7.1.2. The Born Approximation

To determine the scattering amplitudes

$$f^{(+)}(\vec{k},\vec{k}') = -\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}'}\, U(\vec{r}')\, \psi^{(+)}_{\vec{k}}(\vec{r}')$$

and hence the differential cross section

$$\sigma(\theta,\varphi) = |f^{(+)}(\vec{k},\vec{k}')|^2,$$

we must solve the Schrödinger integral equation for $\psi^{(+)}_{\vec{k}}(\vec{r})$ (Lippmann Schwinger equation)

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \psi^{in}_{\vec{k}}(\vec{r}) + \int d^3r'\, G_+(\vec{r},\vec{r}')\, U(\vec{r}')\, \psi^{(+)}_{\vec{k}}(\vec{r}').$$

As usual with such an integral equation we can solve it by iteration. Relabelling $\vec{r}\to\vec{r}'$ and the integration variable $\vec{r}'\to\vec{r}''$ we have

$$\psi^{(+)}_{\vec{k}}(\vec{r}') = \psi^{in}_{\vec{k}}(\vec{r}') + \int d^3r''\, G_+(\vec{r}',\vec{r}'')\, U(\vec{r}'')\, \psi^{(+)}_{\vec{k}}(\vec{r}'')$$

we now substitute this into the RHS of the integral equation

$\Rightarrow$

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \psi^{in}_{\vec{k}}(\vec{r}) + \int d^3r'\, G_+(\vec{r},\vec{r}')\, U(\vec{r}')\, \psi^{in}_{\vec{k}}(\vec{r}')$$

$$+ \int d^3r' \int d^3r''\, G_+(\vec{r},\vec{r}')\, U(\vec{r}')\, G_+(\vec{r}',\vec{r}'') \times$$

$$\times U(\vec{r}'')\, \psi^{(+)}_{\vec{k}}(\vec{r}'') \quad .$$

Then the first 2 terms on the RHS involve known quantities, only the 3rd term still involves $\psi^{(+)}_{\vec{k}}(\vec{r}'')$. Thus we substitute the solution

$$\psi^{(+)}_{\vec{k}}(\vec{r}'') = \psi^{in}_{\vec{k}}(\vec{r}'') + \int d^3r'''\, G_+(\vec{r}'',\vec{r}''') \times$$

$$\times U(\vec{r}''')\, \psi^{(+)}_{\vec{k}}(\vec{r}''')$$

into this term and so on. Thus we obtain the series solution for the wavefunction:

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \psi^{in}_{\vec{k}}(\vec{r}) + \int d^3r_2\, G_+(\vec{r},\vec{r}_2)\, U(\vec{r}_2)\, \psi^{in}_{\vec{k}}(\vec{r}_2)$$

$$+ \int d^3r_2\, d^3r_3\; G_+(\vec{r},\vec{r}_2)\, U(\vec{r}_2)\, G_+(\vec{r}_2,\vec{r}_3) \times$$
$$\times\, U(\vec{r}_3)\, \psi^{in}_{\vec{k}}(\vec{r}_3)$$

$$+ \cdots + \int d^3r_2 \cdots d^3r_n\; G_+(\vec{r},\vec{r}_2)\, U(\vec{r}_2)\, G_+(\vec{r}_2,\vec{r}_3) \times$$
$$\times\, U(\vec{r}_3)\, G_+(\vec{r}_3,\vec{r}_4) \cdots U(\vec{r}_n)\, \psi^{in}_{\vec{k}}(\vec{r}_n)$$

$$+ \cdots .$$

Substituting this into the expression for the scattering amplitude

$$f^{(+)}(\vec{k},\vec{k}') = -\frac{1}{4\pi} \int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\, U(\vec{r})\, \psi^{(+)}_{\vec{k}}(\vec{r})$$

we obtain the <u>Born Expansion</u> for $f^{(+)}(\vec{k},\vec{k}')$:

$$f^{(+)}(\vec{k},\vec{k}') \equiv -\frac{1}{4\pi} \sum_{n=1}^{\infty} f^{(+)}_n(\vec{k},\vec{k}')$$

where

$$\boxed{\begin{aligned} f_n^{(+)}(\vec{k},\vec{k}') = \int d^3r_1 \cdots d^3r_n \, e^{-i\vec{k}'\cdot\vec{r}_1} \times \\ U(\vec{r}_1) G_+(\vec{r}_1,\vec{r}_2) U(\vec{r}_2) G_+(\vec{r}_2,\vec{r}_3) \times \\ \times U(\vec{r}_3) G_+(\vec{r}_3,\vec{r}_4) \cdots U(\vec{r}_n) \, e^{+i\vec{k}\cdot\vec{r}_n} \end{aligned}}$$

where we used $\psi_{\vec{k}}^{in}(\vec{r}) = e^{+i\vec{k}\cdot\vec{r}}$.

