Note that we did not use $2\pi J_1$ to find the flux but $2\pi \sin \theta J_1$, since the cross terms were ignored. This is physically reasonable since we have postulated the incoming wave packet is of finite extent and the detector is considered outside this flux. Also, we have reduced the particle scattering to an (relative) particle being scattered from a potential, we are in the C-M frame.

7.1.1. The Scattering Green Function

We can more rigorously determine the above formula and at the same time relate the asymptotic (large $r$) scattering amplitudes to the short-distance scattering effects of the potential by considering the Green functions for the Schrödinger equation with our scattering boundary conditions. The Schrödinger equation is
\[-10.15\]

\[
\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})
\]

defining \( E = \frac{\hbar^2 k^2}{2m} \geq 0 \) and \( U = \frac{2m}{\hbar^2} V \), this becomes

\[
(\nabla^2 + k^2)\psi(\vec{r}) = U(\vec{r})\psi(\vec{r}),
\]

the Helmholtz equation. The Green function \( G(\vec{r}, \vec{r}') \) for the Helmholtz equation is defined as a solution to

\[
(\nabla^2 + k^2)G(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}'),
\]

\( G(\vec{r}, \vec{r}') \) is a function of \(|\vec{r} - \vec{r}'|\) from this:

Of course, we will have to specify appropriate boundary conditions, or else \( G \) will not be defined uniquely as we know it. The "solution" to the Schrödinger equation can be obtained as an integral equation

\[
\psi(\vec{r}) = \psi_{in}(\vec{r}) + \int d^3r' G(\vec{r}, \vec{r}') U(\vec{r}') \psi(\vec{r}')
\]
where \((\nabla^2 + k^2) \Psi_{\text{in}}^\text{in}(\vec{r}) = 0\) is a solution to the free particle Schrödinger equation, and specifies our incoming particle state, for the plane wave case

\[
\Psi_{\text{in}}^\text{in}(\vec{r}) = C e^{i k \frac{\vec{r} \cdot \vec{r}}{r}}
\]

Note \(k^2 = -\hbar^2\) is fixed, but the direction is undetermined (infinite degree of degeneracy) and corresponds to choosing the incoming flux direction. So \(\Psi_{\text{in}}^\text{in}(\vec{r})\) obeys the Schrödinger equation since

\[
(\nabla^2 + k^2) \Psi_{\text{in}}(\vec{r}) = (\nabla^2 + k^2) \Psi_{\text{in}}^\text{in}(\vec{r}) = 0
\]

\[
\begin{align*}
+ \left\{ \int d^3r' \left[ (\nabla^2 + k^2) G(\vec{r}, \vec{r}') \right] U(\vec{r}') \Psi_{\text{in}}^\text{in}(\vec{r}') \right\} \\
= \delta^3(\vec{r} - \vec{r}')
\end{align*}
\]

\[
= \int d^3r' \delta^3(\vec{r} - \vec{r}') U(\vec{r}') \Psi_{\text{in}}^\text{in}(\vec{r}') = U(\vec{r}) \Psi_{\text{in}}^\text{in}(\vec{r})
\]
To find $G$ recall that
\[
\nabla^2 \frac{1}{|r - r'|} = -4\pi \delta^3(r - r'), \quad \text{while}
\]
\[
\nabla (e^{\pm ik |r - r'|}) = \pm ik e^{\pm ik |r - r'|} \frac{r - r'}{|r - r'|^3}.
\]

Thus
\[
\nabla^2 e^{\pm ik |r - r'|} = -k^2 e^{\pm ik |r - r'|} \pm 2ik e^{\pm ik |r - r'|} \frac{e^{\pm ik |r - r'|}}{|r - r'|}.
\]

