5.3.8. The Wigner-Eckart Theorem and Irreducible Tensor Operators

Recall that we began our discussion of eigenstates of angular momentum by considering the classification of operators under the rotation of space, that is how they commuted with the angular momentum operator $\vec{J}$ since it is the generator of infinitesimal spatial rotations.

Thus we had

1) Scalar operators S so that

$$[\vec{J}, S] = 0$$ or equivalently

under finite rotations since $U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\theta}}$

$$S' \equiv U(R(\vec{\theta})) S U^{-1}(R(\vec{\theta})) = S$$

2) Vector operators $\vec{V}$ were such that

$$[J^i, V^j] = i\hbar \epsilon_{ijk} V^k$$

or under finite rotations

$$V_i' \equiv U(R(\vec{\theta}))\, V_i\, U^{-1}(R(\vec{\theta})) = R^{-1}_{ij}(\vec{\theta})\, V_j$$

$$= V_j\, R(\vec{\theta})_{ji}$$

3) Tensor operators of Rank n transform as products of vector operators

$$T'_{i_1 i_2 \cdots i_n} \equiv U(R(\vec{\theta}))\, T_{i_1 \cdots i_n}\, U^{-1}(R(\vec{\theta}))$$

$$= T_{j_1 \cdots j_n}\, R_{j_1 i_1}(\vec{\theta}) \cdots R_{j_n i_n}(\vec{\theta}),$$

or under infinitesimal transformation commute with $\vec{J}$ as a sum

$$[J_i, T_{i_1 \cdots i_n}] = i\hbar \epsilon_{i i_1 j_1} T_{j_1 i_2 \cdots i_n}$$

$$+ \cdots + i\hbar \epsilon_{i i_n j_n} T_{i_1 \cdots i_{n-1} j_n}.$$

We further found we could define Spinor functions using the Pauli matrix Representation of the rotation operators and hence we ~~may~~ also have spinor operators.

4) Spinor operators, $U_a,\ a=1,2$ transform as spin $\frac{1}{2}$ states so that

$$[J_i, \mathcal{U}_a] = \mathcal{U}_b J^{(\frac{1}{2})i}_{ba} = \mathcal{U}_b \left(\tfrac{\hbar}{2}\sigma^i\right)_{ba}$$

or equivalently for finite rotations

$$\mathcal{U}'_a \equiv U(R(\vec{\theta}))\,\mathcal{U}_a\, U^{-1}(R(\vec{\theta})) = \mathcal{U}_b\, D^{(\frac{1}{2})}_{ba}(R(\vec{\theta}))$$
$$= \mathcal{U}_b \left(e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\right)_{ba}$$

and similarly

5) Rank-n spinor operators

$\mathcal{U}_{a_1\cdots a_n}$ transform as the product of n, rank 1 spinors

$$\mathcal{U}'_{a_1\cdots a_n} \equiv U(R(\vec{\theta}))\,\mathcal{U}_{a_1\cdots a_n}\, U^{-1}(R(\vec{\theta}))$$
$$= \mathcal{U}_{b_1\cdots b_n}\, D^{(\frac{1}{2})}_{b_1a_1}(R(\vec{\theta}))\cdots D^{(\frac{1}{2})}_{b_na_n}(R(\vec{\theta}))$$

or

$$[J^i, \mathcal{U}_{a_1\cdots a_n}] = \tfrac{\hbar}{2}(\sigma^i)_{b_1a_1}\,\mathcal{U}_{b_1a_2\cdots a_n}$$
$$+\cdots+ \tfrac{\hbar}{2}(\sigma^i)_{b_na_n}\,\mathcal{U}_{a_1a_2\cdots a_{n-1}b_n}.$$

In general higher rank tensors contain tensors of lower rank. That is consider the second rank tensor T_{ij}

$$[J_i, T_{jk}] = i\hbar\epsilon_{ijm}T_{mk} + i\hbar\epsilon_{ikm}T_{jm}$$

Suppose we consider the trace of T_{ij}

$$T \equiv \sum_{j=1}^{3} T_{jj} \text{ ; then}$$

$$[J_i, T] = [J_i, T_{jj}]$$

$$= i\hbar\epsilon_{ijm}T_{mj} + i\hbar\epsilon_{ijm}T_{jm}$$

re-labelling the dummy (m, j) indices in the second term we have

$$= i\hbar\epsilon_{ijm}T_{mj} + i\hbar\underbrace{\epsilon_{imj}}_{= -\epsilon_{ijm}}T_{mj}$$

$$= 0.$$

Thus T is a scalar operator

$$[J_i, T] = 0.$$

Similarly consider

$$T_i \equiv \frac{1}{2}\epsilon_{ijk} T_{jk}$$

Then it follows that

$$[J, T_j] = i\hbar \epsilon_{ijk} T_k$$

T_j is a vector or rank 1 tensor operator.

Subtracting these operators from T_{ij}

$$\hat{T}_{ij} \equiv \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij} T_{kk}$$

we are left with an operator $\hat{T}_{ij}$ which contain no lower rank tensors among its components. It is an irreducible rank 2 tensor operator.

More precisely stated, an irreducible tensor operator with $(2j+1)$ components will transform according to the $(2j+1)$-dimensional representation of the angular momentum operator $\underline{\underline{J}}^{(j)}_{MM'}$.

