mussy near or at degenerate energy levels, the B-W scheme will handle these cases more simply.

6.2. Brillouin-Wigner Stationary State Perturbation Theory

As we saw above, the unperturbed energy differences in denominators led to a breakdown of the non-degenerate R-S perturbation theory at or close to degenerate energy levels. In general then we would like to avoid such denominators. At the same time we would like to develop a general expression for the eigenvalue equation applicable even in the degenerate case. Recall in that case we must determine which vectors \( |\phi_n\rangle \) go into at \( \alpha = 0 \).

Thus we can first consider an expansion not in \( \frac{1}{E_0 - E_n} \), but in \( \frac{1}{E_n - E_\alpha} \), which does not blow up at degenerate values, since we assume \( E_n \neq E_\alpha \).
So as usual we begin with the Schrödinger equation
\[ H |\phi_n\rangle = (H_0 + H') |\phi_n\rangle = E_n |\phi_n\rangle. \]

Rather than expanding \( E_n = E_0 + \ldots \) as before we write this as
\[ (E_n - H_0) |\phi_n\rangle = H' |\phi_n\rangle \]
which yields the equation
\[ |\phi_n\rangle = (E_n - H_0)^{-1} H' |\phi_n\rangle \]

Since \( H_0 \) has eigenvalues \( E_n \),
\[
\frac{1}{E_n - H_0}
\]
is well behaved. In order to find a more explicit equation for \( E_n \) as well as \( |\phi_n\rangle \) we project this equation onto the various subspaces of the unperturbed Hamiltonian. In particular we can expand \( |\phi_n\rangle \) in terms of the \( H_0 \) eigen-basis.
\[ |\Psi_n> = \sum_k C_{nk} |\psi_n,k> + \sum_i \sum_{M+N} C_{mi} |\psi_{M+N}>. \]

In the degenerate case we must find an eigenvalue equation for the \( C_{nk} \) at \( \lambda = 0 \).

(Note: \( C_{nk} = C_{nk}(\lambda) \) here, as \( \lambda \to 0 \) the \( C_{nk} \to 0 \), \( m+n \), while the \( C_{nk} \) go to a particular set \( C_{nk} \to 2^{1/n}k \), at \( |\Psi_n> \to |\Psi_{n(0)}> \). For \( \lambda \to 0 \) we denote the component of \( |\Psi_n> \) in the degenerate subspace by \( |\Psi_{n(0)}> \).

\[ |\Psi_{n(0)}> = \lim_{\lambda \to 0} |\Psi_{n(0)}> \]

We can introduce the projector \( P_n \) onto the \( \Psi_n \)-degenerate subspace, call it \( \Psi_{n(0)} \), of eigensubspace \( E_n \).

\[ P_n = \sum_k |\psi_{n,k}><\psi_{n,k}|. \]
So since the \( \{ \psi_{n,l} \} \) are orthonormal we have
\[
P_n P_n = P_n
\]
\[
P_m P_n = 0 \quad \text{if} \quad m \neq n
\]
and summing over all \( n \) we recover completeness
\[
1 = \sum_n P_n.
\]
The projector onto the space orthogonal to \( \mathbb{H}_n \) is the complement of \( \mathbb{H}_n \),
\[
Q_n = 1 - P_n
\]
\[
= \sum_l \sum_{m \neq n} | \psi_{n,l} \rangle \langle \psi_{n,l} |
\]
Note \( Q_n Q_n = Q_n \), \( Q_n P_n = 0 \). By ad
\[
P_n = 1 - Q_n
\]
\[
= 1 - \sum_l \sum_{m \neq n} \langle \psi_{n,l} | \psi_{n,l} \rangle X \langle \psi_{n,l} | P_n | \psi_{n,l} \rangle.
\]

The projection of \( | A_n \rangle \) onto \( \mathbb{H}_n \) is
\[
| A_n \rangle \equiv P_n | A_n \rangle
\]
and as $\lambda \to 0$ we have $\mathbf{1}_{\mathbf{n}} \to \mathbf{1}_{\mathbf{n}}^\dagger \mathbf{K} = \sum_{k=1}^{g_n} \mathbf{q}_{n,k} \mathbf{p}_{n,k}$ as a particular vector in $\mathbf{H}_n$.