Along with $\nabla \frac{1}{|r - r'|} = -\frac{(r - r')}{|r - r'|^3}$ we find
\[
\nabla^2 \frac{e^{\pm ik |r - r'|}}{|r - r'|} = \left( \nabla^2 \frac{1}{|r - r'|} \right) e^{\pm ik |r - r'|}
\]
\[
+ \frac{1}{|r - r'|} \left( \nabla^2 e^{\pm ik |r - r'|} \right)
\]
\[
+ 2 \left( \nabla \frac{1}{|r - r'|} \right) \cdot \left( \nabla e^{\pm ik |r - r'|} \right)
\]
\[
= -4\pi \delta^3(r - r') e^{\pm ik |r - r'|}.
\[-1018-
\begin{align*}
&+ \frac{3}{2} - \frac{k^2}{4\pi} \oint \frac{(r-r')}{|r-r'|^2} \left( \pm ik \frac{(r-r')}{|r-r'|} \right)^2 \times e^{\pm ik |r-r'|} \\
&\Rightarrow (\alpha^2 + k^2) \frac{e^{\pm ik |r-r'|}}{1 - \frac{1}{|r-r'|}} = -4\pi S^3 \delta(r-r')
\end{align*}

Thus we have the **outgoing** $G_{\pm}$ and **incoming** $G_{-}$ Green functions:

\[G_{\pm}(r, r') = -\frac{1}{4\pi} \frac{e^{\pm ik |r-r'|}}{|r-r'|},\]

and correspondingly the outgoing $2^{(+)\frac{1}{2}}$ and incoming $2^{(-)\frac{1}{2}}$ solutions to the **Schrödinger equation**, called **Lippmann-Schwinger equation**:

\[2^{(+)\frac{1}{2}}(r) = 2^{(-)\frac{1}{2}}(r) + \int d^3r' \, G_{\pm}(r, r') \mathcal{M}(r') 2^{(+)\frac{1}{2}}(r').\]

For large distances $r$ with $|r| \gg |r'|$ we have that
\[ |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 - 2r r' \cos \theta + r'^2} = r \left( 1 - \frac{2 r r' \cos \theta}{r^2} + \frac{r'^2}{r^2} \right)^{1/2} \]
\[ = r - \frac{r r'}{r} + O \left( \frac{r'^2}{r^2} \right) \]

Thus,
\[ G_{\pm} \left( \mathbf{r}, \mathbf{r}' \right) \sim - \frac{1}{4 \pi} \frac{e^{\pm i k r}}{r} \]

for asymptotic distance \( r \). Hence the solutions to Schrödinger's equation have the asymptotic form
\[ \varphi_{\pm} (\mathbf{r}) \sim 2 \frac{i}{2} \left( \mathbf{r} \right) - \frac{1}{4 \pi} \frac{e^{\pm i k r}}{r} \int d^3 r'/e \]
\[ \ast U(\mathbf{r}') \varphi_0 (\mathbf{r}) \]

Defining the scattered particle's wave vector \( \mathbf{k}' = \mathbf{k} - \mathbf{k} \) we define the scattering amplitudes
\[ f_{\pm} (\mathbf{k}, \mathbf{k}') \equiv \int d^3 r'/e \ U(\mathbf{r}') \varphi_0 (\mathbf{r}) \]

which are functions of \( \mathbf{k} \) and \( (\theta, \phi) \) independent of \( r \).
Then the asymptotic form of the solution to Schrödinger's equation has the form
\[ \psi_{\pm}(r) \sim \psi_{\pm}^{\infty}(r) + f^{\pm}(k, \theta, \phi) \frac{e^{\pm ikr}}{r} \]

We have that the time dependence of the stationary state is simply
\[ e^{-\frac{i}{\hbar}Et} \]
Thus
\[ \psi_{\pm}(r) \sim \psi_{\pm}^{\infty}(r, \gamma) + \frac{f^{\pm}(k, \theta, \phi)}{r} e^{-\frac{i}{\hbar}(Et+ikr)} \]

Hence \( \psi_{\pm}^{\infty}(r, \gamma) \) contains an outgoing scattered spherical wave while \( \psi_{\pm}(r, \gamma) \) contains an incoming scattered spherical wave.