This is analogous to how the standard basis vectors transform.

$$\vec{J}\,|k,j,m\rangle = \sum_{m'=-j}^{+j} \vec{J}^{(j)}_{m'm}\,|k,j,m'\rangle$$

or

$$U(R(\vec{\theta}))\,|k,j,m\rangle = \sum_{m'=-j}^{+j} |k,j,m'\rangle\, D^{(j)}_{m'm}(R(\vec{\theta})),$$

with $D^{(j)}_{m'm}(R(\vec{\theta})) = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}^{(j)}}\right)_{m'm}$.

Hence we define an irreducible tensor operator of rank (degree) j, $T^{(j)}_{m}$, where $m = -j, -j+1, \ldots, +j$ labels its $(2j+1)$-components, by its transformation law

$$U(R(\vec{\theta}))\,T^{(j)}_{m}\,U^{-1}(R(\vec{\theta})) = \sum_{m'=-j}^{+j} T^{(j)}_{m'}\, D^{(j)}_{m'm}(R(\vec{\theta}))$$

That is $T^{(j)}_{m}$ commutes with $\vec{J}$ according to

$$[\vec{J}, T^{(j)}_{m}] = \sum_{m'=-j}^{+j} T^{(j)}_{m'}\,\vec{J}^{(j)}_{m'm}$$

(Since we will concern ourselves with observables, the operators $T^{(j)}$ must be invariant under 2π rotations; thus we will only consider $j = 0, 1, 2, \ldots$ integers. Thus we need not consider irreducible spinor operators.)

The scalar operator S is a rank 0 irreducible ~~tensor~~ operator

$$[\vec{J}, S] = 0$$

; it transforms according to the spin 0 $j = 0$ representation of $\vec{J}$.

& The vector operator $\vec{V}$ is a rank 1 irreducible tensor operator

$$[\vec{J}, V_j] = \vec{J}^{(1)}_{kj} V_k$$

transforming according to the spin 1, $j = 1$, representation of $\vec{J}$.

Introducing the spherical components (raising & lowering) of $\vec{V}$

$$V_{\pm} \equiv (V_x \pm i V_y)\left(\frac{\mp 1}{\sqrt{2}}\right)$$

$$V_0 \equiv V_z$$

, the commutation

relations become

$$[J_{\pm,0}, V_{\pm,0}] = 0$$

$$[J_+, V_-] = 2\hbar V_z$$

$$[J_-, V_+] = -2\hbar V_z$$

$$[J_z, V_\pm] = \pm\hbar V_\pm \; ; \; [J_\pm, V_z] = \mp\hbar V_\pm .$$

Further since $\vec{J}\,|k,j,m\rangle = \vec{J}^{(j)}_{m'm}\,|k,j,m'\rangle$ is the same form as the irreducible tensor commutation relation

$$[\vec{J}, T^{(j)}_m] = \vec{J}^{(j)}_{m'm}\; T^{(j)}_{m'}$$

we have that

$$J_\pm\,|k,j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\;|k,j,m\pm1\rangle$$

$$J_z\,|k,j,m\rangle = m\hbar\,|k,j,m\rangle$$

and so $[J_\pm, T^{(j)}_m] = \hbar\sqrt{j(j+1) - m(m\pm1)}\; T^{(j)}_{m\pm1}$

$$[J_z, T^{(j)}_m] = \hbar m\, T^{(j)}_m .$$

The irreducible tensor operators have an extremely important property summarized by the

<u>Wigner-Eckart Theorem:</u>

In the standard $\{A, \vec{J}^2, J_z\}$ basis, $\{|k, j, m\rangle\}$, the matrix elements

$\langle k', J, M | T^{(j_2)}_{m_2} | k, j_1, m_1\rangle$ of the m_2^{th}-component of a rank j_2 irreducible tensor operator $T^{(j_2)}$ is given by

$$\langle k', J, M | T^{(j_2)}_{m_2} | k, j_1, m_1\rangle$$
$$= \langle k', J \| T^{(j_2)} \| k, j_1\rangle \langle j_1, j_2; m_1, m_2 | J, M\rangle.$$

The quantity $\langle k', J \| T^{(j_2)} \| k, j_1\rangle$ is called the reduced matrix element of $T^{(j_2)}$.

It is <u>independent</u> of m_1, M, m_2. The only m_1, m_2, M dependence of $\langle k', J, M | T^{(j_2)}_{m_2} | k, j_1, m_1\rangle$ is in the C-G coefficient $\langle j_1, j_2; m_1, m_2 | J, M\rangle$.

Before giving a general proof, consider the simplest example of the theorem, the case where $T = S$ a scalar operator. So

$$S = T^{(j=0)}_{m=0}$$. Thus the theorem states

$$\langle k', J, M | S | k, j_1, m_1 \rangle = \langle k', J \| S \| k, j_1 \rangle \times$$

$$\times \langle j_1, 0; m_1, 0 | J, M \rangle$$

but we are adding $0 = \vec{J}_2$ to $\vec{J}_1$:

$$| j_1, 0; m_1, 0 \rangle = | j_1, m_1 \rangle \otimes | 0, 0 \rangle$$

and $\langle j_1, 0; m_1, 0 | J, M \rangle = \delta_{J j_1} \delta_{M m_1}$.

Thus

$$\langle k', J, M | S | k, j_1, m_1 \rangle = \delta_{J j_1} \delta_{M m_1} \langle k', J \| S \| k, j_1 \rangle$$

The matrix el. is diagonal in J-j_1 and independent of (M, m_1). Hence in $\mathcal{H}(k, j_1)$ it is a constant times the identity.