Thus in our Schrödinger equation, we would like to separate out this projection of $\mathbf{1}_{\mathbf{n}}$ onto $\mathbf{H}_n$ and perturb about it.

$$\mathbf{1}_{\mathbf{n}} = P_n \mathbf{1}_{\mathbf{n}} + \left(1 - P_n\right) \mathbf{1}_{\mathbf{n}}$$

$$= \mathbf{1}_{\mathbf{n}}^\dagger + \left(1 - P_n\right) \mathbf{1}_{\mathbf{n}}$$

but the Schrödinger equation is

$$\mathbf{1}_{\mathbf{n}} = \left(\mathbf{E}_n - \mathbf{H}_o\right)^{-1} \mathbf{H}' \mathbf{1}_{\mathbf{n}}$$

Substituting into the second term on the RHS\Rightarrow

$$\mathbf{1}_{\mathbf{n}} = \mathbf{1}_{\mathbf{n}}^\dagger + \left(1 - P_n\right) \left(\mathbf{E}_n - \mathbf{H}_o\right)^{-1} \mathbf{H}' \mathbf{1}_{\mathbf{n}}$$

Now we note that

$$P_n \mathbf{H}_o = \sum_{k=1}^{g_n} \mathbf{q}_{n,k} \mathbf{p}_{n,k} \mathbf{H}_o \mathbf{q}_{n,k}$$

$$= \mathbf{E}_n \sum_{k=1}^{g_n} \mathbf{q}_{n,k} \mathbf{p}_{n,k}$$

$$= \mathbf{E}_n \mathbf{q}_{n,k}$$
So \[ P_n H_0 = H_0 P_n = E_n^0 P_n \]

Thus \[
(1-P_n)(E_n - H_0)^{-1} = (E_n - H_0)^{-1}(1-P_n)
\]
\[
= (E_n - H_0)^{-1} \sum_{m \pm n} \sum_{l \pm c} \frac{|\psi_{m,l} \rangle \langle \psi_{m,l}|}{E_n - E_m^0}
\]
\[
= (1-P_n)(E_n - H_0)^{-1}
\]

where we used \( H_0 |\psi_{m,l} \rangle = E_n^0 |\psi_{m,l} \rangle \).

Since \( E_n \neq E_m^0 \) by assumption, this is a well defined sum. So the Schrödinger equation in the form above is well defined (naturally). So we need next to extract more explicitly 3 things: 1) an expression for \( E_n \), 2) an equation determining \( |\psi_{m,l} \rangle \) and therefore 2) an equation...
Determining the component of \( |2n\rangle \) to \( \theta_n \).

Considering the last first, the Schrödinger equation:

\[
|2n\rangle = |2n\rangle'' + (1 - \Phi_n) (E_n - H_0) |H' 2n\rangle
\]

\[
= |2n\rangle'' + R_n H' |2n\rangle
\]

with

\[
R_n = (1 - \Phi_n) (E_n - H_0)^{-1}
\]

\[
= \sum_{m=0}^{\infty} \frac{\varphi_m \times \varphi_n}{E_n - E_m}
\]

Can be solved recursively or more formally we can re-write this as:

\[
[1 - R_n H'] |2n\rangle = |2n\rangle''
\]

\[
\Rightarrow \quad |2n\rangle = \left[1 - R_n H' \right]^{-1} |2n\rangle''
\]

\[
= |2n\rangle'' + R_n H' |2n\rangle'' + (R_n H')^2 |2n\rangle'' + \cdots
\]
This is the Brillouin–Wigner Perturbation
expansion for \[ |\alpha^+_n\rangle \]

Note: \[ |\alpha^+_n\rangle = |\alpha^+_n\rangle + |\alpha^+_n\rangle \quad \text{with} \]
\[ |\alpha^+_n\rangle = P_n |\alpha^+_n\rangle \quad \text{and} \quad |\alpha^+_n\rangle = Q_n |\alpha^+_n\rangle \]

Since \( R_n = Q_n (E_n - H_0)^{-1} \) \( R_n \) \text{ have that the above equation gives } \[ |\alpha^+_n\rangle \]

\[ |\alpha^+_n\rangle = Q_n |\alpha^+_n\rangle \]
\[ = \left[ R_n H' + (R_n H')^2 + \ldots \right] |\alpha^+_n\rangle \]
\[ = \frac{R_n H'}{1 - R_n H'} |\alpha^+_n\rangle \]