Correspondingly, \( G_{\pm}(r, \gamma) \) are the outgoing \( +1 \) and incoming \( -1 \) scattering Green functions. Since we are interested in particles scattered...
out to our detector from the scattering center and $2^{(4)}_{1/2}$ and $2^{(4)}_{1/2}$ have the correct boundary conditions for that case. Thus we find

$$f^{(4)}_{1/2}(r) = f^{(4)}_{1/2}(r) \frac{e^{i \varphi}}{r} + \text{other terms}$$

and as earlier, the differential cross section is given as

$$\sigma(\theta, \phi) = |f^{(4)}_{1/2}(k_x, k_z)|^2.$$ 

The scattering amplitudes $f^{(4)}_{1/2}(k_x, k_z)$ are given by

$$f^{(4)}_{1/2}(k_x, k_z) = -\frac{1}{4\pi} \int d^3r' e^{-i \frac{k_x}{2} r'} u(r') \frac{e^{i \varphi}}{r} 2^{(4)}_{1/2}(r')$$

$$\equiv f(\theta, \phi).$$

To determine $f(\theta, \phi)$ we must find the solution to the Schrödinger equation $2^{(4)}_{1/2}(r)$ in the region where the potential $V(r)$ is non-zero. That is, we must solve the integral equation:

\[-021-\]
\[ 2^{(4)}(F) = 2^{(4)}(F) + \int d^3 r' G_+(F, r') U(r') 2^{(4)}(F') \]
\[ = e^{ik_0 r} \]
\[ = 2 \text{ scattered}(F) \]

Of course we will not be able to solve this integral equation exactly and must resort to approximation techniques.

First, let's consider the determination of the Green function by solving the Helmholtz equation by Fourier transform:

\[ G(F, F') = \int \frac{d^3 q}{(2\pi)^3} e^{iq \cdot (F - F')} G(q) \]

So that the Green function equals

\[ (D^2 + k^2) G(F, F') = \delta^3(F - F') \]

becomes

\[ (-q^2 + k^2) G(q) = 1 \]

\[ \Rightarrow G(q) = \frac{-1}{q^2 - k^2} \quad \text{away from} \quad \frac{q^2}{2} = k^2. \]
It still remains to define \( \hat{G}(\vec{q}) \) for \( \frac{q^2}{\epsilon^2} = k^2 \). As will be seen this is equivalent to specifying the coordinate space boundary conditions.

Corresponding to the incoming and outgoing spherical wave boundary conditions, we replace \( k \) by \( k \pm i \epsilon \) where \( \epsilon > 0 \) is a small positive number which we take to zero after the various integrals are performed. Thus we have two Fourier coefficients

\[
\hat{G}_\pm(\vec{q}) = \frac{-1}{\frac{q^2}{\epsilon^2} - (k \pm i \epsilon)^2}
\]

which as we shall see directly give the outgoing and incoming scattering cross-sections \( \sigma_{\pm}(F, F') \).

So

\[
G_\pm(\vec{q}, \vec{q}') = \lim_{\epsilon \to 0^+} \int \frac{d^3q}{(2\pi)^3} \frac{-e^F}{q^2 - (k \pm i \epsilon)^2}
\]

Using spherical polar coordinates for \( \vec{q} \) we leave, with the limit \( \epsilon \to 0^+ \) understood at the end.
\[ G_{\pm}(\epsilon, \pm \epsilon) = -\frac{1}{8\pi^3} \int_0^\infty dq \frac{q^2}{q^2 - (k \pm i\epsilon)^2} \times \]
\[ \times \int_0^{2\pi} d\phi \int_{-1}^{+1} d(cos \theta) e^{i \phi |r - \vec{r}| \cos \theta} \]
\[ = -\frac{1}{4\pi^2} \int_0^\infty dq \frac{q^2}{q^2 - (k \pm i\epsilon)^2} \times \]
\[ \times \frac{1}{i \phi |r - \vec{r}|} (e^{i \phi |r - \vec{r}|} - e^{-i \phi |r - \vec{r}|}) \]
\[ = -\frac{1}{4\pi^2 \epsilon |r - \vec{r}|} \left[ \int_0^\infty dq \frac{qe^{i \phi |r - \vec{r}|}}{q^2 - (k \pm i\epsilon)^2} \right] \]
\[ - \int_0^\infty dq \frac{qe^{-i \phi |r - \vec{r}|}}{q^2 - (k \pm i\epsilon)^2} \]

