VII. Approximation Methods for Continuous Energy Eigenvalues: Potential (elastic) Scattering Theory

In the previous chapter we concerned ourselves with determining the discrete energy eigenvalues of a time independent Hamiltonian. These bound state energies were taken, conventionally to be negative. In general the Hamiltonian also has a positive energy or continuous portion to its spectrum. These values do not correspond to bound states but are referred to as scattering states. Since any value of E is in the spectrum it is not a question of determining it, but rather we choose the energy E in such a situation it usually corresponds to the initial energy of some incoming (projectile) particle. This particle will, when it approaches closely enough, interact with a target particle (or alternately stated, the particles will collide) and go off at some direction. It then is our aim to predict the probability that the particle will, after interacting, be located at some point in space.

This probability of course is related to the scattering state energy eigenfunction. Hence scattering theory will concern itself with predicting the scattering probability through the determination of the eigenfunction for a given (continuum) energy eigenvalue $E > 0$.

In general there are 4 types of collision processes

1) Elastic scattering: Particles A and B collide and keep their same energies relative to the center of mass (and do not alter their internal states). This is represented by $A + B \rightarrow A + B$.

2) Inelastic scattering: Particles A and B change their internal states. That is kinetic energy of A and B can be absorbed by either A or B during the collision to alter their internal state. This is represented by
$A + B \rightarrow A' + B'$.

For example an electron can scatter off a hydrogen atom and leave the hydrogen atom in the same state it was initially, say the ground state. This is elastic scattering. Or it could leave the hydrogen atom in some excited state while changing its energy. This is inelastic scattering.

3) Rearrangement collisions: Particles A and B are exchanged for other particles upon interaction. The outgoing particles may be of a different type than the incoming particles. This often occurs in chemical and nuclear reactions. It is represented by $A+B \rightarrow C+D$.

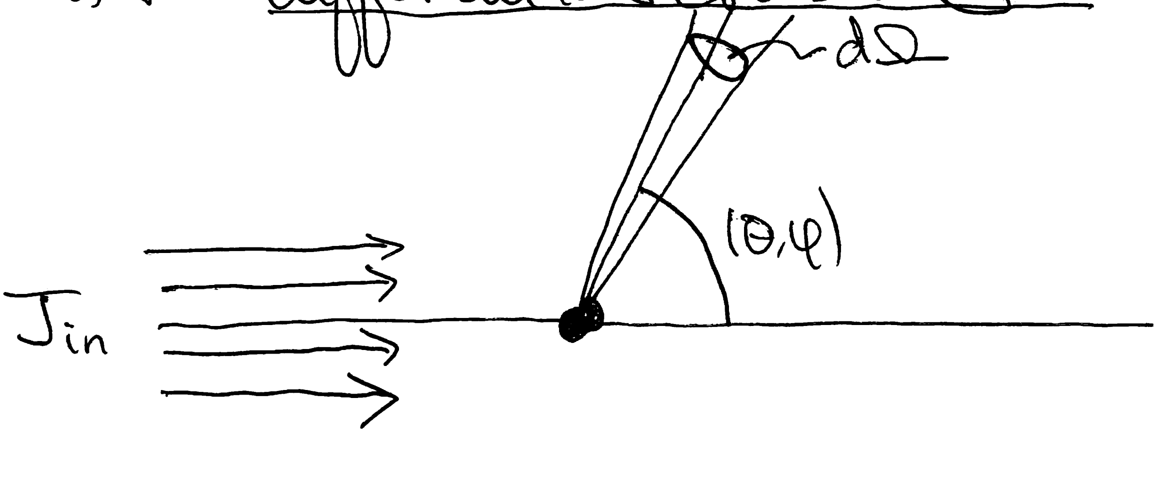
4) Particle Production: The collision of 2 particles produces 3 or more outgoing particles. This is common in elementary particle physics. It is represented by $A+B \rightarrow C+D+E+\dots$.

In these collisions, each final set of particles possible is called a different <u>channel</u>. Scattering theory will also predict the probabilities for each type of channel.

In our case we will deal only with elastic and inelastic scattering. Indeed in a scattering process the angular distribution of the scattered particles is described in terms of a <u>differential cross section</u>, $\sigma(\theta,\varphi)$. Initially we have a flux J_{in} of incoming particles per unit area per unit time incident on the target (or for collision). The number of particles per unit time scattered into a solid angle $d\Omega$ centered about spherical polar angles (θ,φ) is proportional to the incident flux J_{in} and the angular opening $d\Omega$,

$$dn(\theta,\varphi) = \sigma(\theta,\varphi)\, J_{in}\, d\Omega .$$

The constant of proportionality $\sigma(\theta,\varphi)$ is the <u>differential cross section</u>.