(S)

We can see this result directly from the defining commutation relations of a scalar operator

1) $[J_z, S] = 0 \Rightarrow \langle k', j', m' | [J_z, S] | k, j, m \rangle$

$$= 0$$

$$= (m' - m)\hbar \langle k', j', m' | S | k, j, m \rangle$$

$$\Rightarrow \quad \langle k', j', m' | S | k, j, m \rangle = \delta_{m'm} S_{k'k}^{j'jm}$$

2) $[\vec{J}^2, S] = 0$

$$\Rightarrow 0 = \langle k', j', m' | [\vec{J}^2, S] | k, j, m \rangle$$

$$= [j'(j'+1) - j(j+1)]\hbar^2 \langle k', j', m' | S | k, j, m \rangle$$

$$\Rightarrow$$

$$\langle k', j', m' | S | k, j, m \rangle = \delta_{m'm} \delta_{j'j} S_{jm}(k', k)$$

It only remains to show that $S_{jm}(k', k)$ is independent of m.

Consider $[J_+, S] = 0 \Rightarrow$

$$0 = \langle k', j, m+1 | [J_+, S] | k, j, m \rangle$$

$$= \langle k', j, m+1 | J_-^{\dagger} S | k, j, m \rangle$$

$$- \langle k', j, m+1 | S J_+ | k, j, m \rangle$$

$$= \hbar \sqrt{j(j+1) - m(m+1)} \, \langle k', j, m | S | k, j, m \rangle$$

$$- \hbar \sqrt{j(j+1) - m(m+1)} \, \langle k', j, m+1 | S | k, j, m+1 \rangle$$

$\Rightarrow$

$$\langle k', j, m+1 | S | k, j, m+1 \rangle = \langle k', j, m | S | k, j, m \rangle$$

$\Rightarrow$

$$\boxed{S_{j\,m+1}(k', k) = S_{jm}(k', k)}$$

hence $S_{jm}(k', k)$ is independent of m

and

$$\boxed{\langle k', j', m' | S | k, j, m \rangle = \delta_{m'm} \delta_{j'j} \langle k', j' \| S \| k, j \rangle}$$

as the Wigner-Eckart Theorem stated.

Proof: Consider the vectors

$$T^{(j_2)}_{m_2}|k, j_1, m_1\rangle \quad \text{for } m_2 = -j_2, \ldots, +j_2,\ m_1 = -j_1, \ldots, +j_1$$

there are $(2j_2+1)(2j_1+1)$ of them. Further define the linear combinations of them

$$|k', J, M\rangle_T \equiv \sum_{\substack{m_2, m_1 \\ -j_1 \le m_1 \le +j_1 \\ -j_2 \le m_2 \le +j_2}}' T^{(j_2)}_{m_2}|k, j_1, m_1\rangle\langle j_1, j_2; m_1, m_2|J, M\rangle.$$

Applying the C-G orthogonality condition

$$\sum_{\substack{J, M \\ |j_1 - j_2| \le J \le j_1 + j_2 \\ -J \le M \le +J}}' \langle j_1, j_2; m_1, m_2|JM\rangle\langle j_1, j_2; m_1', m_2'|JM\rangle = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

(page -771-), we find the inverse formula

$$T^{(j_2)}_{m_2}|k, j_1, m_1\rangle = \sum_{J = |j_1 - j_2|}^{|j_1 + j_2|}{}' \sum_{M = -J}^{+J}{}' |k', J, M\rangle_T \langle j_1, j_2; m_1, m_2|J, M\rangle$$

Now we use the $J_\pm$ on these states

$$J_+ T^{(j_2)}_{m_2} |k, j_1, m_1\rangle = [J_+, T^{(j_2)}_{m_2}] |k, j_1, m_1\rangle + T^{(j_2)}_{m_2} J_+ |k, j_1, m_1\rangle$$

$$= \hbar\sqrt{j_2(j_2+1) - m_2(m_2+1)}\; T^{(j_2)}_{m_2+1} |k, j_1, m_1\rangle$$

$$+ \hbar\sqrt{j_1(j_1+1) - m_1(m_1+1)}\; T^{(j_2)}_{m_2} |k, j_1, m_1+1\rangle$$

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and so we have

$$J_+ |k', J, M\rangle_T = \sum_{m_1, m_2}{}' T^{(j_2)}_{m_2+1} |k, j_1, m_1\rangle \times$$

$$\times \left\{\hbar\sqrt{j_2(j_2+1) - m_2(m_2+1)}\, \langle j_1, j_2; m_1, m_2 | J, M\rangle\right\}$$

$$+ \sum_{m_1, m_2}{}' T^{(j_2)}_{m_2} |k, j_1, m_1+1\rangle \left\{\hbar\sqrt{j_1(j_1+1) - m_1(m_1+1)} \times\right.$$

$$\left.\times \langle j_1, j_2; m_1, m_2 | J, M\rangle\right\}$$

In the first term let $m_2+1 \to m_2$ so $m_2 \to m_2 - 1$

and in the second term $m_1+1 \to m_1$ so $m_1 \to m_1 - 1$ and m_1, m_2 sum stays the same due to either $\sqrt{\ }$ or Clebsch vanishing.