Secondly we can find an eigenmode

the Schrödinger equation into \[ H' |\alpha^+_n\rangle = (E_n - H_0) |\alpha^+_n\rangle \]

Now operate with \( P_n \)
\[ P_n H' |\alpha^+_n\rangle = P_n (E_n - H_0) |\alpha^+_n\rangle \]
\[ = (E_n - H_0) P_n |\alpha^+_n\rangle \]
\[ \mathcal{P}_n H' \mathcal{P}_n - 1^n > = (E_n - E_{n_0}) \mathcal{P}_n - 1^n > = (E_n - E_{n_0}) 1^n > \]

where we used \( \text{HoP}_n = E_n P_n \).

Relating \( 1^n > \) on the LHS to \( 1^n'' > \) by the equation for \( 1^n > \) :

\[ 1^n > = \sum (1 - R_n H')^{-1} 1^n'' > \]

we have

\[ \left( \mathcal{P}_n H' \left[ 1 - R_n H' \right]^{-1} \right) 1^n'' > = (E_n - E_{n_0}) 1^n'' > \]

Thus we have an eigenvalue equation for the energy shifts

\( (E_n - E_{n_0}) \) and the eigenstates \( 1^n'' > \).

A more direct formula for the energy levels can be obtained by simply projecting the Schrödinger equation onto \( 1^n'' > \) as above.
\[ E_n |2n''\rangle = E_n^0 |2n''\rangle + P_n H |2n\rangle \]

Taking the inner product with any vector \( \langle a_n^0 | H' | 2n\rangle \),

\[
E_n = E_n^0 + \frac{\langle a_n^0 | H' | 2n\rangle}{\langle a_n^0 | 2n\rangle} \]

with \( |a_n^0\rangle \in H_n^0 \) so \( P_n |a_n^0\rangle = |a_n^0\rangle \).

Substituting the expansion for \( P_n \) we have

\[
E_n = E_n^0 + \frac{\langle a_n | H' | 2n''\rangle}{\langle a_n | 2n''\rangle} + \frac{\langle a_n | H' | R_n H' | 2n''\rangle}{\langle a_n | 2n''\rangle} + \cdots
\]

Using the formula for \( R_n \) (page 887)

\[
E_n = E_n^0 + \frac{\langle a_n | H' | 2n''\rangle}{\langle a_n | 2n''\rangle}
+ \sum_l \frac{1}{r_l} \sum_{m,n \leq l} \frac{\langle a_n | H' | \phi_{m,n} \rangle \langle \phi_{m,n} | H' | 2n''\rangle}{\langle a_n | 2n''\rangle \left( E_n - E_n^0 \right)}
\]
Choosing $|\alpha_n\rangle = |\gamma_n\rangle$ and normalizing $\langle\gamma_n | \gamma_n\rangle = 1$, this looks simpler.

$$E_n = E_n^0 + \langle\gamma_n | H' | \gamma_n\rangle$$

$$+ i \int_{\mu \neq n} \frac{g_{\mu n} \langle\gamma_n | H' | \gamma_{\mu}\rangle^2}{(E_n - E_n^0)^2} + \cdots$$

Since it is $(E_n - E_n^0)$ that appears in this formula, there are no difficulties even for degenerate $E_n^0$ eigenvalues. Of course the appearance of $E_n$ on the RHS defines $E_n$ only implicitly. The utility of the formula occurs when evaluating the equations numerically. One starts with an initial value for $E_n$ on the LHS and determines the $E_n$ on the RHS. This process can be continued until the two values converge.
Likewise the equation for $|2n\rangle$:

$$|2n\rangle = |1\rangle - R_n H |12n\rangle$$

is exact - of course we can only solve it approximately. Namely, expanding the inverse to obtain

$$|2n\rangle = |2n\rangle'' + R_n H |2n\rangle'' + (R_n H)^2 |2n\rangle'' + \ldots$$

$$= |2n\rangle'' + \sum_{m \neq n} \sum_{l=1}^{g_m} \frac{\langle \psi_m | H | 2n\rangle''}{E_n - E_m} |\psi_m\rangle''$$

$$+ \sum_{m \neq n} \sum_{l=1}^{g_m} \sum_{l'=1}^{g_m'} \frac{\langle \psi_m | H | \psi_{m'}\rangle' \langle \psi_{m'} | H | 2n\rangle''}{(E_n - E_m)(E_n - E_{m'})} \times |\psi_{m'}\rangle'$$

$$+ \ldots$$

Again it is $(E_n - E_m)$ that occurs in the denominators providing applicability for even degenerate $E_n$ eigenvalues.