Let \( q \to -q \) in the second term.
\[ = +\int_0^\infty dq \frac{qe^{i \phi |r - \vec{r}|}}{q^2 - (k \pm i\epsilon)^2} \]
Thus
\[ G_+ (F, \xi) = \frac{-1}{2\pi i} \int_{C} \frac{e^q}{q^2 - (k \pm i, \xi)^2} \, dq \]

As suggested by the form of the integral, we can evaluate it by the methods of contour integration. The integrand has simple poles given by
\[ 0 = q^2 - (k \pm i, \xi)^2 \]

\[ = [q + (k \pm i, \xi)][q - (k \pm i, \xi)] \]

Thus \( G_+ (F) \) has simple poles at
\[ q = k \pm i, \xi \quad \text{and} \quad q = -k \pm i, \xi \]

Consider \( G_+ \) first, \( G_+ (F) \) has the simple poles at \( q = k + i, \xi \) and \( q = -k - i, \xi \).

Extending \( F \) to the complex plane, we note that for \( F \) in the upper half plane, i.e. \( F = F_r + i\xi \), \( \xi > 0 \),
The exponential becomes
\[ e^{\int_{C} f(z) \, dz} = e^{\int_{C} f(z) \, dz} \]
providing an exponential damping of the integrand. Hence we can close the real axis contour in the upper half plane since the integrand vanishes on the semi-circle at infinity.

\[ \text{Thus} \]
\[ C_+ (q, \bar{q}) = -\frac{1}{2\pi i} \oint_{C} \frac{\rho e^{\int_{C} f(z) \, dz}}{[f-(k+\rho i)][f+(k+\rho i)]} \, \rho \, d\rho \]

\( C \) encloses the simple pole at \( q = k+\rho i \) only, hence by Cauchy's residue...
Theorem we find

\[ G_+ (\mathcal{F}, \mathcal{F}') = \frac{-1}{2\pi i \mathcal{F} - \mathcal{F}' i} \frac{(k + i\epsilon) e^{+i k |\mathcal{F} - \mathcal{F}'|}}{2(k + i\epsilon)} \]

which in the \( \epsilon \to 0^+ \) limit is finite and

\[ G_+ (\mathcal{F}, \mathcal{F}') = -\frac{e^{+i k |\mathcal{F} - \mathcal{F}'|}}{4\pi i \mathcal{F} - \mathcal{F}' i} \]

our previous result on page 1018.

Similarly for \( G_- \), the integrand is analytic everywhere in the upper half plane except at the simple pole

\[ \mathcal{F}' = -k + i\epsilon \]. As before \( \epsilon \)
dumps the integrand in the upper half complex \( \mathcal{F} \)-plane so we can close the contour above the real axis as previously with a semi-circle at \( \infty \).
Hence
\[ G_1(F, \tilde{F}) = \frac{-1}{2\pi |F - \tilde{F}|} \oint_C \frac{d\vec{q}}{2\pi i} \frac{e^{iq} - iq|F - \tilde{F}|}{[q + (k + i\epsilon)]^2[q - (k + i\epsilon)]} \]

Performing the integral by residues
\[ G_1(F, \tilde{F}) = \frac{-1}{2\pi |F - \tilde{F}|} \frac{(-k + i\epsilon) e^{-ik|F - \tilde{F}|}}{2(-k + i\epsilon)} \]

Taking the well-defined \( \epsilon \to 0^+ \) limit, we obtain
\[ G_1(F, \tilde{F}) = \frac{-1}{4\pi} \frac{e^{-ik|F - \tilde{F}|}}{|F - \tilde{F}|} \] again in agreement with previous result on page - 1018.