This becomes

$$J_+|k',J,M\rangle_T = \sum_{m_1,m_2}{}' T^{(j_2)}_{m_2} |k,j_1,m_1\rangle \times$$

$$\times\{\hbar\sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle j_1,j_2;m_1,m_2-1|J,M\rangle$$

$$+\hbar\sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle j_1,j_2;m_1-1,m_2|J,M\rangle\}.$$

But according to p. -773- the $\{\ \}$ terms are part of a C-G recursion formula,

$$\{\ \ \} = \hbar\sqrt{J(J+1)-M(M+1)}\,\langle j_1,j_2;m_1,m_2|J,M+1\rangle$$

Thus bringing out the (J,M) square root $\Rightarrow$

$$J_+|k',J,M\rangle_T = \hbar\sqrt{J(J+1)-M(M+1)}\ \times$$

$$\times \underbrace{\sum_{m_1,m_2}{}' T^{(j_2)}_{m_2} |k,j_1,m_1\rangle\langle j_1,j_2;m_1,m_2|J,M+1\rangle}_{=|k',J,M+1\rangle_T}$$

(but this is how we defined these states)

Thus we find

$$\boxed{\begin{aligned} &J_+|k',J,M\rangle_T = \hbar\sqrt{J(J+1)-M(M+1)}\,|k',J,M+1\rangle_T \\ &\text{and similarly we can show} \\ &J_-|k',J,M\rangle_T = \hbar\sqrt{J(J+1)-M(M-1)}\,|k',J,M-1\rangle_T \\ &\text{and} \\ &J_z|k',J,M\rangle_T = \hbar M|k',J,M\rangle_T \end{aligned}}$$

Hence we conclude that the $(2J+1)$ vectors $|k',J,M\rangle_T$ for fixed J are

1) either all zero, or

2) the (un-normalized) eigenstates of $\{\vec{J}^2, J_z\}$ with (J,M) eigenvalues and are obtained with lowering and raising operators one from the other, like the standard basis vectors.

Hence $\langle k,j,m|k',J,M\rangle_T$ are zero except for $j = J$, $M = m$, and these $(2J+1)$ inner products $\langle k', J, M | k', J, M \rangle_T$ are independent of M as we have shown for the usual standard basis using $J_+J_- = \vec{J}^2 - J_z(J_z - 1)$ & $J_-J_+ = \vec{J}^2 - J_z(J_z+1)$.

Thus the Wigner-Eckart theorem is proven; from $T^{(j_2)}_{m_2} | k, j_1, m_1 \rangle$ we have

$$\langle k', J, M | T^{(j_2)}_{m_2} | k, j_1, m_1 \rangle$$

$$= \sum_{J'=|j_1-j_2|}^{j_1+j_2} \sum_{M'=-J'}^{+J'} \langle k', J, M | k', J', M' \rangle_T \times$$

$$\times \langle j_1, j_2; m_1, m_2 | J', M' \rangle$$

$$= \underbrace{\langle k', J, M | k', J, M \rangle_T}_{} \langle j_1, j_2; m_1, m_2 | J, M \rangle$$

$$\equiv \langle k', J \| T^{(j_2)} \| k, j_1 \rangle \text{ (independent of } m_1, m_2, M)$$

$$= \langle k', J \| T^{(j_2)} \| k, j_1 \rangle \langle j_1, j_2; m_1, m_2 | J, M \rangle.$$

Thus we have an extremely important consequence of the Wigner-Eckart theorem, the <u>selection rules</u> for the operator $T^{(j_2)}_{m_2}$:

<u>Corollary</u>:
For the matrix element $\langle k', J, M | T^{(j_2)}_{m_2} | k, j_1, m_1 \rangle$ to be non-zero, we must simultaneously have

$$m_1 + m_2 = M$$

$$|j_1 - j_2| \le J \le j_1 + j_2$$

which by the symmetry of the triangle rule we can write as

$$|J - j_1| \le j_2 \le J + j_1 \quad .$$

These rules follow immediately from the fact that the C-G is non-zero only for these conditions.

Just as we considered products of states to obtain the C-G expansion

$$|k_1, k_2; j_1, j_2; J, M\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |k_1, k_2; j_1, j_2; m_1, m_2\rangle \times$$
$$\times \langle j_1, j_2; m_1, m_2 | J, M\rangle$$

and its inverse

$$|k_1,k_2;j_1,j_2;m_1,m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{+J} |k_1,k_2;j_1,j_2;J,M\rangle \times$$

$$\times \langle j_1,j_2;m_1,m_2|J,M\rangle$$

and just as we considered the product of an irreducible tensor $T^{(j_2)}_{m_2}$ with a state $|k,j_1,m_1\rangle$ to obtain the Wigner-Eckart expansion

$$T^{(j_2)}_{m_2}|k,j_1,m_1\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{+J} |k',J,M\rangle_T \langle j_1,j_2;m_1,m_2|J,M\rangle$$

and

$$|k',J,M\rangle_T = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} T^{(j_2)}_{m_2}|k,j_1,m_1\rangle\langle j_1,j_2;m_1,m_2|J,M\rangle$$

