Thus we find

$$\frac{\sin\theta_{cm}}{\sin\theta_{lab}}\left|\frac{d\theta_{cm}}{d\theta_{lab}}\right| = \frac{(1+2\beta\cos\theta_{cm}+\beta^2)^{3/2}}{|1+\beta\cos\theta_{cm}|}.$$

Therefore the cross sections in the lab and CM frames are related by

$$\boxed{\sigma_{lab}(\theta_{lab},\varphi_{lab}) = \sigma_{CM}(\theta_{cm},\varphi_{cm})\frac{(1+2\beta\cos\theta_{cm}+\beta^2)^{3/2}}{|1+\beta\cos\theta_{cm}|}}.$$

Theoretically it is usually simpler to work in the C-M frame.

It now becomes our task to relate the wavefunctions of the incoming and outgoing states to the cross section.

7.1. Potential Scattering

We begin by considering two interacting particles described by the Hamiltonian

$$H = +\frac{1}{2m_1}\vec{P}_1^{\,2} + \frac{1}{2m_2}\vec{P}_2^{\,2} + V(\vec{r}_1 - \vec{r}_2) .$$

As usual we change to the CM and relative coordinates & momenta

$$\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$ (page -172-)

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2$$

$$\vec{p} = \frac{m_2}{m_1+m_2}\vec{P}_1 - \frac{m_1}{m_1+m_2}\vec{P}_2 .$$

The commutator's become

$$[P_i, X_j] = -i\hbar\delta_{ij} \text{ and}$$

$$[p_i, x_j] = -i\hbar\delta_{ij} .$$

So the Hamiltonian reduces to

$$H = \frac{1}{2M}\vec{P}^{\,2} + \frac{1}{2m}\vec{p}^{\,2} + V(\vec{r})$$

with $M = m_1 + m_2$ the total mass and

$$m = \frac{m_1 m_2}{m_1+m_2}$$, the reduced mass.

The time evolution of the system is easily found by expressing the wavefunction in terms of the stationary states of the Hamiltonian. Thus Schrödinger's equation with the CM motion separated out, reduces to

$$\left(\frac{1}{2m}\vec{p}^{\,2} + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})$$

(where $\Psi(\vec{r},\vec{R}) = \phi(\vec{R})\psi(\vec{r})$ and the CM moves as a free particle of momentum $\hbar\vec{K}$

$$\phi(\vec{R}) = e^{i\vec{K}\cdot\vec{R}}\,)$$

Since we are interested in the relative motion we ignore the CM free motion and re-label

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$ as the (relative) Hamiltonian.

Here $E > 0$ is the (specified) energy of the relative motion.

As we know there are infinite solutions to such a differential equation, we must specify the boundary conditions appropriate to the physical situation (experiment) being described in order to pick out the relevant solution. We will consider potentials that fall off rapidly at large distance r. Thus for large distance the Schrödinger equation will be a free equation

$$\left(\frac{\vec{p}^{\,2}}{2m} - E\right)\psi(\vec{r}) \sim 0 \quad \text{as } r \to \infty .$$

Included in $\psi(\vec{r})$ must be the incident free particle flux J_{in} as well as the radially outgoing scattered flux J_{out}. Thus in general

$$\psi(\vec{r}) = \psi_{in}(\vec{r}) + \psi_{scattered}(\vec{r})$$

$\psi_{in}(\vec{r})$ describes the incoming flux of (non-interacting) particles and $\psi_{scatt.}(\vec{r})$ the scattered particles. Clearly the incoming particles have a flux of finite extant and are described by some wave packet with definite energy E. As we know dealing with wave packets is quite cumbersome so we will idealize the incoming state to be described by a plane wave. If any mathematical or physical ambiguities arise, we can be more careful and resort to wavepackets. Thus

$$\psi_{in}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$$

where $E = \frac{\hbar^2 \vec{k}^2}{2m}$ the incoming energy, which is constant, of the particle. Again the potential will be considered as finite in extent; thus asymptotically $\psi(\vec{r})$ is a solution to the free equation