For non-degenerate $E_n$ we have that

$$|2n\rangle'' = |\psi_n\rangle''$$

and

$$E_n = E_n^0 + \frac{\langle \psi_n | H | \psi_n \rangle'}{1 - R_n H |\psi_n\rangle}$$
\[ E_n = E_0 + \langle \phi_n | H' | \phi_n \rangle \]
\[ + \sum_i \sum_{l=1}^{q_m} \left( \frac{K_{\phi_{i\alpha} | H' | \phi_{i\alpha}}}{E_n - E_0} \right)^2 + \ldots \]

The R-S expansion can be obtained if we further expand \( E_n = E_0 + \lambda \varepsilon^{(1)}_n + \cdots \).

So

\[ \lambda \varepsilon^{(1)}_n + \lambda^2 \varepsilon^{(2)}_n + \cdots = \langle \phi_n | \hat{\lambda} | \phi_n \rangle + \sum_i \sum_{l=1}^{q_m} \lambda^2 \frac{K_{\phi_{i\alpha} | H' | \phi_{i\alpha}}^2}{E_0 - E_0} + \ldots \]

Recovering the result we found earlier. For higher order corrections this is necessary.

For degenerate eigenvalues \( E_0 \) we must determine the \( \phi_{i\alpha} \) state as well as \( E_n - E_0 \).
Recalling the eigenvalue equation on page 889:

\[
P_n \mathbf{H} \left\{ \frac{1}{1-R_n \mathbf{H}} \right\} \left| \textbf{1} \right\rangle_n = (E_n - E_0) \left| \textbf{1} \right\rangle_n
\]

where

\[
\left| \textbf{1} \right\rangle_n = \sum_{k=1}^{N_{\text{inf}}} C_{nk} \left| \Psi_{n,k} \right\rangle
\]

we have

\[
\sum_{k=1}^{N_{\text{inf}}} \left\langle \Psi_{n,k} | \mathbf{H} \left\{ \frac{1}{1-R_n \mathbf{H}} \right\} | \Psi_{n,k} \right\rangle C_{nk} = (E_n - E_0) C_{n1}
\]

\[
\sum_{k=1}^{N_{\text{inf}}} \left\{ \left\langle \Psi_{n,k} | \mathbf{H} \left\{ \frac{1}{1-R_n \mathbf{H}} \right\} | \Psi_{n,k} \right\rangle + \sum_{m=1}^{N_{\text{inf}}} \sum_{j=1}^{N_{\text{inf}}} \frac{\left\langle \Psi_{n,k} | \mathbf{H} \left\{ \frac{1}{1-R_n \mathbf{H}} \right\} | \Psi_{m,j} \right\rangle \left\langle \Psi_{m,j} | \mathbf{H} \left\{ \frac{1}{1-R_n \mathbf{H}} \right\} | \Psi_{n,k} \right\rangle}{E_n - E_m} \right\} C_{nk} = (E_n - E_0) C_{n1}
\]

As before this is a complicated equation since \(E_n\) appears on the LHS, thus we can only solve it approximately.

