mV}{\hbar^2 q}\frac{d}{dq}\left(\frac{\sin(qa)}{q}\right) \quad \left(\begin{array}{c}\sim\text{Spherical}\\ \text{Bessel}\\ \text{function}\end{array}\right)$$

$$= \left(\frac{2m}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)V\left[\frac{3}{(qa)}\underbrace{\left(\frac{\sin(qa)}{(qa)^2} - \frac{\cos(qa)}{(qa)}\right)}_{=j_1(qa)}\right]$$

So $\boxed{\tilde{U}(\vec{q}) = \left(\frac{2mV}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)\left[\frac{3}{(qa)} j_1(qa)\right]}$

The scattering amplitude is given by, in the Born approximation,

$$f^{(+)}_{Born}(\vec{k},\vec{k}') = f_{Born}(\vec{q})$$

$$= -\frac{1}{4\pi}\tilde{U}(\vec{q})$$

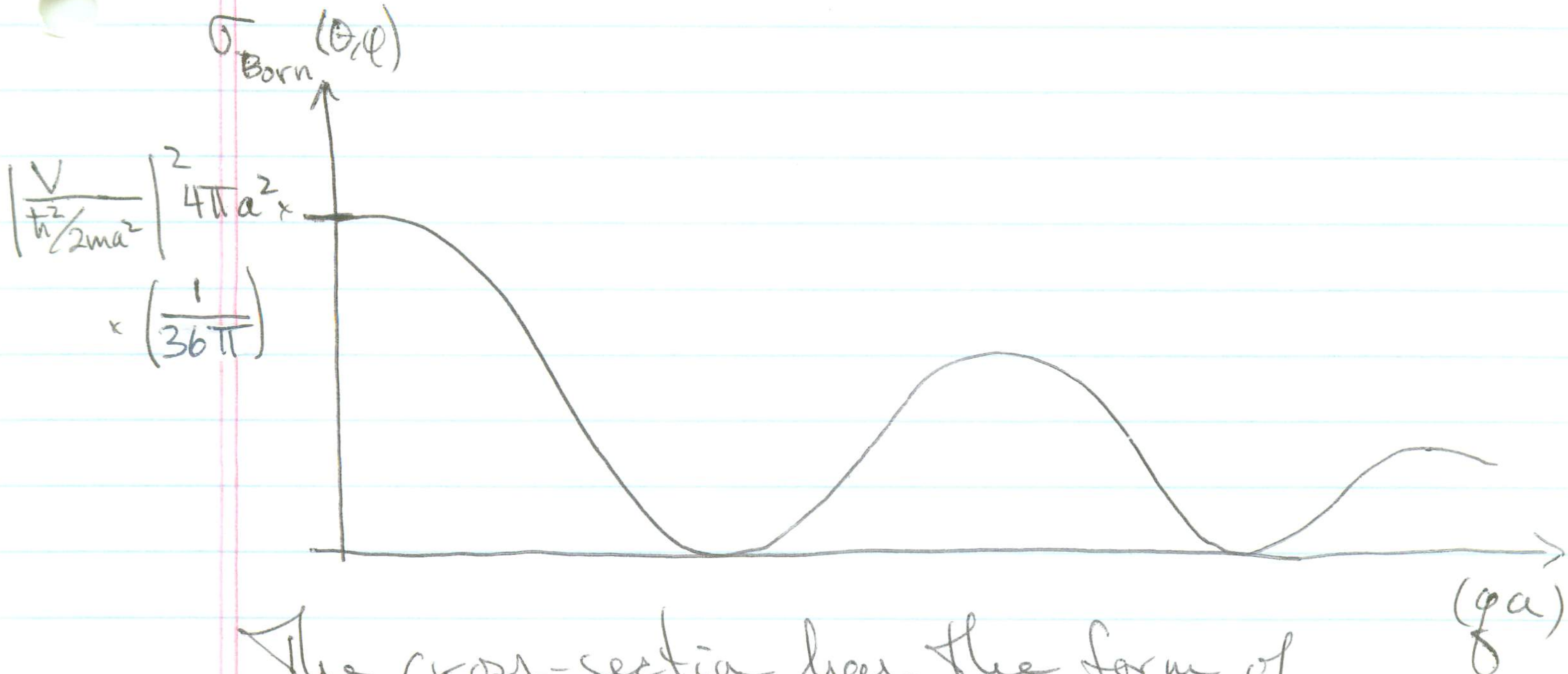
$$\boxed{f_{Born}(\vec{q}) = -\frac{1}{4\pi}\left(\frac{2mV}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)\left[\frac{3}{(qa)}j_1(qa)\right]}$$

The factor $\left[\frac{3}{(qa)}j_1(qa)\right]$ is called the form factor; it equals $+1$ at $q=0$ and oscillates as a function of q.