Suppose the particle detector is located in the direction (θ,φ) at a distance r from the target and well outside the incident beam of particles. If the detector subtends solid angle $d\Omega$ it will receive $dn = J_{in}\,\sigma\,d\Omega$ scattered particles per unit time. The area of the detector is $(r^2 d\Omega)$ then we have a flux of outgoing scattered particles at the detector of

$$J_{out} = dn/r^2 d\Omega$$

$$= J_{in}\,\sigma(\theta,\varphi)\frac{d\Omega}{r^2 d\Omega}.$$

Hence the differential cross section is also given by

$$\sigma(\theta,\varphi) = \frac{r^2 J_{out}}{J_{in}}.$$

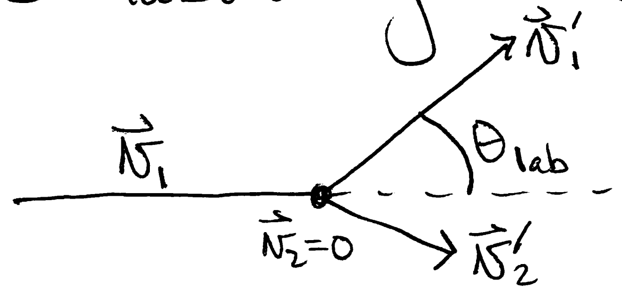
$\sigma(\theta,\varphi)$ has units of area. We also have that J_{out} will decrease like $\frac{1}{r^2}$ so that $\sigma(\theta,\varphi)$ is actually independent of r. By summing over all angular directions we obtain the <u>total cross section</u>

$$\sigma = \int_{4\pi} d\Omega\, \sigma(\theta,\varphi)$$

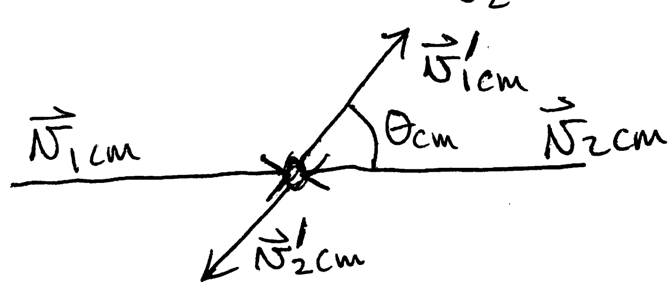
$$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\, \sigma(\theta,\varphi) .$$

Note that the cross section $\sigma(\theta,\varphi)$ is given in terms of a ratio of fluxes, quantities that are easily measured as well as calculated. Of course this could be classical or quantum mechanics, we have not specified any dynamics as yet.

Finally we have been treating the target particle as if it were stationary, but in fact particles often collide with each other with equal momentum. The two most common frames in which to work then are the "laboratory" and "CM" frames.



lab frame



CM frame

In the lab frame the incoming particle has mass m_1 velocity $\vec{v}_1$ while the target particle of mass m_2 is at rest. The velocity of the center of mass is then

$$\vec{V} = \left(\frac{m_1}{m_1+m_2}\right)\vec{v}_1 .$$

In the CM frame, the center of mass is at rest. Thus we subtract $\vec{V}$ from all velocities. So the incoming particle 1 now has velocity

$$\vec{v}_{1cm} = \vec{v}_1 - \vec{V} = \left(\frac{m_2}{m_1+m_2}\right)\vec{v}_1$$

while the target particle 2 now has velocity

$$\vec{v}_{2cm} = -\left(\frac{m_1}{m_1+m_2}\right)\vec{v}_1 \quad (= -\vec{V}) .$$

Now the final velocity of particle 1 in the 2 frames is related by

$$v_1' \cos\theta_{lab} = v_{1cm}' \cos\theta_{cm} + V$$

$$v_1' \sin\theta_{lab} = v_{1cm}' \sin\theta_{cm} .$$

Taking the ratio, we have

$$\tan\theta_{lab} = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \frac{V}{v'_{1cm}}} \quad \text{(general)}$$

For <u>elastic</u> <u>scattering</u> the energies of the particles and therefore their speeds, relative to the CM, are unchanged. Thus

$$v_{1cm} \equiv v'_{1cm} = \frac{m_2}{m_1+m_2} v_1 \quad \text{(elastic)}$$

hence
$$\frac{V_0}{v'_{1cm}} = \frac{m_1}{m_2} \quad \text{and}$$

$$\boxed{\tan\theta_{lab} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + \frac{m_1}{m_2}} \quad \text{(elastic)}}$$