One approximation is the R-S scheme:

Then we expand in powers of \(\lambda\)
So we find to order \( \lambda^2 \) (note: \( C^{(1)}_{nl} = 2lne \))

\[
\sum_{h=1}^{Qn} \left[ \lambda \hat{A}^{(1)}_{lh} + \lambda^2 \hat{A}^{(2)}_{lh} \right] [2lne + \lambda C^{(1)}_{nl}]
\]

\[
= \left[ \lambda E^{(1)}_{nl} + \lambda^2 E^{(2)}_{nl} \right] [2lne + \lambda C^{(1)}_{nl}]
\]

\[
\Rightarrow \sum_{h=1}^{Qn} \hat{A}^{(1)}_{lh} 2lne = E^{(1)}_{nl} 2lne
\]

\[
\hat{A}^{(2)}_{lh} 2lne = E^{(2)}_{nl} + \left( \hat{A}^{(1)}_{lh} - E^{(1)}_{nl} \delta_{lh} \right) C^{(1)}_{nl}
\]

These are just our previous equations in R-S degenerate perturbation theory.

Of course the B-W expansion for the energy

\[
E_n = E_0 + \langle \phi_n | H | \phi_n \rangle
\]

\[
+ \sum_{m \neq n} \sum_{l=0}^{Qn} \frac{|\langle \phi_n | H | \phi_m \rangle|^2}{(E_n - E_m)} + \ldots
\]

and the eigenstate

\[
|\phi_n \rangle = \sum_l \sum_{\phi_m} R_{nl} |\phi_m \rangle + \ldots
\]
Although more difficult to solve than the RS approximation (we are summing an infinite [large] number of RS terms), they yield more accurate approximations to the energy levels.

Since these equations only implicitly define \( E_n \) due to its appearance on both sides of the equation, they are known as self-consistent determinations of \( E_n \). In fact, this is how we described the use of the energy level equation. We guess a value for \( E_n \), plug it into the equation, and see if the \( E_n \) we calculate is the same. If not, we continue the process until the results converge, determined.

Finally, let's consider a simple example comparing the RS and B-W schemes, which begins as a non-degenerate system whose parameters we can adjust to make degenerate.

**Example:** Consider a 2-level system with free Hamiltonian \( H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \) and
perturbing Hamiltonian $H' = \begin{bmatrix} 0 & N \\ N^\ast & 0 \end{bmatrix}$ so that $H = H_0 + H'$.

This problem can be solved exactly. The energy eigenvalues are

$$\det \begin{vmatrix} \epsilon_i - E & N \\ N^\ast & \epsilon_j - E \end{vmatrix} = 0$$

$$\Rightarrow E^2 - (\epsilon_i + \epsilon_j)E + \epsilon_i \epsilon_j - |N|^2 = 0$$

$$\Rightarrow E = \frac{1}{2} (\epsilon_i + \epsilon_j) \pm \frac{1}{2} \sqrt{(\epsilon_i - \epsilon_j)^2 + 4|N|^2}$$

Thus the exact energy eigenvalues are

$$E_1 = \frac{1}{2} (\epsilon_i + \epsilon_j) + \frac{1}{2} \sqrt{(\epsilon_i - \epsilon_j)^2 + 4|N|^2}$$

$$E_2 = \frac{1}{2} (\epsilon_i + \epsilon_j) - \frac{1}{2} \sqrt{(\epsilon_i - \epsilon_j)^2 + 4|N|^2}$$

The unperturbed Hamiltonian $H_0$ has energy eigenvalues

$$E_1^0 = \epsilon_i$$

$$E_2^0 = \epsilon_j$$
The unperturbed eigenstates are

\[ | \psi_1 \rangle = |0\rangle ; \quad | \psi_2 \rangle = |1\rangle. \]

Since \( \langle \psi_2 | H' | \psi_2 \rangle = 0 \), we must use

second order R-S perturbation theory

to find, for instance,

\[
E_1^{\text{R-S}} = E_1 + \langle \psi_1 | H' | \psi_1 \rangle
+ \frac{| \langle \psi_1 | H' | \psi_2 \rangle |^2}{E_1^0 - E_2^0}
= E_1 + \frac{| 1 0 \rangle \langle 0 1 | \langle 1 1 |^2}{E_1 - E_2}

E_1^{\text{R-S}} = E_1 + \frac{| 1 0 \rangle \langle 0 1 |}{E_1 - E_2} \quad (\text{likewise we find } E_2^{\text{R-S}} = E_2 - \frac{1 0 |^2}{E_1 - E_2})

Clearly for almost degenerate unperturbed energy levels or degenerate ones.

\( E_1 \neq E_2 \) this becomes nonsensical.

We must use the complicated degenerate R-SPT.