The differential cross section for scattering into the solid angle $d\Omega$ is

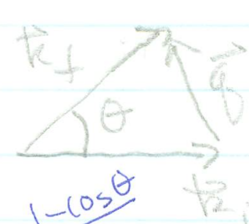
$$\sigma_{Born}(\theta,\varphi) = \left|f_{Born}(\vec{q})\right|^2$$

$$= \underbrace{\left|\frac{V}{(\hbar^2/2ma^2)}\right|^2}_{\substack{\text{(dimensionless)} \\ \text{energy ratio}}} \underbrace{4\pi a^2}_{\substack{\text{area} \\ \text{broad side of} \\ \text{a barn}}} \underbrace{\left|\frac{3\,j_1(qa)}{6\sqrt{\pi}\,(qa)}\right|^2}_{\substack{\text{form factor} \\ \text{(dimensionless)}}}$$



The cross-section has the form of a diffraction pattern due to the abrupt change of $V(\vec{r})$ with r.

Recall $q = 2k|\sin\frac{\theta}{2}|$, so as the particle detector is moved to different angles θ, the number of scattering events per unit time oscillates. At the zeroes of $j_1(qa)$ the number of events measured for that θ vanishes.



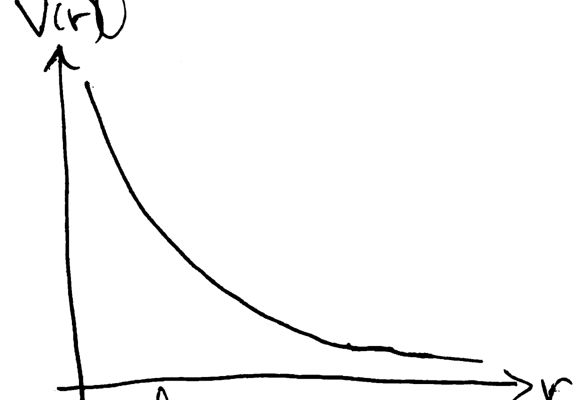
$$q^2 = k^2 + k^2 - 2k^2\cos\theta$$
$$= 2k^2(1-\cos\theta)$$
$$= 4k^2\sin^2\theta/2$$
$$\Rightarrow q = 2k|\sin\theta/2|$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

2) Scattering from a Yukawa potential:

The Yukawa potential is defined as

$$V(\vec{r}) = V \frac{a}{r} e^{-r/a}$$



with $V>0$, $a>0$ being the units of length.

As usual the Born approximation scattering amplitude is given by the Fourier transform of $V(\vec{r})$

$$f_{Born}(\vec{q}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \tilde{V}(\vec{q})$$

with

$$\tilde{V}(\vec{q}) = \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})$$

$$= Va \int_0^\infty dr\, r^2 \frac{e^{-r/a}}{r} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\, e^{-iqr\cos\theta}$$

$$= Va \int_0^\infty dr\, r e^{-r/a} (2\pi)\left(\frac{1}{-iqr}\right)\left(e^{-iqr} - e^{+iqr}\right)$$

$$= \frac{2\pi i}{q} Va \int_0^\infty dr \left[e^{-\left(\frac{1}{a}+iq\right)r} - e^{-\left(\frac{1}{a}-iq\right)r} \right]$$

$$= \frac{2\pi i}{q} Va \left[\frac{1}{\frac{1}{a}+iq} - \frac{1}{\frac{1}{a}-iq} \right]$$