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we can consider the product of <u>2</u> irreducible tensor operators. In general the product is not irreducible; just like the product of 2 spin ½ states is not irreducible it is the <u>sum</u> of a spin 0 and a spin 1 state. We can use the Clebschs again to find the sum of irreducible ~~tensor~~ operators that are included in the product. The analogy is for $\vec{J} = \vec{J}_1 + \vec{J}_2$ we have that $|k_1, k_2; j_1, j_2; J, M\rangle$ was an total angular momentum eigenstate. The C-G coefficients told us how to extract from the product eigen states of $\vec{J}_1$, $\vec{J}_2$ this total $\vec{J}$ eigenstate. In general we had

$$\vec{J}\,|k_1,k_2;j_1,j_2;J,M\rangle = \sum_{m_1=-j_1}^{+j_1}\sum_{m_2=-j_2}^{+j_2}\langle j_1,j_2;m_1,m_2|J,M\rangle \times$$

$$\times\left\{\vec{J}_1|k_1,j_1,m_1\rangle \otimes |k_2,j_2,m_2\rangle + |k_1,j_1,m_1\rangle \otimes \vec{J}_2|k_2,j_2,m_2\rangle\right\}$$

$$= \sum_{m_1=-j_1}^{+j_1}\sum_{m_2=-j_2}^{+j_2}\langle j_1,j_2;m_1,m_2|J,M\rangle \times$$

$$\times\left\{\sum_{m_1'=-j_1}^{+j_1}\sum_{m_2'=-j_2}^{+j_2} \vec{J}^{(j_1)}_{m_1'm_1}\otimes\delta_{m_2'm_2} + \delta_{m_1'm_1}\otimes\vec{J}^{(j_2)}_{m_2'm_2}\right\}$$

$$\times\left(|k_1,j_1,m_1'\rangle \otimes |k_2,j_2,m_2'\rangle\right)$$

While $\vec{J}|k_1,k_2;j_1,j_2;J,M\rangle = \sum_{M'=-J}^{+J} \vec{J}^{(J)}_{M'M} |k_1,k_2;j_1,j_2;J,M'\rangle$.

Similarly we considered the Wigner-Eckart theorem, in general

$$\vec{J}\left(T^{(j_2)}_{m_2}|k,j_1,m_1\rangle\right) = [\vec{J},T^{(j_2)}_{m_2}]|k,j_1,m_1\rangle + T^{(j_2)}_{m_2}\vec{J}|k,j_1,m_1\rangle$$

and $[\vec{J},T^{(j_2)}_{m_2}] = \sum_{m_2'=-j_2}^{+j_2} \vec{J}^{(j_2)}_{m_2'm_2} T^{(j_2)}_{m_2'}$ as we have by definition; So

$$\vec{J}\left(T^{(j_2)}_{m_2}|k,j_1,m_1\rangle\right) = \sum_{m_1'=-j_1}^{+j_1}\sum_{m_2'=-j_2}^{+j_2} \times$$

$$\times\left\{\vec{J}^{(j_1)}_{m_1'm_1}\otimes\delta_{m_2'm_2} + \delta_{m_1'm_1}\otimes\vec{J}^{(j_2)}_{m_2'm_2}\right\}\times$$

$$\times\left(T^{(j_2)}_{m_2'}|k,j_1,m_1'\rangle\right)$$

The Clebschs tell us how to combine these states to find an eigenstate of $\vec{J}$ again; namely

$$|k', J, M\rangle_T = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \left(T^{(j_2)}_{m_2} |k, j_1, m_1\rangle\right) \times$$
$$\times \langle j_1, j_2; m_1, m_2 | J, M\rangle .$$

Thus we can use the C-G coefficients to extract a irr. tensor operator from a product. Let

$U^{(j_1)}_{m_1}$ ($V^{(j_2)}_{m_2}$) be the m_1 (m_2) components of a rank j_1 (j_2) irreducible tensor operator. We have that

$$[\vec{J}, U^{(j_1)}_{m_1} V^{(j_2)}_{m_2}] = [\vec{J}, U^{(j_1)}_{m_1}] V^{(j_2)}_{m_2} + U^{(j_1)}_{m_1} [\vec{J}, V^{(j_2)}_{m_2}]$$

$$= \sum_{m_1'=-j_1}^{+j_1} \sum_{m_2'=-j_2}^{+j_2} \left\{ \vec{J}^{(j_1)}_{m_1' m_1} \otimes \delta_{m_2' m_2} + \delta_{m_1' m_1} \otimes \vec{J}^{(j_2)}_{m_2' m_2} \right\} U^{(j_1)}_{m_1'} V^{(j_2)}_{m_2'}$$

Thus we know how to combine the different components of $U^{(j_1)}V^{(j_2)}$ to form an irreducible tensor operator of rank J i.e. so that

$$[\vec{J}, T_M^{(J)}] = \sum_{M'=-J}^{+J} \vec{J}^{(J)}_{M'M} \, T^{(J)}_{M'} .$$

Let the irred. tensor operator be defined by

$$T_M^{(J)} \equiv \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} U^{(j_1)}_{m_1} V^{(j_2)}_{m_2} \langle j_1 j_2; m_1, m_2 | J, M \rangle$$

(where $M = m_1 + m_2$ and $|j_1 - j_2| \le J \le j_1 + j_2$).

Then indeed $[\vec{J}, T_M^{(J)}] = \sum_{M'=-J}^{+J} \vec{J}^{(J)}_{M'M} \, T^{(J)}_{M'}$, as required of a rank J irreducible tensor operator.