For \( \frac{1 0 |}{E_1 - E_2} \ll 1 \), this is correct to second order for \( E_1 \) as seen by expanding the exact energy eigenstate.
Consider the degenerate R-S perturbed energy. So \( E_1 = E_2 = E \) and \( |\psi_1\rangle \) both have energy \( E \), \( H_0 |\psi_2\rangle = E |\psi_2\rangle \).

Thus the eigenstates of the system \( |\psi^\prime\rangle \) are given by

\[
|\psi^\prime\rangle = C_1 |\psi_1\rangle + C_2 |\psi_2\rangle
\]

and we must use the 1st order R-S equation to find \( C_1, C_2 \) and \( E^{(1)} \). This is simply the equation on page -845--

\[
\sum_{k=1}^{2} H'_k C_k = (E - E^0) C_k
\]

with \( E^0 = E \) and \( H'_k = \langle \psi_k | H' | \psi_k \rangle = \begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix} \)

So

\[
\begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = (E - E) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix} N C_2 \\ N^* C_1 \end{pmatrix} = (E - E) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow E - E = \pm \text{101}
\]

Thus the interacting energy levels are
\[ E_1 = \epsilon + 1/\delta \]
\[ E_2 = \epsilon - 1/\delta \]

The exact results.

The eigenstates corresponding to these energies are found from the above equation
\[ N C_2 = (E-\epsilon) C_1 \]

Then, with \( |2_{1/2}^n \rangle \) corresponding to \( E_2 \), we have
\[ C_{12} = \frac{E_1-\epsilon}{N} \quad C_{11} = \frac{1/\delta}{N} \quad C_{11} \]
\[ C_{22} = \frac{E_2-\epsilon}{N} \quad C_{21} = -\frac{1/\delta}{N} \quad C_{21} \]

Normalizing \( \langle 2_{1/2}^n | 2_{1/2}^n \rangle = 1 \) \( \Rightarrow \)
\[ |C_{11}|^2 + |C_{12}|^2 = 1 = |C_{21}|^2 + |C_{22}|^2 \]
\[ \Rightarrow \quad C_{11} = \sqrt{\frac{1}{2}} = C_{21} \]
So
\[ |\psi_1^+\rangle = c_{11} |\psi_1\rangle + c_{12} |\psi_2\rangle \]
\[ = \frac{1}{\sqrt{2}} \left( \frac{|1\rangle + |0\rangle}{\sqrt{2}} \right) \quad \text{corresponding to energy } E_1 \quad \text{and} \]
\[ |\psi_2^-\rangle = c_{21} |\psi_1\rangle + c_{22} |\psi_2\rangle \]
\[ = \frac{1}{\sqrt{2}} \left( \frac{|1\rangle - |0\rangle}{\sqrt{2}} \right) \quad \text{corresponding to energy } E_2. \]

Note: Since these energies are exact, \( |\psi_1^+\rangle \) and \( |\psi_2^-\rangle \) are exact.

\[ H |\psi_1^+\rangle = E_1 |\psi_1^+\rangle. \]

And clearly \( \langle \psi_1^- | \psi_2^+ \rangle = 0 \) as it must.
On the other hand B-W perturbation theory gives
\[
E^{B-W}_1 = E_1 + \frac{\langle \Phi_1 | H' | \Phi_1 \rangle}{E_1 - E_2} + \frac{\langle \Phi_1 | H' | \Phi_2 \rangle}{E_1 - E_2}^2
\]
\[
= E_1 + \frac{\langle 10 | \Phi_1 | 0 \rangle^2}{E_1 - E_2}
\]
\[
E^{B-W}_1 = E_1 + \frac{|V|^2}{E_1 - E_2}
\]

and likewise
\[
E^{B-W}_2 = E_2 + \frac{|V|^2}{E_2 - E_1}
\]

Thus multiplying these equations out we see that
\[
E^{B-W}_1 = E_1 \pm \frac{|V|^2}{2}
\]
\[
E^{B-W}_2 = E_2 \pm \frac{|V|^2}{2}
\]

The exact solutions.

In the degenerate \( E_1 = E_2 = \epsilon \) limit
\[
E_1 = \epsilon \pm 15|V|, \text{ the exact result.}
\]