So, as usual, there is no need to calculate the differential cross section in both frames, we usually calculate in the CM frame and then relate it to lab frame cross section by a transformation. The relation is determined from the fact that the number of particles scattered into a particular angular cone is the same in either frame. Also the incident flux (# of particles per unit area per unit time) is the same in both frames, thus

$$\sigma_{lab}(\theta_{lab},\varphi_{lab})\,d\Omega_{lab} = \sigma_{CM}(\theta_{CM},\varphi_{CM})\,d\Omega_{CM}$$

with $\varphi_{lab} = \varphi_{CM}$ and $\tan\theta_{lab} = \dfrac{\sin\theta_{cm}}{\cos\theta_{cm}+\beta}$

$\left(\beta \equiv \dfrac{V}{v_{1cm}}\right)$. Thus

$$\sigma_{lab}(\theta_{lab},\varphi_{lab}) = \sigma_{CM}(\theta_{cm},\varphi_{cm})\,\frac{\sin\theta_{cm}}{\sin\theta_{lab}}\left|\frac{d\theta_{cm}}{d\theta_{lab}}\right|$$

Now from $\tan\theta_{lab} = \dfrac{\sin\theta_{cm}}{\cos\theta_{cm}+\beta}$, we have taking the differential

$$(1+\tan^2\theta_{lab})\,d\theta_{lab} = \left[\frac{\cos\theta_{cm}}{\cos\theta_{cm}+\beta} + \frac{\sin^2\theta_{cm}}{(\cos\theta_{cm}+\beta)^2}\right] \times d\theta_{cm}$$

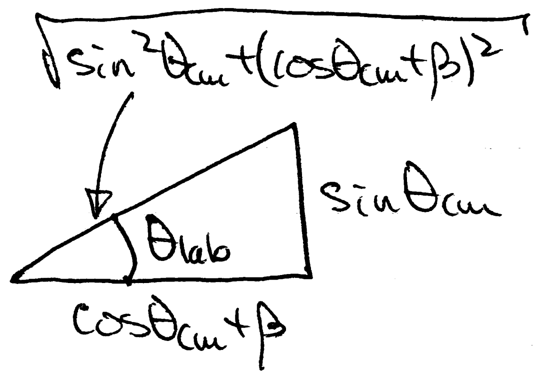
So

$$\frac{d\theta_{cm}}{d\theta_{lab}} = \frac{(\cos\theta_{cm}+\beta)^2\left[1+\tan^2\theta_{lab}\right]}{\left(\cos\theta_{cm}(\cos\theta_{cm}+\beta)+\sin^2\theta_{cm}\right)}$$

$$= \frac{(\cos\theta_{cm}+\beta)^2\left[\frac{(\cos\theta_{cm}+\beta)^2+\sin^2\theta_{cm}}{(\cos\theta_{cm}+\beta)^2}\right]}{(1+\beta\cos\theta_{cm})}$$

$$\frac{d\theta_{cm}}{d\theta_{lab}} = \frac{(1+2\beta\cos\theta_{cm}+\beta^2)}{(1+\beta\cos\theta_{cm})}$$

Further



so

$$\sin\theta_{lab} = \frac{\sin\theta_{cm}}{\sqrt{\sin^2\theta_{cm}+(\cos\theta_{cm}+\beta)^2}}$$

$$\Rightarrow \frac{\sin\theta_{cm}}{\sin\theta_{lab}} = \sqrt{1+2\beta\cos\theta_{cm}+\beta^2}$$

Thus we find

$$\frac{\sin\theta_{cm}}{\sin\theta_{lab}}\left|\frac{d\theta_{cm}}{d\theta_{lab}}\right| = \frac{(1+2\beta\cos\theta_{cm}+\beta^2)^{3/2}}{|1+\beta\cos\theta_{cm}|}.$$

Therefore the cross sections in the lab and CM frames are related by

$$\boxed{\sigma_{lab}(\theta_{lab},\varphi_{lab}) = \sigma_{CM}(\theta_{cm},\varphi_{cm})\frac{(1+2\beta\cos\theta_{cm}+\beta^2)^{3/2}}{|1+\beta\cos\theta_{cm}|}}.$$

Theoretically it is usually simpler to work in the C-M frame.

It now becomes our task to relate the wavefunctions of the incoming and outgoing states to the cross section.

7.1. Potential Scattering

We begin by considering two interacting particles described by the Hamiltonian