Example: We can make a scalar operator $S = T^{(J=0)}_{M=0}$ from 2 rank j irred. operators by

$$S = \sum_{m=-j}^{+j} U^{(j)}_{+m} V^{(j)}_{-m} \langle j, j; m, -m | 0, 0 \rangle$$

The C-G coefficient for the singlet state $|0,0\rangle$ is given by

$$\langle j,j;m,-m|0,0\rangle = (2j+1)^{-1/2}(-1)^{j-m}$$

Since for $m=j$ the coefficient is positive by convention; and $|0,0\rangle$ is normalized ~~to~~ 1; since there are $(2j+1)$ vectors $|j,m\rangle \otimes |j,-m\rangle$ we divide by $\frac{1}{\sqrt{2j+1}}$ and they are symmetric under interchange for j = integer. Hence

$$\boxed{S = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m} U^{(j)}_{+m} V^{(j)}_{-m}}\,.$$

If $U^{(j)}, V^{(j)}$ are vector operators, $j=1$, then

$$S = \frac{-1}{\sqrt{2\cdot 1+1}} \sum_{m=-1}^{+1} (-1)^{m} U^{(1)}_{m} V^{(1)}_{-m}$$

$$= \frac{-1}{\sqrt{3}}\left(-U^{(1)}_{-1}V^{(1)}_{+1} + U^{(1)}_{0}V^{(1)}_{0} - U^{(1)}_{+1}V^{(1)}_{-1}\right)$$

~~But~~ let $U^{(1)}_{\pm} = (U_x \pm i U_y)\frac{\pm 1}{\sqrt{2}}$

$U^{(1)}_0 = U_z$, same for $V^{(1)}$

Thus

$$S = -\frac{1}{\sqrt{3}}\left[U_x V_x + U_y V_y + U_z V_z\right]$$

and we have $S = -\frac{1}{\sqrt{3}} \vec{U}\cdot\vec{V}$, just what we would expect.

Finally let's consider the case of determining the matrix elements of the operator $\vec{J}\cdot V$ where $\vec{J}$ is the ∡ momentum operator and $\vec{V}$ is a vector operator. $\vec{J}\cdot\vec{V}$ is a scalar operator, clearly, and its matrix ~~are~~ is diagonal in the standard basis,

$$\langle k', j, m | \vec{J}\cdot\vec{V} | k, j, m\rangle$$
$$= \sum_{m'=-j=-1}^{+j=+1} \langle k', j, m | (-1)^{m'} J^{(1)}_{m'} V^{(1)}_{-m'} | k, j, m\rangle$$

where $J^{(1)}_{\pm} = J_x \pm iJ_y$; $J_z = J_0$

and now $V^{(1)}_{\pm} \equiv \frac{\pm 1}{2}(V_x \pm iV_y)$; $V^{(1)}_0 \equiv V_z$

inserting a complete set of states

$$= \sum_{k''} \sum_{J,M} \sum_{m'=-j}^{+j} (-1)^{m'} \langle k', j, m | J^{(1)}_{m'} | k'', J, M \rangle \times \langle k'', J, M | V^{(1)}_{-m'} | k, j, m \rangle$$

Now $\langle k', j, m | J^{(1)}_{m'} | k'', J, M \rangle \propto \delta_{k'k''} \delta_{jJ}$

So

$$= \sum_{M=-j}^{+j} \sum_{m'=-j}^{+j} (-1)^{m'} \langle k', j, m | J^{(1)}_{m'} | k', j, M \rangle \times \langle k', j, M | V^{(1)}_{-m'} | k, j, m \rangle$$

further $\langle k', j, m | J^{(1)}_{m'} | k', j, M \rangle$ is independent of k'

So

$$\langle k', j, m | \vec{J} \cdot \vec{V} | k, j, m \rangle = \sum_{M=-j}^{+j} \sum_{m'=-j}^{+j} (-1)^{m'} \langle j, m | J^{(1)}_{m'} | j, M \rangle \times \langle k', j, M | V^{(1)}_{-m'} | k, j, m \rangle.$$

Next we apply the Wigner-Eckart theorem to the $\vec{V}$ matrix elements

$$\langle k',j,M|V^{(1)}_{-m'}|k,j,m\rangle = \langle k',j\|V^{(1)}\|k,j\rangle \times$$
$$\times \langle j,1;m,-m'|j,M\rangle$$

So

$$\langle k',j,m|\vec{J}\cdot\vec{V}|k,j,m\rangle$$
$$= \sum_{M=-j}^{+j}\sum_{m'=-j}^{+j} \overbrace{(-1)^{m'}\langle j,m|J^{(1)}_{m'}|j,M\rangle\langle j,1;m,-m'|j,M\rangle}^{\text{independent of } k,\, k' \text{ and } \vec{V}}$$
$$\times \underbrace{\langle k',j\|V^{(1)}\|k,j\rangle}_{\text{independent of } m,\, m',\, M}$$

$$\equiv C_{jm}\ \langle k',j\|V^{(1)}\|k,j\rangle$$

where C_{jm} is independent of k, k' and $\vec{V}$.