In particular the <u>first order Born term</u> (the Born approximation) is simply

$$f_{Born}^{(+)}(\vec{k},\vec{k}') = -\frac{1}{4\pi}\int d^3r_1 \, e^{-i\vec{k}'\cdot\vec{r}_1} U(\vec{r}_1) \, e^{+i\vec{k}\cdot\vec{r}_1}$$

$$= -\frac{1}{4\pi}\int d^3r_1 \, U(\vec{r}_1) \, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}_1}.$$

Defining $\vec{q} = \vec{k}' - \vec{k}$, the <u>momentum</u> <u>transfer</u> we have that $f_{Born}^{(+)}(\vec{k},\vec{k}')$ is a function of the momentum transfer $\vec{q}$ only

$$f^{(+)}_{\text{Born}}(\vec{k},\vec{k}') \equiv f_{\text{Born}}(\vec{q}) = -\frac{1}{4\pi}\int d^3r\, U(\vec{r})\, e^{-i\vec{q}\cdot\vec{r}}.$$

Indeed this is just the Fourier Transform of the potential

$$\boxed{\begin{aligned} & f_{\text{Born}}(\vec{q}) = -\frac{1}{4\pi}\tilde{U}(\vec{q}) \\ & \text{where} \\ & \tilde{U}(\vec{q}) = \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, U(\vec{r}) \\ & \qquad = \frac{2m}{\hbar^2}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r}). \end{aligned}}$$

(Note that

$$\vec{q}^{\,2} = (\vec{k}'-\vec{k})^2 = \vec{k}'^2 + \vec{k}^2 - 2\vec{k}\cdot\vec{k}'$$

but $\vec{k}' = k\frac{\vec{r}}{r} \Rightarrow \vec{k}'^2 = k^2 = \vec{k}^2$

and

$$\vec{k}\cdot\vec{k}' = k^2\,\hat{k}\cdot\hat{r} = k^2\cos\theta$$

So

$$\vec{q}^{\,2} = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$$

So

)

Note that the differential cross-section in the Born approximation is simply related to the Fourier transform of the potential

$$\sigma_{Born}(\theta,\varphi) = |f_{Born}(\vec{q})|^2$$
$$= \frac{1}{(4\pi)^2} |\tilde{U}(\vec{q})|^2 .$$

Hence measuring the cross-section tells us something about the potential in the theory since

$$U(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \tilde{U}(\vec{q})$$
$$= -4\pi \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} f_{Born}(\vec{q}).$$

Indeed this is used to probe the structure of the fundamental interactions of matter (i.e. to determine $V(\vec{r})$) in all branches of physics. Of course the accuracy of the results depends on the validity of the Born approximation for the scattering process and potential under consideration.

Each term in the Born expansion brings in an additional factor of the potential. Thus if the potential is weak we expect such a series solution to $\psi^{(+)}_{\vec{k}}$ to converge. A more accurate criterion for the validity of the Born expansion is obtained from the demand that the scattered wavefunction is small compared to the incoming wavefunction.

$$\left|\psi^{(+)}_{\vec{k}}(\vec{r}) - \psi^{in}_{\vec{k}}(\vec{r})\right| << \left|\psi^{in}_{\vec{k}}(\vec{r})\right| = 1$$

$$\Rightarrow \quad \left|\int d^3r' G_+(\vec{r},\vec{r}')\, U(\vec{r}')\, \psi^{(+)}_{\vec{k}}(\vec{r}')\right| << 1 \, .$$

An estimate of this can be found from the Born approximation $\psi^{(+)}_{\vec{k}\,Born}(\vec{r}') = \psi^{in}_{\vec{k}}(\vec{r}')$ and $G_+(\vec{r},\vec{r}') = \frac{-1}{4\pi} \frac{e^{+ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$ at $\vec{r}=0$

$$\left|\int d^3r' \frac{1}{4\pi} \frac{e^{ikr'}}{r'} U(\vec{r}')\, e^{i\vec{k}\cdot\vec{r}'}\right| << 1 \, .$$

Choosing the $\vec{r}'$ coordinates so that $\vec{k}\cdot\vec{r}' = kr'\cos\theta'$ we have in the case of central potentials $U(\vec{r}) = U(r)$

$$\left|\int d^3r' \frac{1}{4\pi} \frac{e^{ikr'}}{r'} U(r') e^{ikr'\cos\theta'}\right|$$

$$= \left|\int_0^\infty dr' \frac{r' e^{ikr'}}{4\pi} U(r') \int_{-1}^{+1} d(\cos\theta') e^{ikr'\cos\theta'} \int_0^{2\pi} d\varphi\right|$$

$$= \left|\frac{1}{2}\int_0^\infty dr' \frac{e^{ikr'}U(r')}{ik}\left(e^{ikr'} - e^{-ikr'}\right)\right|$$

$$\boxed{= \frac{1}{2k}\left|\int_0^\infty dr' \left(e^{2ikr'} - 1\right) U(r')\right| \ll 1 .}$$

For low energies $k \sim 0$, then $e^{2ikr'} - 1 \approx 2ikr'$ and the inequality becomes

$$\left|\int_0^\infty dr' \, r' U(r')\right| \ll 1 .$$

Thus for Coulomb potentials, $U(r) \sim \frac{1}{r}$, the inequality is violated. The Born approximation for Coulomb scattering at low energy must be dealt with more carefully.