$$-\frac{\hbar^2}{2m}\nabla^2 \psi(\vec{r}) = E\psi(\vec{r})$$

Since the incoming particle plane wave has energy E so does the scattered wave. Also the outgoing wave must have a flux that drops off like $\frac{1}{r^2}$ since the # of particles through a sphere should be constant. Thus

$$\psi_{scatt}(\vec{r}) \sim f(\theta,\phi)\frac{e^{ikr}}{r} \quad \text{as } r \to \infty$$

where the angular dependence of this outgoing spherical wave is contained in $f(\theta,\varphi)$. Such a spherical wave is a solution of the free Schrödinger equation with $E>0$ for large r (page -216- to -221-). Thus the asymptotic boundary condition for scattering is

$$\psi(\vec{r}) \sim e^{i\vec{k}_0\cdot\vec{r}} + \frac{f(\theta,\varphi)e^{ikr}}{r}$$

as $r \to \infty$. $\underline{f(\theta,\varphi)}$ is called the <u>scattering amplitude</u>.

Recall for bound states we required $\psi(\vec{r})$ to be normalizable, which led to its vanishing at large r. This could only be satisfied for specific discrete energy eigenvalues, which, for asymptotically vanishing $V(\vec{r})$ were negative. For the scattering problem E takes a continuum of positive values and the scattered wave goes like $\frac{e^{ikr}}{r}$ for large r, which is <u>not</u> normalizable.

This does not produce a problem

since the cross-section is defined as a ratio of fluxes.

Indeed the flux of particles is given by the probability current density $\vec{J} = \frac{\hbar}{2im}\left[\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi)^*\psi\right]$.

For the incident flux of particles we have the plane wave probability current density ($\psi_{in} = e^{i\vec{k}\cdot\vec{r}}$)

$$\vec{J}_{in} = \frac{\hbar}{2im}\left[\psi_{in}^* \vec{\nabla}\psi_{in} - (\vec{\nabla}\psi_{in}^*)\psi_{in}\right]$$

$$= \frac{\hbar\vec{k}}{m}$$

a uniform flux.

For the outgoing flux we have that

$$\psi_{scatt} \sim \frac{f(\theta,\varphi)\, e^{ikr}}{r}, \quad so$$

that the radial component of

The probability current for the scattered wave

$$\left(\vec{J}_{out}\right)_r = \frac{\hbar}{2im}\left(\psi^*_{out}\frac{\partial}{\partial r}\psi_{out} - \left(\frac{\partial}{\partial r}\psi^*_{out}\right)\psi_{out}\right)$$

$$= \frac{\hbar k}{m}\frac{|f(\theta,\varphi)|^2}{r^2} \quad \text{for large } r.$$

Thus the differential cross section is given by

$$\sigma(\theta,\varphi) = \frac{r^2\left(\vec{J}_{out}\right)_r}{|\vec{J}_{in}|}$$

$$\boxed{\sigma(\theta,\varphi) = |f(\theta,\varphi)|^2}$$

Thus we only need to determine the asymptotic form of the scattered wavefunction to find the cross-section. Since we have ignored any internal degrees of freedom of the particles here, this is elastic scattering. The above can be easily modified for inelastic scattering as we will see.

Note that we did not use $\psi(\vec{r})$ to find the flux but ψ_{in} and ψ_{scatt}. The cross terms were ignored. This is physically reasonable since we have that the incoming wave packet is of finite extent and the scattering detector is considered outside this flux. Also we have reduced the 2 particle scattering to one (relative) particle being scattered from a potential, we are in the C-M frame.

7.1.1. The Scattering Green Function

We can more rigorously determine the above formula and at the same time relate the asymptotic (large r) scattering amplitudes $f(\theta,\varphi)$ to the short distance scattering effects of the potential by considering the Green function for the Schrödinger equation with our scattering boundary conditions. The Schrödinger equation is