Thus we can find C_{jm} using any vector operator; let $\vec{V}=\vec{J}$

Then

$$\underbrace{\langle k', j, m | \vec{J}^2 | k, j, m \rangle}_{= j(j+1)\hbar^2 \delta_{kk'}} = c_{jm} \langle k', j \| J^{(1)} \| k, j \rangle$$

and we have

$$\langle k', j, m | J^{(1)}_{m_2} | k, j, m_1 \rangle = \langle k', j \| J^{(1)} \| k, j \rangle \times$$
$$\times \langle j, 1; m_1, m_2 | j, m \rangle$$

So using C-G coefficients and $J_{\pm, 0}$ we have

$$\langle j, 1; j, 0 | j, j \rangle = \sqrt{\frac{j}{j+1}} \implies$$
$$\langle k', j \| J^{(1)} \| k, j \rangle = \hbar \sqrt{j(j+1)}\, \delta_{k'k} .$$

Thus

$$c_{jm} = \hbar \sqrt{j(j+1)} . \quad \text{Thus}$$

$$\boxed{\langle k', j, m | \vec{J} \cdot \vec{V} | k, j, m \rangle = \hbar \sqrt{j(j+1)} \langle k', j \| \vec{V} \| k, j \rangle}$$

So we can use this to find the general matrix elements of $\vec{V}$.

The Wigner - Eckart theorem states

$$\langle k', J, M | V^{(1)}_{m_2} | k, j_1, m_1 \rangle = \langle k', J \| \vec{V} \| k, j_1 \rangle \times$$
$$\times \langle j_1, 1; m_1, m_2 | J, M \rangle$$

$$\langle k', J, M | J^{(1)}_{m_2} | k, j_1, m_1 \rangle = \langle k', J \| \vec{J} \| k, j_1 \rangle \times$$
$$\times \langle j_1, 1; m_1, m_2 | J, M \rangle$$

(zero unless $k = k'$; $J = j_1$ for $\vec{J}$)

Thus

$$\langle k', J, M | V^{(1)}_{m_2} | k, j_1, m_1 \rangle = \frac{\langle k', J \| \vec{V} \| k, j_1 \rangle}{\langle k, j_1 \| \vec{J} \| k, j_1 \rangle} \times$$
$$\times \langle k, j_1, M | J^{(1)}_{m_2} | k, j_1, m_1 \rangle$$

---

or for $J = j_1 = j$ and $\langle k, j \| \vec{J} \| k, j \rangle = \hbar \sqrt{j(j+1)}$
we have

$$\langle k', j, M | V^{(1)}_{m_2} | k, j, m_1 \rangle = \frac{\langle k', j \| \vec{V} \| k, j \rangle}{\hbar \sqrt{j(j+1)}} \times$$
$$\times \langle k, j, M | J^{(1)}_{m_2} | k, j, m_1 \rangle$$

From above we find

$$\boxed{\langle k',j,M|V^{(1)}_{m_2}|k,j,m\rangle = \frac{\langle k',j,m|\vec{J}\cdot\vec{V}|k,j,m\rangle}{\hbar^2 j(j+1)} \times \langle k,j,M|J^{(1)}_{m_2}|k,j,m\rangle}$$

That is the matrix elements of any vector operator ~~are~~ proportional to the matrix elements of the angular momentum operator

$$\boxed{\langle k',j,m'|\vec{V}|k,j,m\rangle = \frac{\langle k',j|\vec{J}\cdot\vec{V}|k,j\rangle}{\hbar^2 j(j+1)} \times \langle k,j,m'|\vec{J}|k,j,m\rangle}$$

This is another way to state the Wigner-Eckart theorem for vector operators.

Wigner-Eckart Theo. for Vector Operators

$$\vec{V} = c_{kj} \vec{J} \quad \text{on each } \mathcal{H}(k,j)$$

$$\Rightarrow \quad \vec{V}\cdot\vec{J} = c_{kj} \vec{J}^2$$

$$\Rightarrow \quad \langle k',j,m' | \vec{J}\cdot\vec{V} | k,j,m \rangle = c_{kj} \langle k',j,m' | \vec{J}^2 | k,j,m \rangle$$

$$= c_{kj}\, \delta_{kk'}\, \hbar^2 j(j+1)\, \delta_{mm'}$$

$$\Rightarrow \quad c_{kj} = \frac{\langle k,j,m | \vec{J}\cdot\vec{V} | k,j,m \rangle}{\hbar^2 j(j+1)} \Rightarrow$$

$$\boxed{\vec{V} = \frac{\langle k,j,m | \vec{J}\cdot\vec{V} | k,j,m \rangle}{\hbar^2 j(j+1)} \vec{J}}$$

<u>Proof</u>: $J_\pm = J_x \pm i J_y \qquad V_z = V_0$

$V_\pm = V_x \pm i V_y \qquad J_z = J_0$

Vector op. $\Rightarrow \quad [J_i, V_j] = i\hbar \epsilon_{ijk} V_k$

$$\Rightarrow \left.\begin{aligned} &[J_+, V_-] = 2\hbar V_0 \\ &[J_-, V_+] = -2\hbar V_0 \\ &[J_z, V_\pm] = \pm\hbar V_\pm \\ &[J_\pm, V_0] = \mp\hbar V_\pm \end{aligned}\right\} \quad \text{all others} = 0$$

1) So $[J_0, V_0] = 0 \Rightarrow$

$$0 = \langle k', j', m' | [J_0, V_0] | k, j, m \rangle$$

$$\Rightarrow \quad 0 = \hbar (m' - m) \langle k', j', m' | V_0 | k, j, m \rangle$$

$$\Rightarrow \quad \langle k', j', m' | V_0 | k, j, m \rangle \propto \delta_{mm'}$$

2) $[J_0, V_\pm] = \pm \hbar V_\pm$

$$\Rightarrow \langle k', j', m' | V_\pm | k, j, m \rangle \propto \delta_{m', m\pm 1}$$

3) use $[J_+, V_+] = 0 \Rightarrow$ after work, use Complete set of states

$$\Rightarrow \quad \frac{\langle k, j, m+1 | V_+ | k, j, m \rangle}{\langle k, j, m+1 | J_+ | k, j, m \rangle} = C^+_{k,j} = \text{const. indep. of } m$$

$$\Rightarrow \quad \langle k, j, m' | V_+ | k, j, m \rangle = C^+_{k,j} \langle k, j, m' | J_+ | k, j, m \rangle$$

4) $[J_-, V_-] = 0 \Rightarrow$ same with V_-, $C^-_{k,j}$, J_-.