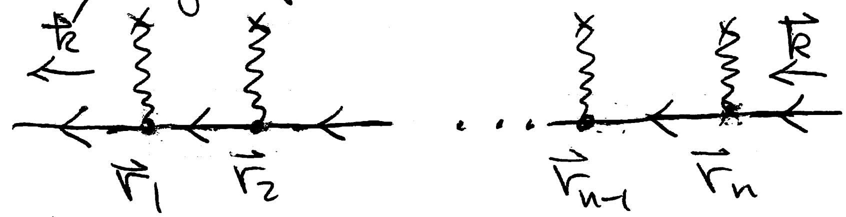
For high energies $k \sim \infty$, the exponential $e^{2ikr'}$ oscillates rapidly and by the Riemann-Lebesgue lemma $\int_0^\infty dr' \, e^{2ikr'} U(r') = 0$. Thus the validity criterion becomes

$$\left| \int_0^\infty dr' \, U(r') \right| << 2k ,$$

Clearly for high energy the inequality holds and the Born approximation is valid, provided the integral converges.

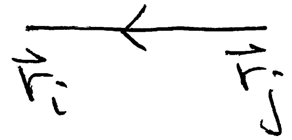
Finally, it is often convenient to represent the terms in the Born expansion by Feynman graphs.

The n^{th} Born term $f_n^{(+)}(\vec{k},\vec{k}')$ is represented by the graph

$f_n^{(+)}(\vec{k},\vec{k}') \longleftrightarrow$ 

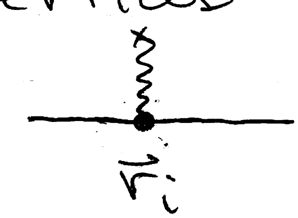
Graphical Elements, lines and vertices, correspond in $f_n^{(+)}$ to (propagators) Green function G_+ and potential $U(\vec{r})$. The incoming wave function is represented by the line with momentum $\vec{k}$, the outgoing wavefunction by the line with momentum $\vec{k}'$

1) Internal Lines



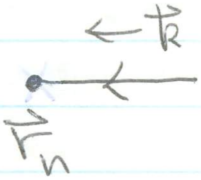
$\longleftrightarrow \quad G_+(\vec{r}_i,\vec{r}_j)$

2) Vertices

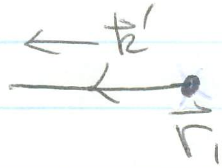


$\longleftrightarrow \quad U(\vec{r}_i)$

3) External Lines

Incoming 

$\longleftrightarrow \; e^{+i\vec{k}\cdot\vec{r}_n}$

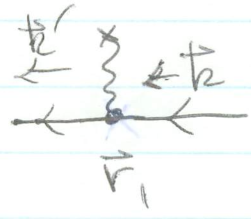
Outgoing 

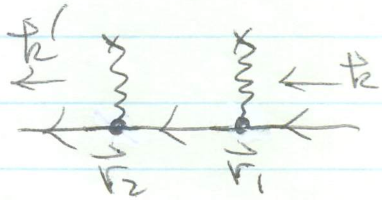
$\longleftrightarrow \; e^{-i\vec{k}'\cdot\vec{r}_1}$

At each vertex, sum over all possible space coordinates

4) Integrate $\int d^3 r_i$ for each vertex.

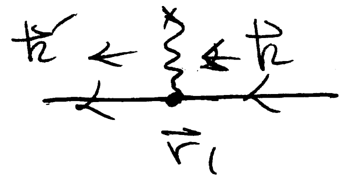
With these rules we can represent the Born expansion as

$-4\pi f^{(+)}(\vec{k},\vec{k}') =$ 

$+$ 

$+ \cdots$

The Born approximation is given by the simplest graph in the expansion.

$$-4\pi f^{(+)}_{Born}(\vec{k},\vec{k}') = $$ 

$$= \int d^3r_1\, e^{-i\vec{k}'\cdot\vec{r}_1}\, U(\vec{r}_1)\, e^{+i\vec{k}\cdot\vec{r}_1}$$

$$= \int d^3r_1\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}_1}\, U(\vec{r}_1)$$

as we found earlier.

7.6.3. Examples of Scattering from Potentials

1) Scattering from a potential barrier

$$V(\vec{r}) = V\,\Theta(a-r)$$

$$V>0, \quad r \geq 0.$$

So $U(\vec{r}) = \frac{2m}{\hbar^2} V\,\Theta(a-r)$ and

the Fourier transform is simply that of the step function