$$= \frac{2\pi i}{q} Va \left[\frac{-2iqa^2}{1+(qa)^2} \right]$$

So

$$\boxed{\tilde{V}(\vec{q}) = \frac{4\pi a^2 (Va)}{1+(qa)^2}}$$

Thus the scattering amplitude is

$$f_{Born}(\vec{q}) = -\left(\frac{Va}{\hbar^2/2ma}\right) \frac{a}{1+(qa)^2}$$

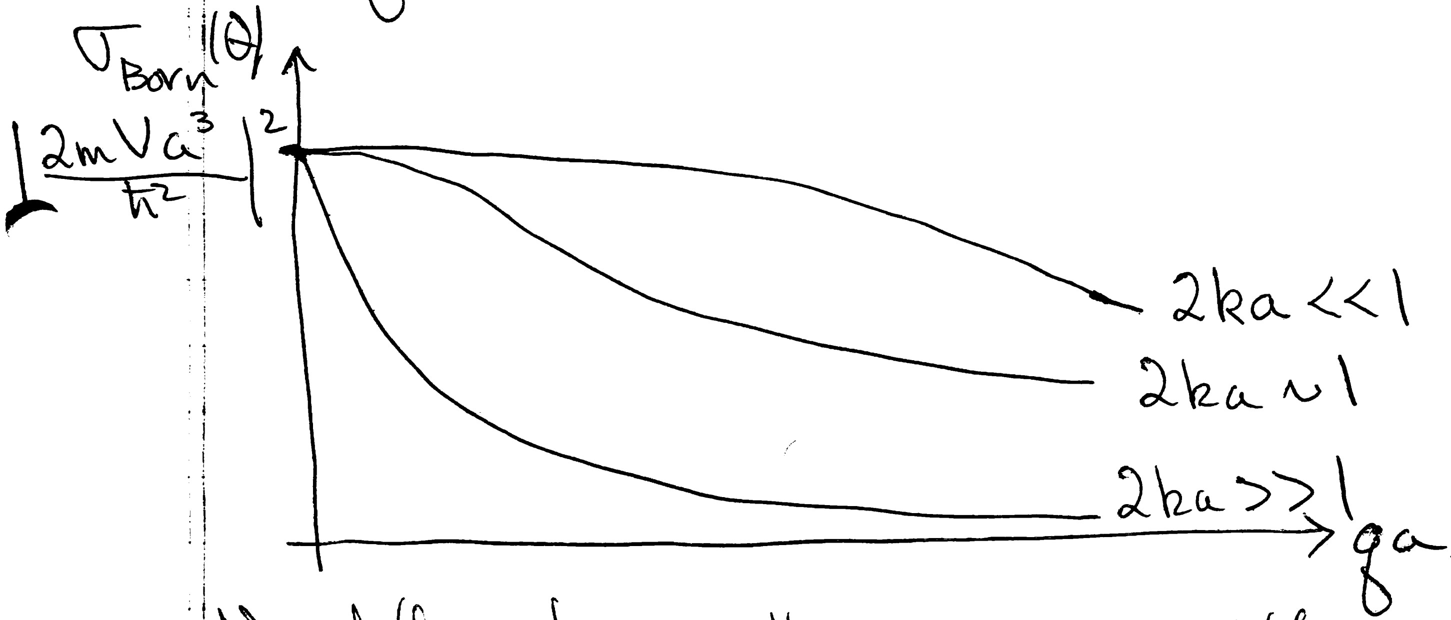
and so the Born approximation differential cross-section is

$$\sigma_{Born}(\theta,\varphi) = |f_{Born}(\vec{q})|^2$$

$$\boxed{\sigma_{Born}(\theta,\varphi) = \left|\frac{Va}{\hbar^2/2ma}\right|^2 \frac{a^2}{(1+(qa)^2)^2}}$$

Again we have that $q^2 = 4k^2 \sin^2\frac{\theta}{2}$ and the C-M energy is $E = \frac{\hbar^2 k^2}{2m}$

So $q^2 = \frac{8mE}{\hbar^2}\sin^2\frac{\theta}{2}$.



No diffraction pattern occurs in this case since the Yukawa potential is a smoothly varying function of r. Also σ_{Born} is not zero for any finite (qa).

The cross-section for Coulomb potential scattering can be obtained by letting $a \to \infty$ but keeping $Va = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}$, fixed then the Yukawa potential goes over to the Coulomb potential

$$V(\vec{r}) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r}.$$

In this limit we obtain the Rutherford scattering formula

$$\boxed{\begin{aligned}\sigma_{Born}(\theta) &= \left|\frac{2m Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2}\right|^2 \frac{1}{q^4} \\ &= \frac{(Z_1 Z_2 e^2 m)^2}{\underbrace{(4\pi\epsilon_0)^2 \hbar^4 \qquad 4k^4}_{16m^2E^2}} \frac{1}{\sin^4\frac{\theta}{2}}\end{aligned}}$$

$$= \left(\frac{Z_1 Z_2 e^2}{(4\pi\epsilon_0) 4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$