5) $[J_-, V_+] = -2\hbar V_z \Rightarrow \langle k, j, m' | V_z | k, j, m \rangle = C^+_{k,j} \langle k, j, m' | J_z | k, j, m \rangle$

5') $[J_+, V_-] = 2\hbar V_z \Rightarrow \quad = C^-_{k,j} \langle \ \rangle$

$$\Rightarrow C^+ = C^- = C_{k,j}$$

$$\Rightarrow \boxed{\langle k,j,m' | \vec{V} | k,j,m \rangle = C_{k,j} \langle k,j,m' | \vec{J} | k,j,m \rangle}$$

$$\boxed{\underline{\text{i.e.}} \text{ on } \mathcal{H}(k,j) \quad \vec{V} = C_{k,j} \vec{J}}$$

$$C_{k,j} = \frac{\langle k,j,m | \vec{V}\cdot\vec{J} | k,j,m \rangle}{\hbar^2 j(j+1)} \longleftarrow \text{indep. of } m.$$

$$\text{So } \langle k,j,m | \vec{J}\cdot\vec{V} | k,j,m \rangle = \frac{1}{2j+1} \sum_{m=-j}^{+j} \langle k,j,m | \vec{J}\cdot\vec{V} | k,j,m \rangle$$

$$\equiv \langle \vec{J}\cdot\vec{V} \rangle_{k,j}.$$

average value.

Lastly, consider the case of an atom whose magnetic moment $\vec{\mu}$ has a piece proportional to the spin operator $\vec{S}$ and a piece proportional to the orbital angular momentum operator $\vec{L}$

$$\vec{\mu} = \frac{e\hbar}{2mc}\left[g_L \vec{L} + g_s \vec{S}\right]$$

for		g_L	g_S
	protons	$+1$	2 (2.79)
	neutrons	0	2 (-1.91)
	electrons	$+1$	$2\left(1+\frac{\alpha}{2\pi}+\cdots\right)$

The matrix elements of $\vec{\mu}$ in a state of definite $(\vec{J}^2, J_z)$ where $\vec{J} = \vec{L}+\vec{S}$ are by the above theorem proportional to the matrix elements of $\vec{J}$.

$$\langle k,j,m'|\vec{\mu}|k,j,m\rangle \equiv \frac{-e}{2mc} g_{eff} \langle j,m'|\vec{J}|j,m\rangle$$

from above we have that

$$g_{eff} = \frac{\langle k,j,m|\vec{\mu}\cdot\vec{J}|k,j,m\rangle}{\hbar^2 j(j+1)}$$

$$= \frac{\langle k,j,m|(g_L \vec{L}\circ\vec{J} + g_S \vec{S}\circ\vec{J})|k,j,m\rangle}{\hbar^2 j(j+1)}$$

(Clearly, once we use the result that matrix elements of $\vec{\mu}$ are proportional to matrix elements of $\vec{J}$ we have

$$\vec{\mu} = c\vec{J} \quad . \text{ Thus } \vec{\mu}\circ\vec{J} = c\vec{J}^2$$

but the $\mathcal{H}(k,j)$ matrix elements have $\vec{J}^2$ eigenvalue $j(j+1)\hbar^2$ Thus we have in $\mathcal{H}(k,j)$ that

$$\vec{\mu} = \frac{\langle\vec{\mu}\circ\vec{J}\rangle}{\hbar^2 j(j+1)}\vec{J} \Bigg).$$

Now $\vec{L}\circ\vec{J} = \frac{1}{2}(\vec{J}^2 + \vec{L}^2 - \vec{S}^2)$ and

$$\vec{S}\circ\vec{J} = \frac{1}{2}(\vec{J}^2 + \vec{S}^2 - \vec{L}^2) \quad \text{so}$$

$$g_{eff} = \frac{\langle k,j,m|\{(g_L+g_S)\vec{J}^2 + (g_L-g_S)(\vec{L}^2-\vec{S}^2)\}|k,j,m\rangle}{2\hbar^2 j(j+1)}$$

Now recall $|k,j,m\rangle = |k;l,s;j,m\rangle$, so assuming the atom is also in an eigenstate of $\vec{L}^2, \vec{S}^2$, we have

$$\boxed{\begin{aligned} 2g_{eff} &= (g_L+g_s) + (g_L-g_s)\frac{[l(l+1)-s(s+1)]}{j(j+1)} \\ &= \frac{(g_L+g_s)j(j+1) + (g_L-g_s)[l(l+1)-s(s+1)]}{j(j+1)} \\ &= g_L\frac{[j(j+1)+l(l+1)-s(s+1)]}{j(j+1)} \\ &\quad + g_s\frac{[j(j+1)+s(s+1)-l(l+1)]}{j(j+1)} \end{aligned}}$$

g_{eff} is the atomic Landé g-factor.